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Homogenization of Periodic Structures With Holes

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In the last decade, the concept of Γ-convergence of functionals has been widely investigated; to clarify its link with mechanics, consider the following simple consequence of Γ-convergence.

Assume (F_h) is a sequence of functionals defined, say, on H_0^1, and all satisfying

(1) $$F_h(u) \geq \int |Du|^2 \, dx \ ,$$

and let F_∞ be another functional. If the sequence (F_h) is $\Gamma^-(L^2)$-converging to F_∞, then for every $g \in H^{-1}$ and every minimizing sequence (u_h) of

$$F_h(u) + \langle g, u \rangle$$

we may select a subsequence (u_{h_k}) converging in L^2 to a minimum point of

$$F_\infty(u) + \langle g, u \rangle \ .$$

If the functionals F_h are the free energies of some materials, then we may well say that the material represented by F_h tends to behave as the one associated with F_∞.

The usefulness of Γ-convergence lies in the several compactness results available, although the identification of the limit F_∞ is sometimes not straightforward.

Most frequently in the literature, Γ-convergence appears in the context of homogenization: suppose you are given a structure, in the unit cube Q, whose free energy is given by

$$F_1(u) = \int_Q f(x, Du) \, dx \ ,$$

and repeat it periodically in the space. Rescale everything of a factor $1/h$, and you obtain in the unit cube a tighter structure whose energy is

$$F_h(u) = \int_Q f(hx, Du) \, dx \ ,$$

where f is now 1-periodic in x. It is likely that, looking at this structure from "very far", it will seem a homogeneous material.

Every homogenization result thus consists in proving that (F_h) converges, in the Γ-sense, to some functional F_∞, and identifying F_∞ as an energy functional:

$$F_\infty(u) = \int_Q \phi(Du) \, dx \ .$$

In this field, many results have been obtained, especially in the scalar case, i.e., u takes its values in $\mathbf{R}$, or when condition (1) is attained through the coerciveness of f:

$$f(x,\xi) \geq |\xi|^2 \quad \text{for all } x \ .$$

This condition, unfortunately, rules out many interesting cases: in order to represent an inhomogeneous elastic structure with holes, the integrand

$$f : Q \times \mathbf{R}^9 \to \mathbf{R}$$

must satisfy only

$$f(x,\xi) \geq |\xi|^2 \quad \text{only if } x \notin H \ , \tag{2}$$

where the hole H is a subset of Q with nice boundary and well contained in Q.
A homogenization result under these assumptions is contained in [1], where in addition the dependence of f on u is not through the gradient, but through the strain tensor $e(u)$: if f is convex in ξ and satisfies (2) and

$$0 \leq f(x,\xi) \leq c(1+|\xi|^2) \tag{3}$$

then the functionals

$$F_h(u) = \int_Q f(hx, e(u))\, dx$$

$\Gamma^-(L^2)$-converge to the homogenized functional

$$F_\infty(u) = \int_Q \phi(e(u))\, dx \ ,$$

where $\phi(\xi)$ is given by

$$\inf\left\{ \int_Q f(x, e(u))\, dx \ : \ u \in H^1_{loc}(\ R^3),\ u - \xi x \text{ is } Q\text{-periodic} \right\} .$$

As a matter of fact, the result is more general: for example, in conditions (2) and (3) we may have a generic growth $p > 1$, instead of 2. The main tools employed in the proof are an abstract Γ-compactness theorem, various forms of Korn's inequality and the following interesting extension lemma:

Let $p > 1$ and let Ω, ω be bounded open subsets of $\mathbf{R}^n$ with lipschitz boundary, such that $\omega \subseteq \Omega$. Then there exists a constant $c(\Omega,\omega)$ such that for every $u \in W^{1,p}(\omega;\mathbf{R}^n)$ there exists $\tilde{u} \in W^{1,p}(\Omega;\mathbf{R}^n)$ such that

$$\tilde{u} = u \ \text{in} \omega$$

$$\int_\Omega |e(\tilde{u})|^p\, dx \leq c(\Omega,\omega) \int_\omega |e(u)|^p\, dx \ .$$

Moreover $c(t\Omega, t\omega) = c(\Omega,\omega)$ for every $t > 0$.

REFERENCES

[1] Acerbi, E. & Percivale, D.: *Homogenization of non coercive functionals: periodic materials with soft inclusions.* Submitted to Applied Mathematics and Optimization.

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