Limit Problems for some Linear and Nonlinear Systems

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This paper contains the summary of a talk given by the author at the Loyola University of Chicago, during a workshop on "Weak Convergence Methods in Nonlinear PDEs".

We will examine some limit problems for elliptic systems with discontinuous coefficients: assume $\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma$, where Ω_1 and Ω_2 are open and Σ is the common part of their boundaries, and consider the problem

$$(1) \qquad \begin{cases} Au = \varphi & \text{in } \Omega_1 \\ Bu = \varphi & \text{in } \Omega_2 \\ \text{transmission conditions} & \text{on } \Sigma \\ \text{boundary conditions} & \text{on } \partial\Omega, \end{cases}$$

where A and B are elliptic operators.

We are interested in the behaviour of the solutions to problem (1) when both Ω_2 shrinks to Σ and B varies — its coefficients might vanish or explode. Therefore, we deal with a *sequence* of problems, and we seek their limit (in a suitable sense).

Many noteworthy cases may be considered in which Ω_1 is surrounded by Ω_2, that is, $\partial\Omega_2 = \Sigma \cup \partial\Omega$: In this situation, one may expect that the solutions to (1) approach the solution of a "limit problem" of the type

$$\begin{cases} Au = \varphi & \text{in } \Omega_1 \\ \text{new boundary conditions} & \text{on } \partial\Omega_1, \end{cases}$$

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where the new boundary conditions incorporate what is left of the former ones, of B, Ω_2 and the transmission conditions. Some examples in which this actually occurs (the meaning of "approach" and "limit" still need a precise definition) are:

the torsion of a (linearly elastic) bar with cross-section Ω_1, reinforced with a thin shirt Ω_2 of much harder material — here, both A and B are second-order operators; see e. g.

- Caffarelli & Friedman, *Rocky Mountain J. Math.* **10,** 1980.
- Brezis, Caffarelli & Friedman, *Ann. Mat. Pura Appl.* **123,** 1980.
- Acerbi & Buttazzo, *Ann. Inst. Henri Poincaré* **3,** 1986.

the buckling of a (linearly elastic) shell Ω_1 surrounded by a thin annulus Ω_2 of very soft material — A and B are fourth-order operators; see e. g.

- Acerbi & Buttazzo, *Arch. Rational Mech. Anal.* **92,** 1986.

We remark that in these examples problem (1) is a PDE, not a system. In the last two papers, the point of view is that of calculus of variations, i.e., minimum points of functionals instead of solutions of differential equations. This we also do henceforward.

We concentrate on a second family of problems: precisely, we study some cases when Ω_2 is *inside* Ω_1.

Take a smooth, compact $(n-1)$-dimensional manifold Σ of $\mathbb{R}^n$, an open set Ω which encloses Σ, and an ε-neighbourhood of Σ:

$$\Sigma_\varepsilon = \{\sigma + t\nu(\sigma) : \sigma \in \Sigma, |t| < \varepsilon\}.$$

Define on $W^{1,p}(\Omega;\mathbb{R}^n)$

$$F_\varepsilon(u) = \int_{\Omega\setminus\Sigma_\varepsilon} f\big(x, e(u)\big)\,dx + c_\varepsilon \int_{\Sigma_\varepsilon} f\big(x, e(u)\big)\,dx,$$

where $e(u)$ is the linearized strain tensor, $f(x,\cdot)$ is a p-homogeneous convex function satisfying

$$|z^*|^p \le f(x,z) \le c(1 + |z^*|^p)$$

with $p > 1$, depending only on the symmetric part $z^* = (z + z^T)/2$ of z, and well continuous in x.

A useful notion in this situation is Γ-convergence, of which little knowledge is needed here: what matters is that if F_ε, F_0 are real functions on a topological space X, then
$F_\varepsilon \to^\Gamma F_0$ is equivalent to

$$\begin{cases} \text{for every } x_\varepsilon \to x \text{ we have } F_0(x) \le \liminf F_\varepsilon(x_\varepsilon); \\ \text{for every } x \text{ there exists } x_\varepsilon \to x \text{ such that } F_0(x) = \lim F_\varepsilon(x_\varepsilon); \end{cases}$$

whereas $F_\varepsilon \to^\Gamma F_0$ implies

$$\begin{cases} \text{if } x_\varepsilon \text{ is a minimum point for } F_\varepsilon \text{ and } x_\varepsilon \to x \text{ then } x \text{ is a minimum point for } F_0; \\ \text{if } C : X \to \mathbb{R} \text{ is continuous then } (F_\varepsilon + C) \to^\Gamma (F_0 + C). \end{cases}$$

Therefore, if we prove that $F_\varepsilon \to^\Gamma F_0$ in the weak topology of $W^{1,p}$ we have that the solutions of

$$\min\{F_\varepsilon(u) + \int_\Omega (|u|^p + gu)\,dx\} \tag{2}$$

converge to the solution of

$$\min\{F_0(u) + \int_\Omega (|u|^p + gu)\,dx\}; \tag{3}$$

thus, Γ-convergence is a convenient way to say that (3) is the limit of problems (2).

A physical model for case (2) above is the inclusion of a thick slab Σ_ε with Young modulus proportional to c_ε in a fixed, "soft" body Ω: then we may say that the limit functional F_0 (if it exists) governs the inclusion of a plate, or a membrane, Σ, in the body Ω. As physicists maintain that the energy of a membrane with thickness ε is proportional to ε, that of a plate to ε^3, we will study the case $c_\varepsilon = 1/\varepsilon^\lambda$ for various exponents λ. To state the result, we introduce:

the tangential derivative $\quad \delta u = Du - (Du\,\nu) \otimes \nu$

the tangential strain $\quad e_\tau(u) = [(I - \nu \otimes \nu)Du]^*$,

and finally the "tangential part" of f:

$$f_\tau(\sigma, z) = \min\{f\big(\sigma, z + \xi \otimes \nu(\sigma)\big) : \xi \in \mathbb{R}^n\}.$$

It turns out that the first interesting exponent is $\lambda = 1$, where we obtain the Γ-limit

$$F_0^1(u) = \begin{cases} \int_\Omega f\big(x, e(u)\big)\,dx + 2\int_\Sigma f_\tau\big(\sigma, e_\tau(u)\big)\,d\sigma & \text{if } u - \langle u, \nu\rangle\nu \in W^{1,p}(\Sigma) \\ +\infty & \text{otherwise.} \end{cases}$$

The second critical exponent is $\lambda = p + 1$, which amounts to 3 in the physical case of quadratic-growth energy, and the limit is

$$F_0^{p+1}(u) = \begin{cases} \int_\Omega f\big(x, e(u)\big)\,dx + \frac{2}{p+1}\int_\Sigma f_\tau(\sigma, \nu\,\delta\delta u)\,d\sigma & \text{if } \langle u, \nu\rangle \in W^{2,p}(\Sigma),\ e_\tau(u) = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Mathematically, this is more surprising than the case above, since for $\lambda = 1$ the limit contains only the (tangential) first derivatives of u, whereas here we have got second derivatives from functionals F_ε depending only on the first derivatives.

To have a glimpse at where this second differential comes from, we remark that to prove Γ-convergence we must 1) find a sequence $v_\varepsilon \rightharpoonup u$ such that $F_0(u) = \lim F_\varepsilon(v_\varepsilon)$, and 2) show that for all $u_\varepsilon \rightharpoonup u$ we have $F_0(u) \le \liminf F_\varepsilon(u_\varepsilon)$.

Step 1) : we *try* with

$$v_\varepsilon = \big(u(\sigma) + t\varphi(\sigma) + \frac{t^2}{2}\eta(\sigma)\big)\theta_\varepsilon + u(x)(1 - \theta_\varepsilon),$$

where θ_ε is a cut-off function vanishing outside $\Sigma_{2\varepsilon}$. Then, close to Σ, we have essentially to deal with

$$(4) \qquad \frac{1}{\varepsilon^{p+1}} \int_{\Sigma_\varepsilon} f\big(\sigma, \delta u + \varphi \otimes \nu + t(-\delta u\, \delta\nu + \delta\varphi + \eta \otimes \nu)\big)\, dx.$$

If $\delta u + \varphi \otimes \nu \neq 0$, the measure ε of Σ_ε cannot compete with ε^{-p-1}, and $\lim F_\varepsilon(v_\varepsilon) = +\infty$. Therefore we must take

$$(5) \qquad \delta u + \varphi \otimes \nu = 0,$$

and (homogeneity + integration in t) we get from (4)

$$\frac{2}{p+1} \int_\Sigma f(\sigma, -\delta u\, \delta\nu + \delta\varphi + \eta \otimes \nu)\, d\sigma.$$

By (5), the argument of f reduces to

$$\nu\, \delta\delta u + \nu \otimes (\eta - \nu\, \delta u\, \delta\nu),$$

and the first step follows since η is free (recall what f_τ is).

Step 2) is quite lengthy (we will come back to it later), but it is essential in that without it we cannot be sure that our choice of v_ε was the smartest one.

A (more intelligible) corollary: take the usual energy density

$$f(z) = \frac{\lambda}{2}(\operatorname{tr} z)^2 + \mu |z^*|^2.$$

For some Σ, the function f_τ may be explicited, and $\nu\, \delta\delta$ calculated; e. g., in the good old flat case (substituting E, σ for λ, μ) we get for plates the limit

$$\int_\Omega f\big(e(u)\big)\, dx\, dy\, dz + 2^3 \frac{E}{24(1-\sigma^2)} \int\!\!\int_\Sigma [\Delta u_z - 2(1-\sigma) \det D^2 u_z]\, dx\, dy,$$

provided the horizontal displacement is a rigid motion in the plane (this comes from the condition $e_\tau = 0$).

Again, some references:

- Caillerie, *Math. Meth. Appl. Sci.* **2,** 1980.
- Ciarlet & Destuynder, *J. Mécanique* **18,** 1979.
- Acerbi, Buttazzo & Percivale, *J. reine angew. Math.* **386,** 1988.

Before going on, an objection is due: we are primarily concerned with the behaviour of u on the manifold Σ, and we speak of weak $W^{1,p}$ convergence in Ω; it would seem a quite poor result, but in step 2 we actually prove that a stronger convergence takes place.

Anyway, we surely cannot rely on weak $W^{1,p}$ convergence in Ω when investigating *wires* in $\mathbb{R}^3$; denote by Σ the one-dimensional string

$$\Sigma = \{(x,0,0) : 0 \leq x \leq 1\}$$

and by Σ_ε the three-dimensional cylinder

$$\Sigma_\varepsilon = \{(x,y,z) : 0 \leq x \leq 1, y^2 + z^2 \leq \varepsilon^2\}.$$

As we said, we expect some trouble if we make use of Ω. Then, we won't use it. This presents us another problem: to use Γ-convergence, all our functionals must be defined on the *same* space X, whereas here the domain of F_ε will consist of functions defined on Σ_ε. Then, we will not use Γ-convergence either.

The functionals we consider on Σ_ε are

$$F_\varepsilon(u) = \int_{\Sigma_\varepsilon} f(Du)\,dx,$$

where f is a function — not required to be convex, which gives the nonlinearity of the system — satisfying:

a) $f : \mathbb{R}^9 \to \mathbb{R} \cup \{+\infty\}$ is continuous

b) if $\det \xi \leq 0$ then $f(\xi) = +\infty$

c) for all $\delta > 0$ there exists c_δ such that if $\det \xi \geq \delta$ then $f(\xi) \leq c_\delta(1 + |\xi|^p)$

d) $f(\xi) \geq c|\xi|^p - c'$.

Here, dependence on x too can be added with little effort; also, the shape of Σ_ε need not be exactly cylindrical. We remark that a,…,d are very reasonable assumptions if we have in mind an application to the physical situation, as b) prevents interpenetration of matter.

Denote by $(a|b|c)$ the matrix whose columns are a, b, c, and define the appropriate tangential part of f as

$$f_\tau(a) = \min\{f(a|b|c) : b, c \in \mathbb{R}^3\};$$

then f_τ is continuous and

$$|a| \geq \delta \quad \Rightarrow \quad f_\tau(a) \leq c_\delta(1 + |a|^p),$$

so that the *convex envelope* f_τ^{**} of f_τ satisfies

$$f_\tau^{**}(a) \leq c(1 + |a|^p)$$

for all a; we may then define on $W^{1,p}(\Sigma;\mathbb{R}^3)$

$$F_0(u) = \int_\Sigma f_\tau^{**}(u')\,dt.$$

Remark that the functionals F_ε are defined only on a part of C^1 or little more, so that we will not speak of minimum points for them. Now we define for every function $v \in L^1(\Sigma_\varepsilon)$ the "normal average" as a function of $x \in \Sigma$:

$$\tilde{v}(x) = \frac{1}{\pi\varepsilon^2} \int_{\{y^2+z^2 \le \varepsilon^2\}} v(x,y,z)\, dy\, dz,$$

i.e., the average of v on a disk orthogonal to Σ at x. It is all too easy to see vhere the functionals F_ε go: as the domain vanishes, the energies vanish too, at least with speed ε^2: this is why we (take any continuous $g : \mathbb{R}^3 \to \mathbb{R}^3$ and) define

$$G_\varepsilon(u) = \frac{F_\varepsilon(u) + \int_{\Sigma_\varepsilon} (|u|^p + gu)\, dx}{\pi\varepsilon^2}$$

$$G_0(u) = F_0(u) + \int_\Sigma (|u|^p + gu)\, dt.$$

Our result is then the following: *let $u_\varepsilon \in C^1(\Sigma_\varepsilon; \mathbb{R}^3)$ be a minimizing sequence for G_ε, i.e.,* $\lim[G_\varepsilon(u_\varepsilon) - \inf G_\varepsilon] = 0$. *Then:*

(1) the sequence $\widetilde{u_\varepsilon}$ is weakly compact in $W^{1,p}(\Sigma; \mathbb{R}^3)$;

(2) if u_0 is a limit point of $\widetilde{u_\varepsilon}$, then it is a minimum point for G_0.

First, a remark: suppose f satisfies also

e) f is frame-indifferent, i.e., $f(Q\xi) = f(\xi)$ for every positive orthogonal matrix Q;

f) $f(I) = \min f = 0$.

Under assumptions a,...,f the functions f_τ depends only on $|a|$ and

$$f_\tau^{**}(a) = 0 \quad \text{whenever } |a| \le 1;$$

this means that for a string (as it should be) the energy is positive under traction, zero under compression.

This applies in particular to a physically interesting class of functions (powers of the gradient plus convex functions of the determinant), as e.g.

$$f(\xi) = |\xi|^2 + (\det \xi)^{-1/3} - 9,$$

which yields

$$f_\tau^{**}(a) = \begin{cases} |a|^2 + 8|a|^{-1/4} - 9 & \text{if } |a| \ge 1 \\ 0 & \text{if } |a| \le 1. \end{cases}$$

A point in the proof of the result is worth a look, although it is very easy in our situation: we prove the analogous of step 2 of the case of plates. We must take any sequence v_ε such that $\widetilde{v_\varepsilon} \rightharpoonup u$, and prove that $F_0(u) \le \liminf F_\varepsilon(v_\varepsilon)/\pi\varepsilon^2$.

One begins with

$$\frac{1}{\pi\varepsilon^2} F_\varepsilon(v_\varepsilon) \ge \fint_{\Sigma_\varepsilon} f_\tau(D_{x_1} v_\varepsilon)\, dx \ge \fint_{\Sigma_\varepsilon} f_\tau^{**}(D_{x_1} v_\varepsilon)\, dx;$$

since f_τ^{**} is subdifferentiable, and $|\partial f_\tau^{**}(a)| \leq c(1+|a|^{p-1})$, we have

$$\cdots \;\geq \int_\Sigma f_\tau^{**}(u')\,dt + \fint_{\Sigma_\varepsilon} \langle \partial f_\tau^{**}(u'), D_{x_1}v_\varepsilon - u' \rangle\,dx$$

$$= F_0(u) + \int_\Sigma \langle \partial f_\tau^{**}(u'), \widetilde{v_\varepsilon}' - u' \rangle\,dt.$$

In the last integral, the first half is in $L^{p'}$ and the second converges weakly to zero in L^p, and this step is proved.

This is the way which is generally followed in proof of results alike — but usually it comes out to be precisely the hardest part. In the case of plates, a suitable normal average of the functions v_ε and their gradients and strains is defined, and step 2 consists essentially in proving that these averages converge to something which leads to the limit function u.

The complete proofs for the result on nonlinear strings may be found in

• Acerbi, Buttazzo & Percivale, *SISSA preprint*, 1988.