A Model for Mixtures of Micromagnetic Materials
allowing Existence and Regularity

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Abstract. In this paper a realistic model for mixtures of magnetic materials is presented, and an application to an optimal design problem is also given.

1. Energies involved

In the Weiss-Landau-Lifschitz model of the micromagnetic theory (which we do not want to discuss here in detail, referring the reader e.g. to [6]), the total energy associated with a single crystal (or "grain") of a magnetic material is given by the sum of several contributions. Every point (every atom in the real, discrete world) in the body $\Omega \subset \mathbb{R}^3$ is magnetized, i.e., it generates a magnetic field, and we may describe it with a vector field

$$\mathbf{m} : \Omega \to \mathbb{R}^3$$

called *magnetization.* Below a certain temperature characteristic of the material (Curie temperature) the modulus of the magnetization is constant,

$$|\mathbf{m}| \equiv m^s \qquad \text{in } \Omega \ ,$$

where the *magnetic saturation* intensity m^s is also a characteristic of the material. Throughout the paper it is assumed that we are below the Curie temperature of each material employed.

The first energy term is the *magnetic exchange energy,* generated by the wish of adjacent points to share the same orientation of $\mathbf{m}$: since $|\mathbf{m}|$ is constant, the exchange energy depends on the gradient matrix $\nabla\mathbf{m}$ through a four-indices tensor A, also characteristic of the material, and is in general given by $\int_\Omega \langle A\,\nabla\mathbf{m}, \nabla\mathbf{m}\rangle \, dx$. A very good approximation, which is commonly assumed, is that A is close to being a multiple of the identity, thus we set

$$\text{Exch} = \int_\Omega a|\nabla\mathbf{m}|^2 \, dx \ ,$$

where the constant a is another characteristic of the material: when a is small the exchange energy is low, and we are in presence of a "soft" magnetic material

as e.g. iron; when a is large, as in permanent magnets, the magnetic material is
"hard".

A second character on the energy stage is the *anysotropic energy:* due to the
structure of the crystal, there are some alignments of the magnetization $\mathbf{m}$ (the
easy axes) which are preferred with respect to others: there is one direction (and
its opposite) for uniaxial crystals, there are more for different symmetry groups.
This is generally described as the integral of a polynomial in $\mathbf{m}$, but we generalize
it here as

$$\text{Anys} = \int_\Omega \phi(\mathbf{m})\,dx\ ,$$

where the continuous function

$$\phi : \partial B_{m^s} \to [0, +\infty[$$

depends on the material. In order to describe (in a qualitative way) the interplay
of these two energies, rescale for a while the function ϕ by setting $\kappa = \max \phi$ and
$\psi = \phi/\kappa$, so the two energy terms are

$$\text{Exch} + \text{Anys} = \int_\Omega [a|\nabla m|^2 + \kappa \psi(\mathbf{m})]\,dx\ .$$

Now consider a rod of uniaxial crystal, and assume that at the two ends the
magnetization is along the easy axis, but points in the two opposite directions: the
anysotropic energy wants $\mathbf{m}$ to stay preferably in these two directions, whereas
the exchange energy term favours a slow transition. What conceivebly happens is
that $\mathbf{m}$ will stay almost constant in two extremal portions of the rod, and there
will be a transition layer between the two orientations; having a large and κ small
leads to a slow transition, thus a thick interface, and having a small and κ large
makes for sudden transition and narrow interface. According to one of the theories
about how the transition occurs, it is easy to compute that the thickness of the
interface is proportional to $\sqrt{a/\kappa}$ and that the total energy contribution of the
layer is proportional to

$$\sqrt{a\kappa}\ \cdot\ \text{area of cross-section;}$$

in other theories the numbers are different, but the area of the cross-section re-
mains. The fact that this really happens may be seen in any magnetic material:
the grain ends up divided into a lot of islands where the orientation of $\mathbf{m}$ is con-
stant, the *magnetic domains,* separated by thin layers where all the transitions
take place, the *Bloch walls* (according to other theories, there are other kinds of
walls, such as e.g. *Néel walls,* to which some of our considerations apply only on
a large-scale level). To get an idea of the dimensions involved, the diameter of
magnetic domains is in the range of one tenth of a micron to millimeters, whereas
the thickness of the walls is of about 10 to 100 atomic layers. We remark that
many times, when doing explicit computations, the sum Exch + Anys is replaced
by a constant (or a function of the magnetizations on the two sides) times the area
of the middle section of the wall. We will later generalize the model in order to
include all these variations.

Another, easy term in the expression of the energy is due to the *Zeeman field*, the influence of the external magnetic field $\mathbf{f}$ (to simplify the notation we dropped a few constants):

$$\text{Ext} = -\int_\Omega \mathbf{f} \cdot \mathbf{m} \, dx \ ,$$

of which little has to be said: $\mathbf{m}$ will try to align with $\mathbf{f}$.

The last term which is classically considered is the magnetostatic, or *demagnetizing energy*: $\mathbf{m}$ itself generates a magnetic field in the whole space, $\mathbf{h}[\mathbf{m}]$, which the magnetostatic equations give as

$$\begin{cases} \text{curl } \mathbf{h} = \mathbf{0} & \text{in } \mathbb{R}^3 \\[2mm] \text{div } (\mathbf{h} + \mathbf{m}\mathbb{I}_\Omega) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{1}$$

These are to be interpreted, in a weak sense, as

$$\begin{cases} \mathbf{h} \in L^2(\mathbb{R}^3; \mathbb{R}^3) \ , & \text{curl } \mathbf{h} = \mathbf{0} \ , \\[2mm] \displaystyle\int_{\mathbb{R}^3} \mathbf{h} \cdot \mathbf{v} \, dx = -\int_\Omega \mathbf{m} \cdot \mathbf{v} \, dx & \forall \mathbf{v} \in L^2(\mathbb{R}^3; \mathbb{R}^3) \text{ such that curl } \mathbf{v} = \mathbf{0} \ ; \end{cases}$$

then the demagnetizing energy (which is a nonlocal term) is given by

$$\text{Demag} = \int_{\mathbb{R}^3} |\mathbf{h}[\mathbf{m}]|^2 \, dx \ .$$

Since (1) holds in $\mathbb{R}^3$, the generated magnetic field is zero if $\mathbf{m}$ is divergence-free and at the same time tangent to the boundary of Ω, whereas it is large if $\mathbf{m}$ has constant direction: thus the demagnetizing energy has large effects and heavily interferes with the exchange and anysotropic energies, which have opposite wishes regarding the alignment of $\mathbf{m}$. It is not the purpose of this paper to describe the interesting microstructure problems arising from this situation, for which we refer e.g. to [5].

The mapping $\mathbf{m} \mapsto \mathbf{h}[\mathbf{m}]$ has some interesting properties (see [4]): it is linear, continuous from $L^p(\Omega)$ to $L^p(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} |\mathbf{h}[\mathbf{m}]|^2 \, dx = -\int_\Omega \mathbf{m} \cdot \mathbf{h}[\mathbf{m}] \, dx \ ;$$

from this formula we deduce in particular that

$$\mathbf{m} \mapsto \int_{\mathbb{R}^3} |\mathbf{h}[\mathbf{m}]|^2 \, dx \qquad \text{is continuous from } L^2(\Omega; \mathbb{R}^3) \text{ to } \mathbb{R} \ .$$

Also, although the term is nonlocal, some local estimates may be recovered: if two magnetizations agree outside a ball, i.e. if $\mathbf{m} = \mathbf{m}'$ outside $B_\varrho \subset \Omega$, we have

$$\int_{\mathbb{R}^3} |\mathbf{h}[\mathbf{m}] - \mathbf{h}[\mathbf{m}']|^2 \, dx = \int_{\mathbb{R}^3} |\mathbf{h}[\mathbf{m} - \mathbf{m}']|^2 \, dx \leq c \|\mathbf{m} - \mathbf{m}'\|_{L^2}^2 \leq c\varrho^3$$

since $|\mathbf{m}| = |\mathbf{m}'| = m^s$; also, by the continuity of ϕ on the compact set ∂B_s

$$\int_\Omega |\phi(\mathbf{m}) - \phi(\mathbf{m}')| \, dx \leq c\varrho^3 \ :$$

both estimates are extremely useful when proving regularity.

The total energy associated with a magnetization $\mathbf{m}$ of a single crystal is given by the sum of the four terms we discussed, i.e.,

$$\begin{aligned} E(\mathbf{m}) \quad &= \quad \text{Exch} + \text{Anys} + \text{Ext} + \text{Demag} \\ &= \quad \int_\Omega [a|\nabla\mathbf{m}|^2 + \phi(\mathbf{m}) - \mathbf{f} \cdot \mathbf{m}] \, dx + \int_{\mathbb{R}^3} |\mathbf{h}[\mathbf{m}]|^2 \, dx \ . \end{aligned}$$

A self-evident defect of this model is that it only applies to a single crystal (it has other defects, the main of which is that it is valid only for approximately insulating materials, since conduction electrons play an important rôle in the magnetic theory of conductors).

2. Mixtures and their energy

We now turn to *mixtures* of magnetic materials: we begin by considering a body Ω made of two crystals Ω_1 and Ω_2 of different materials, separated by a smooth surface Σ; we stress the fact that by "different" we mean that the two grains may also be made of the same "stuff" but with crystallographic axes orientated in different directions. In general, as we have seen, a magnetic material is identified by just three quantities: the exchange constant a, the magnetic saturation m^s, and the anysotropy function ϕ, which contains all the necessary crystallographic information, thus we are confronted with two materials identified by the triples (a_1, m_1^s, ϕ_1) and (a_2, m_2^s, ϕ_2) respectively. The energy contribution of the magnetizations $\mathbf{m}_1$ and $\mathbf{m}_2$ of the two grains is thus given by

$$\begin{aligned} V(\mathbf{m}_1, \mathbf{m}_2) \quad &= \quad \int_{\Omega_1} [a_1|\nabla\mathbf{m}_1|^2 + \phi_1(\mathbf{m}_1) - \mathbf{f} \cdot \mathbf{m}_1] \, dx \\ &\quad + \int_{\Omega_2} [a_2|\nabla\mathbf{m}_2|^2 + \phi_2(\mathbf{m}_2) - \mathbf{f} \cdot \mathbf{m}_2] \, dx \\ &\quad + \int_{\mathbb{R}^3} |\mathbf{h}|^2 \, dx \ , \end{aligned}$$

where $\mathbf{h} = \mathbf{h}[\mathbf{m}_1 \mathbb{I}_{\Omega_1} + \mathbf{m}_2 \mathbb{I}_{\Omega_2}]$.

In addition to this volume energy, an extra energy term on Σ has to be considered; the density of this surface energy is given by the sum of two terms: a positive constant $\alpha_{1,2}$, depending on the two materials and due to chemical and electric disturbances in the lattice atoms, and a non-negative function β depending on the two materials, on the traces $\text{Tr } \mathbf{m}_1$ and $\text{Tr } \mathbf{m}_2$ of the magnetizations on the two sides of Σ, and possibly also on the normal vector ν to Σ. The function

$\beta_{1,2}(\mathrm{Tr}\,\mathbf{m}_1, \mathrm{Tr}\,\mathbf{m}_2, \nu)$ keeps track of the magnetic disturbances across Σ, and the surface term is

$$S(\mathbf{m}_1, \mathbf{m}_2) = \int_\Sigma [\alpha_{1,2} + \beta_{1,2}(\mathrm{Tr}\,\mathbf{m}_1, \mathrm{Tr}\,\mathbf{m}_2, \nu)]\,d\mathcal{H}^2 \, ,$$

so the total energy is

$$E(\mathbf{m}_1, \mathbf{m}_2) = V(\mathbf{m}_1, \mathbf{m}_2) + S(\mathbf{m}_1, \mathbf{m}_2) \, .$$

If the number of crystals grows, so does the shape of E; to keep it readable, we make for a while a simplification, which we will later drop to return to a general formulation: we assume $\beta = 0$ and $\alpha = 1$, so the surface term reduces to the area of the interface.

It is now easy to describe the case of K grains: we have K magnetic materials, each characterized by a triple (a_i, m_i^s, ϕ_i) and occupying an open subset Ω_i of Ω; the sets Ω_i are pairwise disjoint and their union is all of Ω up to a 2-dimensional set Σ; if we denote by $\mathbf{m}_i$ the magnetization in Ω_i, the energy is given by

$$\sum_1^K \int_{\Omega_i} [a_i|\nabla\mathbf{m}_i|^2 + \phi_i(\mathbf{m}_i) - \mathbf{f}\cdot\mathbf{m}_i]\,dx + \int_{\mathbb{R}^3} |\mathbf{h}[\textstyle\sum_1^K \mathbf{m}_i \mathbb{I}_{\Omega_i}]|^2\,dx + \mathcal{H}^2(\Sigma) \, , \quad (2)$$

where we recall that

$$|\mathbf{m}_i| = m_i^s \quad \text{in } \Omega_i \, , \qquad \mathbf{m}_i \in W^{1,2}(\Omega_i; \mathbb{R}^3) \, . \tag{3}$$

3. A new form of the energy

We now turn our attention to an optimal design problem, the analysis of which will put under a new light the energy above, will force us to modify it once more, and will lead to a great simplification of the shape of the energy.

Fix a bounded open set $\Omega \subset \mathbb{R}^3$ and a vector field $\mathbf{f} \in L^?(\Omega; \mathbb{R}^3)$,**GUARDARE** and assume you have an infinite amount of each of K different magnetic materials, with which you want to fill Ω in order to minimize the energy (2): thus you want to find K pairs $(\Omega_i, \mathbf{m}_i)$ satisfying all the conditions above, and noticeably the constraints (3), which minimize the energy among all such sets of pairs. The trained eye will see at first glance that this problem is not well posed, because the energy is not semicontinuous: in particular, a magnetization $\mathbf{m}_i$ which is discontinuous along a surface $\Sigma \subset \Omega_i$ is not an admissible competitor, as it violates the Sobolev condition in (3), but it may be easily approached by a sequence of admissible competitors with equibounded energy, simply by fattening Σ into an open set Σ' and adding this to Ω_j for some $j \neq i$ (extend $\mathbf{m}_j$ to Σ' as a constant). This then leads to a finite relaxed energy for the discontinuous function we selected, although it is not in $W^{1,2}$; we remark that this is in some sense analogous to Gibbs' phenomenon in fluids (see [7]): if you want the magnetization to be discontinuous inside a crystal, break the crystal and coat the fracture faces with a different material.

The structure of the relaxed energy, which allows inner discontinuities but penalizes them, may be physically interpreted as keeping into account the possible magnetic disarrangements ("magnetic cracks") inside a crystal, or as a simplification (which we already met) of the energy of a Bloch wall.

Since we have been forced to consider a relaxed energy which no longer forces the magnetizations to be in $W^{1,2}$ inside each grain, but allows jumps, it is natural to take as an ambient space that of special functions of bounded variation, SBV, for whose definition and properties we refer to [2]. This setting allows us to write the energy in a different form, but this is no simpler than before: indeed we cannot charge all surface terms on the jump set of the overall magnetization, because some parts of the surface may then be missing: to convince ourselves, recall that in two adjacent grains Ω_i and Ω_j one may well have $m_i^s = m_j^s$, thus the magnetization might have no jump across the interface, although some energy has to be taken into account (due to the electric disturbances we mentioned). Also, it is impossible to write the energy in a compact form.

We then rescale the magnetizations $\mathbf{m}_i$ in order to obtain an auxiliary magnetization field which will contain all the information, and which will allow us to write the energy in an easy, implicit form. We set for $i = 1, \ldots, K$

$$\mathbf{u}_i = i \cdot \frac{\mathbf{m}_i}{m_i^s} , \qquad \mathbf{u} = \sum_1^K \mathbf{u}_i \mathbb{I}_{\Omega_i} ,$$

so that in particular

$$\mathbf{u} \in SBV(\Omega; \mathbb{R}^3) ,$$

$$\Omega_i = \{ x : |\mathbf{u}| = i \} \quad \text{and} \quad \mathbb{I}_{\Omega_i}(x) = \left[2 - \left(|\mathbf{u}(x)| - (i-1) \right)^+ \right]^+ := \mu_i \big(|\mathbf{u}(x)| \big) ,$$

$$\mathbf{m} = \mathbf{m}(\mathbf{u}) = \Big(\sum_1^K \frac{m_i^s}{i} \mathbb{I}_{\Omega_i} \Big) \mathbf{u} = \Big(\sum_1^K \frac{m_i^s}{i} \mu_i(|\mathbf{u}|) \Big) \mathbf{u} := \lambda(|\mathbf{u}|) \mathbf{u}$$

and the jump set $J_{\mathbf{u}}$ of $\mathbf{u}$ consists exactly of the union of the interfaces between grains and the inner magnetick cracks. We remark that given $\mathbf{u}$ one easily deduces $\mathbf{m}$ and may also decide whether a jump of $\mathbf{u}$ represents an interface or an inner crack: the former is also a jump of $|\mathbf{u}|$, the second is not.

We may now rescale all other factors: take any bounded, positive, continuous function a satisfying

$$a : [0, +\infty[\to]0, +\infty[, \qquad a(i) = (m_i^s)^2 \, a_i \quad \text{for } i = 1, \ldots, K$$

and we have

$$\text{Exch} = \sum_1^K \int_{\Omega_i} a_i |\nabla \mathbf{m}_i|^2 \, dx = \int_\Omega a(|\mathbf{u}|) |\nabla \mathbf{u}|^2 \, dx$$

(the extension of the function a outside the points $1, \ldots, K$ is not really necessary, but then this energy is defined on all SBV); analogously, take any bounded, non-negative, continuous function satisfying

$$\phi : \mathbb{R}^3 \to [0, +\infty[\, , \qquad \phi\big|_{\partial B_i}(\mathbf{z}) = \phi_i(m_i^s \mathbf{z}) \quad \text{for } i = 1, \ldots, K$$

and

$$\text{Anys} = \sum_1^K \int_{\Omega_i} \phi_i(\mathbf{m}_i) \, dx = \int_{\Omega} \phi(\mathbf{u}) \, dx \, .$$

Now, since the mapping $\mathbf{u} \mapsto \mathbf{m}(\mathbf{u}) = \lambda(|\mathbf{u}|)\mathbf{u}$ is continuous in every L^p, so is the mapping

$$\mathbf{u} \mapsto \mathbf{h}[\mathbf{m}(\mathbf{u})] := \tilde{\mathbf{h}}[\mathbf{u}]$$

(although it is no longer linear because λ is not), and we may write the whole energy as

$$\mathcal{E}(\mathbf{u}) = \int_{\Omega} \big[a(|\mathbf{u}|)|\nabla\mathbf{u}|^2 + \phi(\mathbf{u}) - \mathbf{f}\cdot\mathbf{m}(\mathbf{u}) - \tilde{\mathbf{h}}[\mathbf{u}]\cdot\mathbf{m}(\mathbf{u})\big]\, dx + \int_{J_{\mathbf{u}}} \gamma(\mathbf{u}^+, \mathbf{u}^-, \nu)\, d\mathcal{H}^2 \, ,$$

where the function γ encompasses all the surface terms we met before. We make on γ the subadditivity assumptions that are customary to have subadditivity (see [2] p.???), and due to the discussion above we also set

$$\gamma \geq \gamma_0 > 0 \, .$$

The optimal design problem then becomes

$$\min\Big\{\mathcal{E}(\mathbf{u}) : \mathbf{u} \in SBV(\Omega; \mathbb{R}^3), \ |\mathbf{u}| \in \{1, \ldots, K\} \text{ in } \Omega\Big\}$$

a Mumford-Shah-type problem with a constraint on K different surfaces (see e.g. [3] for a single constraint in a simpler setting).

In a forthcoming paper it is proved that $\mathcal{E}$ is semicontinuous and the minimum exists (also in the fixed-volume-fraction case), and a regularity theorem for the solution and the interfaces is given.

References

[1] E. Acerbi, I. Fonseca, R. Mingione: paper in preparation.

[2] L. Ambrosio, N. Fusco, D. Pallara: Functions of bounded variation and free discontinuity problems, Clarendon Press, Oxford University Press, New York, 2000.

[3] M. Carriero, A. Leaci: S^k-valued maps minimizing the L^p norm of the gradient with free discontinuities, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **18** (1991), 321–352.

[4] A. DeSimone: Energy minimisers for large ferromagnetic bodies, Arch. Rational Mech. Anal. **125** (1993), 99–143.

[5] A. DeSimone, R.V. Kohn, S. Müller, F. Otto: Magnetic microstructures - a paradigm of multiscale problems, ICIAM 99 (Edinburgh), 175–190, Oxford Univ. Press, Oxford, 2000.

[6] A. Hubert, R. Schäfer: Magnetic domains, Springer Verlag, Berlin, Heidelberg, New York, 1998.

[7] L. Modica: Gradient theory of phase transitions with boundary contact energy, Ann. Inst. H. Poincaré Anal. Non Linéaire **4** (1987), 487–512.

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