Limit Problems for Plates Surrounded by Soft Material

EMILIO ACERBI & GIUSEPPE BUTTAZZO

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I. Introduction

Consider an inhomogeneous clamped plate D, submitted to an external force $g(x)$. The (small) vertical displacement $u(x)$ solves the minimum problem

$$\min\left\{\int_D \left[\frac{E(x)}{1-\sigma^2(x)}\,(|\Delta u|^2 - 2(1-\sigma(x))\det D^2u) + g(x)\,u\right] dx : u \in H_0^2(D)\right\},$$

where E and σ are the Young modulus and the Poisson coefficient respectively, and D^2u denotes the 2×2 matrix of second derivatives of u. We study a plate having a central part Ω surrounded by an increasingly narrow annulus Σ_ε made of an increasingly soft material (*i.e.* the Young modulus E_ε tends to zero in Σ_ε). The free energy of the plate is then

$$F_\varepsilon(u) = \int_\Omega \frac{E}{1-\sigma^2}\,(|\Delta u|^2 - 2(1-\sigma)\det D^2u)\,dx + \int_{\Sigma_\varepsilon} \frac{E_\varepsilon}{1-\sigma_\varepsilon^2}\,(|\Delta u|^2 - 2(1-\sigma_\varepsilon)\det D^2u)\,dx. \tag{1.1}$$

We study in particular the behavior as $\varepsilon \to 0$ of the solution u_ε of

$$\min\left\{F_\varepsilon(u) + \int_{\Omega\cup\Sigma_\varepsilon} g(x)\,u\,dx : u \in H_0^2(\bar\Omega \cup \Sigma_\varepsilon)\right\}. \tag{1.2}$$

If r_ε is the width of Σ_ε, we may have different limit problems depending on the relation between r_ε and E_ε: let $\sigma_0 = \lim_{\varepsilon\to 0}\sigma_\varepsilon$ and set

$$G(u) = \int_\Omega \frac{E}{1-\sigma^2}\,(|\Delta u|^2 - 2(1-\sigma)\det D^2u)\,dx.$$

Then, if $E_\varepsilon \gg r_\varepsilon$, the limit problem is

$$\min\{G(u) : u \in H_0^2(\Omega)\}$$

(clamped plate); if $\lim\limits_{\varepsilon\to 0} E_\varepsilon/r_\varepsilon = M \neq 0$, the limit problem is

$$\min\left\{G(u) + M \int_{\partial\Omega} \frac{1}{1-\sigma_0^2}\left|\frac{\partial u}{\partial\nu}\right|^2 ds : u \in H^2(\Omega) \cap H_0^1(\Omega)\right\}$$

(the plate becomes simply supported, but a price is paid for having non-zero normal derivative); if $r_\varepsilon \gg E_\varepsilon \gg r_\varepsilon^3$ the limit problem is

$$\min\{G(u) : u \in H^2(\Omega) \cap H_0^1(\Omega)\}$$

(simply supported plate); if $\lim\limits_{\varepsilon\to 0} E_\varepsilon/r_\varepsilon^3 = L$ the limit problem is

$$\min\left\{G(u) + 12L \int_{\partial\Omega} \frac{1}{1-\sigma_0^2}|u|^2\, ds : u \in H^2(\Omega)\right\}$$

(the plate may take off, but a price is paid for having a non-zero boundary value). In all four cases, if $\lim\limits_{\varepsilon\to 0} E_\varepsilon/r_\varepsilon^3 > 0$, we prove that the solutions u_ε converge in $L^2(\mathbb{R}^2)$ to the solution of the limit problem.

The foregoing example is a particular case of our theorem [II.3] and our results in section IV in which we consider the general energy integral

$$F_\varepsilon(u) = \int_\Omega g_\varepsilon(x, u, Du, D^2u)\, dx + \varepsilon \int_{\Sigma_\varepsilon} f_\varepsilon(x, D^2u)\, dx$$

where g_ε is quasi-convex in D^2u and f_ε is convex in D^2u. In addition the functions g_ε and f_ε need not be quadratic, but they satisfy coerciveness and growth conditions of the form

$$|D^2u|^p \leqq g_\varepsilon(x, u, Du, D^2u) \leqq c(1 + |D^2u|^p),$$
$$|D^2u|^p \leqq f_\varepsilon(x, D^2u) \leqq c(1 + |D^2u|^p)$$

with $p > 1$.

A similar problem in the case of membranes (*i.e.* when the energy integral contains only the first derivatives Du and not D^2u) has been studied by several authors: see for example [2], [3], [4], [8] if the energy is a quadratic form, and [1] in the general case.

II. Notations and Statement of the Result

We use the following symbols:

Ω	a bounded open subset of $\mathbb{R}^n$, with $C^{2,1}$ boundary;
ν	the outward normal vector to Ω;
δ	the function $\delta(x) = \text{dist}(x, \overline{\Omega})$;
h	a smooth function from $\partial\Omega$ into $]0, +\infty[$;

$\{r_\varepsilon\}_{\varepsilon>0}$	a set of positive real numbers such that $\lim_{\varepsilon\to 0} r_\varepsilon = 0$;
Σ_ε	the set $\{\sigma + t\nu(\sigma) : \sigma \in \partial\Omega,\ 0 < t < r_\varepsilon h(\sigma)\}$;
Ω_ε	the set $\overline{\Omega} \cup \Sigma_\varepsilon$;
p	a real number greater than 1;
f	a function from $\mathbb{R}^n \times \mathbb{R}^{n\times n}$ into $[0, +\infty[$;
G	a functional from $W^{2,p}(\Omega)$ into $[0, +\infty[$.

By the regularity assumptions on $\partial\Omega$, the mapping $(\sigma, t) \mapsto \sigma + t\nu(\sigma)$ is invertible on Σ_ε if ε is small enough; in particular the point $\sigma(x) \in \partial\Omega$ of minimum distance from $x \in \Sigma_\varepsilon$ is a regular function of x. We shall write $h(x)$ and $\nu(x)$ for $h(\sigma(x))$, $\nu(\sigma(x))$. We make the following assumptions on the function f:

(2.1) the function $f(x, z)$ is continuous in x and convex in z;

(2.2) for all $x \in \mathbb{R}^n$, $z \in \mathbb{R}^{n\times n}$

$$|z|^p \leqq f(x, z) \leqq c(1 + |z|^p);$$

(2.3) there is a non-negative continuous function $\gamma(x, z)$ which is convex and p-homogeneous as a function of z and satisfies

$$\sup\{|f(x, z) - \gamma(x, z)| : x \in \mathbb{R}^n\} \leqq \varrho(|z|)\,(1 + |z|^p)$$

for all $z \in \mathbb{R}^{n\times n}$, where $\varrho : [0, +\infty[\to [0, +\infty[$ is a continuous, decreasing function which vanishes at infinity.

As for the functional G, we suppose that

(2.4) G is lower semicontinuous in the topology $L^p(\Omega)$;

(2.5) G is continuous in the strong topology of $W^{2,p}(\Omega)$;

(2.6) $G(u) \geqq \int_\Omega |D^2 u|^p\, dx$ for every $u \in W^{2,p}(\Omega)$.

If $u \in L^p(\mathbb{R}^n)$ is such that $u|_\Omega \in W^{2,p}(\Omega)$, we write simply $G(u)$ instead of $G(u|_\Omega)$. We remark that conditions (2.4), (2.5), (2.6) are fulfilled by a broad class of functionals, for example the integrals $\int_\Omega g(x, u, Du, D^2u)\, dx$ where $g(x, s, s', s'')$ is a Carathéodory function convex (or quasi-convex in the sense of MORREY [7]) in s'' and satisfying

$$|s''|^p \leqq g(x, s, s', s'') \leqq c(1 + |s''|^p).$$

For every $u \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$ set

$$F_\varepsilon(u) = \begin{cases} G(u) + \varepsilon \displaystyle\int_{\Sigma_\varepsilon} f(x, D^2u)\, dx & \text{if} \quad u \in W_0^{2,p}(\Omega_\varepsilon) \\ +\infty & \text{otherwise}. \end{cases}$$

We wish to characterize the Γ-limit of F_ε in the topology $L^p(\mathbb{R}^n)$, depending on the behavior of r_ε. Indeed, it is well known that the Γ-convergence of a sequence of functionals is strictly related to the convergence of their minimum points and minimum values: more precisely, let X be a metric space, let $(F_\varepsilon)_{\varepsilon>0}$ be mappings from X into $\overline{\mathbb{R}}$, and let $x \in X$. We set

$$\Gamma^-(X) \liminf_{\varepsilon\to 0} F_\varepsilon(x) = \inf\left\{\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) : x_\varepsilon \to x \text{ in } X\right\},$$

$$\Gamma^-(X) \limsup_{\varepsilon\to 0} F_\varepsilon(x) = \inf\left\{\limsup_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) : x_\varepsilon \to x \text{ in } X\right\}.$$

If these two Γ-limits are the same at x, their common value will be denoted by

$$\Gamma^-(X) \lim_{\varepsilon\to 0} F_\varepsilon(x).$$

Theorem [II.1] (see [5], Theorem 2.3). *If $\Phi: X \to \mathbb{R}$ is continuous, then*

$$\Gamma^-(X) \liminf_{\varepsilon\to 0} (\Phi + F_\varepsilon)(x) = \Phi(x) + \Gamma^-(X) \liminf_{\varepsilon\to 0} F_\varepsilon(x),$$

$$\Gamma^-(X) \limsup_{\varepsilon\to 0} (\Phi + F_\varepsilon)(x) = \Phi(x) + \Gamma^-(X) \limsup_{\varepsilon\to 0} F_\varepsilon(x).$$

Theorem [II.2] (see [5], Theorem 2.6). *Assume that*

(i) *the family (F_ε) is equicoercive*, i.e., *for every $c > 0$ there is a compact subset K_c of X such that*

$$\{x \in X : F_\varepsilon(x) \leqq c\} \subseteqq K_c \text{ for every } \varepsilon > 0;$$

(ii) *for every $x \in X$, $\Gamma^-(X) \lim_{\varepsilon\to 0} F_\varepsilon(x)$ exists.*

Set $F = \Gamma^-(X) \lim_{\varepsilon\to 0} F_\varepsilon$. Then F has a minimum on X and $\min_X F = \lim_{\varepsilon\to 0} \left(\inf_X F_\varepsilon\right)$; moreover if $\lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0} \left(\inf_X F_\varepsilon\right)$ and $x_\varepsilon \to \hat{x}$ in X, then $\hat{x}$ is a minimum point for F.

We now state the main result: set

$$K_p = 2^p \left(\frac{2p-1}{p-1}\right)^{p-1};$$

for every $u \in W^{2,p}(\Omega)$ and $L \in [0, +\infty[$ we define

$$G'_L(u) = G(u) + LK_p \int_{\partial\Omega} |u(\sigma)|^p h^{1-2p}(\sigma)\, \gamma(\sigma, \nu(\sigma) \otimes \nu(\sigma))\, dH^{n-1}(\sigma).$$

Let $u \in W^{2,p}(\Omega)$ and $M \in [0, +\infty]$. If $M < +\infty$ we define

$$G''_M(u) = \begin{cases} G(u) + M \int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}(\sigma)\right|^p h^{1-p}(\sigma)\, \gamma(\sigma, \nu(\sigma) \otimes \nu(\sigma))\, dH^{n-1}(\sigma) & \text{if } u \in W_0^{1,p}(\Omega) \\ +\infty & \text{otherwise;} \end{cases}$$

if $M = +\infty$ we define

$$G''_\infty(u) = \begin{cases} G(u) & \text{if } u \in W_0^{2,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

In Section III we shall prove

Theorem [II.3]. *Assume that* (2.1), ..., (2.6) *hold and that both* $\lim_{\varepsilon\to 0} \varepsilon/r_\varepsilon^{2p-1} = L \in [0, +\infty]$ *and* $\lim_{\varepsilon\to 0} \varepsilon/r_\varepsilon^{p-1} = M \in [0, +\infty]$ *exist. Then for every* $u \in W^{2,p}(\Omega)$, $\Gamma^-(L^p(\mathbb{R}^n)) \lim_{\varepsilon\to 0} F_\varepsilon(u)$ *exists, and*

(i) *if* $L < +\infty$, *then* $F = G'_L$;

(ii) *if* $L = +\infty$, *then* $F = G''_M$.

Moreover if $L > 0$ *and* $g \in L^q(\mathbb{R}^n)$, *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then from every sequence of minimum points of*

$$F_\varepsilon(u) + \int_{\Omega_\varepsilon} gu\, dx$$

we may extract a subsequence converging in $L^p(\mathbb{R}^n)$ *to a minimum point of*

$$F(u) + \int_\Omega gu\, dx.$$

III. Proof of the Result

In what follows the letter c will denote any positive constant, and if no confusion is possible we will not write the variables x and σ in the integrals. We shall later need the following lemma:

Lemma [III.1[. *Let* $b \in L^\infty(0, 1)$, $a \in C(\mathbb{R}^n)$ *and* $u_\varepsilon \to u$ *strongly in* $W^{1,p}(\mathbb{R}^n)$. *If we set* $\bar{b} = \int_0^1 b(t)\, dt$, *then*

$$\lim \frac{1}{r_\varepsilon} \int_{\Sigma_\varepsilon} |u_\varepsilon(x)|^p a(x)\, b\left(\frac{\delta(x)}{r_\varepsilon h(x)}\right) dx$$

$$= \bar{b} \int_{\partial\Omega} |u(\sigma)|^p a(\sigma)\, h(\sigma)\, dH^{n-1}(\sigma).$$

Proof. Let $v \in W^{1,p}(\mathbb{R}^n)$; then

$$\int_{\partial\Omega} dH^{n-1}(\sigma) \int_0^{r_\varepsilon h(\sigma)} ||v(\sigma + tv(\sigma))|^p - |v(\sigma)|^p|\, dt$$

$$\leqq c \int_{\partial\Omega} dH^{n-1}(\sigma) \int_0^{r_\varepsilon h(\sigma)} |v(\sigma + tv(\sigma)) - v(\sigma)| \cdot [|v(\sigma + tv(\sigma))|^{p-1} + |v(\sigma)|^{p-1}]\, dt$$

$$\leqq c \int_{\partial\Omega} dH^{n-1}(\sigma) \int_0^{r_\varepsilon h(\sigma)} \left[\int_0^{r_\varepsilon h(\sigma)} |Dv(\sigma + sv(\sigma))|\, ds\right]$$
$$\times [|v(\sigma + tv(\sigma))|^{p-1} + |v(\sigma)|^{p-1}]\, dt$$

$$\leqq c \int_{\partial\Omega} dH^{n-1}(\sigma) \left[\int_0^{r_\varepsilon h(\sigma)} |Dv(\sigma + sv(\sigma))|^p\, ds\right]^{1/p}$$
$$\times r_\varepsilon^{\frac{p-1}{p}} \left[\int_0^{r_\varepsilon h(\sigma)} [|v(\sigma + tv(\sigma))|^p + |v(\sigma)|^p]\, dt\right]^{\frac{p-1}{p}} r_\varepsilon^{1/p}$$

$$\leqq cr_\varepsilon \left[\int_{\Sigma_\varepsilon} |Dv|^p\, dx\right]^{1/p} \left[\int_{\Sigma_\varepsilon} |v|^p\, dx + r_\varepsilon \int_{\partial\Omega} |v|^p\, dH^{n-1}\right]^{\frac{p-1}{p}},$$

with c independent of v. This inequality with $v = u_\varepsilon$ yields

$$\lim_{\varepsilon\to 0} \left|\frac{1}{r_\varepsilon} \int_{\Sigma_\varepsilon} [|u_\varepsilon(x)|^p - |u_\varepsilon(\sigma(x))|^p]\, a(x)\, b\left(\frac{\delta(x)}{r_\varepsilon h(x)}\right) dx\right| = 0. \tag{3.1}$$

Because $a(x)$ is assumed continuous,

$$\lim_{\varepsilon\to 0} \left|\frac{1}{r_\varepsilon} \int_{\Sigma_\varepsilon} |u_\varepsilon(\sigma(x))|^p\, [a(x) - a(\sigma(x))]\, b\left(\frac{\delta(x)}{r_\varepsilon h(x)}\right) dx\right| = 0; \tag{3.2}$$

finally, since $u_\varepsilon \to u$ in $L^p(\partial\Omega)$,

$$\lim_{\varepsilon\to 0} \frac{1}{r_\varepsilon} \int_{\partial\Omega} dH^{n-1}(\sigma)\, |u_\varepsilon(\sigma)|^p\, a(\sigma) \int_0^{r_\varepsilon h(\sigma)} b\left(\frac{t}{r_\varepsilon h(\sigma)}\right) dt \tag{3.3}$$
$$= \lim_{\varepsilon\to 0} \bar{b} \int_{\partial\Omega} |u_\varepsilon|^p\, ah\, dH^{n-1} = \bar{b} \int_{\partial\Omega} |u|^p\, ah\, dH^{n-1},$$

and the conclusion follows by (3.1), (3.2) and (3.3). □

We divide the proof of Theorem [II.3] into several steps. For every $u \in W^{2,p}(\Omega)$, set

$$F^+(u) = \Gamma^-(L^p(\mathbb{R}^n)) \limsup_{\varepsilon\to 0} F_\varepsilon(u),$$

$$F^-(u) = \Gamma^-(L^p(\mathbb{R}^n)) \liminf_{\varepsilon\to 0} F_\varepsilon(u).$$

In the first "critical" case $\varepsilon \sim r_\varepsilon^{2p-1}$ (i.e. $0 < L < +\infty$) we prove separately the two inequalities $F^+ \leqq G_L'$ and $G_L' \leqq F^-$; analogously, we prove that

$F^+ \leqq G''_M$ and $G''_M \leqq F^-$ in the second "critical" case $\varepsilon \sim r_\varepsilon^{p-1}$ (i.e. $0 < M < +\infty$). The result in the remaining cases will be deduced easily.

Case $\varepsilon \sim r_\varepsilon^{2p-1}$, $F^+ \leqq G'_L$.

Let $u \in W^{2,p}(\Omega)$: the regularity of $\partial\Omega$ lets us suppose that $u \in W^{2,p}(\mathbb{R}^n)$. For every $\varepsilon > 0$ set

$$\varphi_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \bar{\Omega}, \\ \Phi\left(\dfrac{\delta(x)}{r_\varepsilon h(x)}\right) & \text{if } x \in \Sigma_\varepsilon, \\ 0 & \text{if } x \notin \Omega_\varepsilon \end{cases}$$

where Φ: $[0,1] \to \mathbb{R}$ is the solution of the minimum problem

$$\min\left\{\int_0^1 |\psi''(t)|^p\,dt : \psi \in W^{2,p}(0,1),\ \psi(0) = 1,\ \psi(1) = \psi'(0) = \psi'(1) = 0\right\}.$$

Some easy computation shows that

$$\Phi(t) = \frac{p-1}{p} 2^{\frac{p}{p-1}} |t - \tfrac{1}{2}|^{\frac{p}{p-1}} (t - \tfrac{1}{2}) - \frac{2p-1}{p}(t - \tfrac{1}{2}) + \tfrac{1}{2},$$

$$\int_0^1 |\Phi''(t)|^p\,dt = K_p$$

and

$$|\Phi''(t)|^{p-2}\,\Phi''(t) = K_p(t - \tfrac{1}{2}). \tag{3.4}$$

In particular, the function Φ is of class C^2, and by our assumptions on $\partial\Omega$ and on the function h we have in Σ_ε

$$\begin{aligned} \left|D\varphi_\varepsilon - \frac{1}{r_\varepsilon}\Phi'\left(\frac{\delta}{hr_\varepsilon}\right)\frac{\nu}{h}\right| &\leqq c, \\ \left|D^2\varphi_\varepsilon - \frac{1}{r_\varepsilon^2}\Phi''\left(\frac{\delta}{hr_\varepsilon}\right)\frac{\nu \otimes \nu}{h^2}\right| &\leqq \frac{c}{r_\varepsilon}. \end{aligned} \tag{3.5}$$

Setting $u_\varepsilon = u\varphi_\varepsilon$ we have $u_\varepsilon \in W_0^{2,p}(\Omega_\varepsilon)$ and $u_\varepsilon \to u \cdot 1_\Omega$ in $L^p(\mathbb{R}^n)$. By the convexity of f we have for every $t \in (0, 1)$

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &\leqq G(u) + \varepsilon \int_{\Sigma_\varepsilon} \Big[t f(x, \frac{1}{t} u\, D^2\varphi_\varepsilon) \\ &\quad + (1-t) f\left(x, \frac{1}{1-t}(Du \otimes D\varphi_\varepsilon + D\varphi_\varepsilon \otimes Du + \varphi_\varepsilon\, D^2 u)\right)\Big]\,dx \\ &\leqq G(u) + t\varepsilon \int_{\Sigma_\varepsilon} f\left(x, \frac{1}{t} u\, D^2\varphi_\varepsilon\right) dx \\ &\quad + c\varepsilon\,[\text{meas}\,(\Sigma_\varepsilon) + r_\varepsilon^{-p}(1-t)^{-p} \|u\|^p_{W^{2,p}(\mathbb{R}^n)}]; \end{aligned}$$

therefore

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leqq G(u) + t \limsup_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} f\left(x, \frac{1}{t} u\, D^2\varphi_\varepsilon\right) dx. \tag{3.6}$$

Fix $R > 0$ and set $A_{R,\varepsilon} = \{x \in \Sigma_\varepsilon : |u\, D^2\varphi_\varepsilon| < Rt\}$; then by (2.3), (3.5)

$$\varepsilon \int_{\Sigma_\varepsilon} f\left(x, \frac{1}{t} u\, D^2\varphi_\varepsilon\right) dx \leqq \varepsilon \int_{\Sigma_\varepsilon} \gamma\left(x, \frac{1}{t} u\, D^2\varphi_\varepsilon\right) dx + \varepsilon \int_{A_{R,\varepsilon}} \varrho(0)\,(1 + R^p)\, dx$$

$$+ \varepsilon \int_{\Sigma_\varepsilon \setminus A_{R,\varepsilon}} \varrho(R) \left(1 + \left|\frac{cu}{tr_\varepsilon^2}\right|^p\right) dx \leqq \frac{\varepsilon}{t^p} \int_{\Sigma_\varepsilon} \gamma(x, u\, D^2\varphi_\varepsilon)\, dx$$

$$+ \varepsilon\varrho(0)\,(2 + R^p) \operatorname{meas}(\Sigma_\varepsilon) + \varrho(R) \frac{c\varepsilon}{t^p r_\varepsilon^{2p}} \int_{\Sigma_\varepsilon} |u|^p\, dx.$$

Now let $\varepsilon \to 0$, apply Lemma [III.1], and then let $R \to +\infty$; we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} f\left(x, \frac{1}{t} u\, D^2\varphi_\varepsilon\right) dx \leqq t^{-p} \limsup_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, u\, D^2\varphi_\varepsilon)\, dx. \tag{3.7}$$

As above, by (3.5) and by the assumptions on γ we have for every $s \in (0, 1)$

$$\varepsilon \int_{\Sigma_\varepsilon} \gamma(x, u\, D^2\varphi_\varepsilon)\, dx \leqq c(1-s)^{1-p} \frac{\varepsilon}{r_\varepsilon^p} \int_{\Sigma_\varepsilon} |u|^p\, dx$$

$$+ \frac{\varepsilon}{r_\varepsilon^{2p} s^{p-1}} \int_{\Sigma_\varepsilon} \frac{\gamma(x, \nu \otimes \nu)}{h^{2p}} |u|^p \left|\Phi''\left(\frac{\delta}{hr_\varepsilon}\right)\right|^p dx,$$

and by Lemma [III.1]

$$\limsup_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, u\, D^2\varphi_\varepsilon)\, dx \leqq L s^{1-2p} K_p \int_{\partial\Omega} |u|^p\, h^{1-2p} \gamma(\sigma, \nu \otimes \nu)\, dH^{n-1}. \tag{3.8}$$

Using (3.6), (3.7), (3.8) and letting $t \to 1$, $s \to 1$, we obtain

$$F^+(u) \leqq \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leqq G'_L(u). \quad \square$$

Case $\varepsilon \sim r_\varepsilon^{2p-1}$, $G'_L \leqq F^-$.

Take $u \in W^{2,p}(\Omega)$; again we may assume that $u \in W^{2,p}(\mathbb{R}^n)$. Letting $u_\varepsilon \in W_0^{2,p}(\Omega_\varepsilon)$ be such that $u_\varepsilon \to u 1_\Omega$ in $L^p(\mathbb{R}^n)$, we have to prove that

$$G'_L(u) \leqq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon);$$

hence we may suppose

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty,$$

whence

$$\int_\Omega |D^2 u_\varepsilon|^p\, dx + \varepsilon \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^p\, dx \leqq c. \tag{3.9}$$

In particular, without loss of generality we may assume that $u_\varepsilon \to u$ weakly in $W^{2,p}(\Omega)$.

By the semicontinuity of G it will suffice to prove that

$$\liminf_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} f(x, D^2u_\varepsilon)\, dx \geqq LK_p \int_{\partial\Omega} |u|^p\, h^{1-2p}\gamma(\sigma, \nu \otimes \nu)\, dH^{n-1}.$$

Fix $R > 0$ and set $B_{R,\varepsilon} = \{x \in \mathbb{R}^n : |D^2u_\varepsilon| < R\}$; by (2.3), (3.9)

$$\begin{aligned}\varepsilon \int_{\Sigma_\varepsilon} f(x, D^2u_\varepsilon)\, dx &\geqq \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, D^2u_\varepsilon)\, dx \\ &\quad - \varepsilon \int_{\Sigma_\varepsilon\cap B_{R,\varepsilon}} \varrho(0)\,(1 + R^p)\, dx - \varepsilon \int_{\Sigma_\varepsilon\setminus B_{R,\varepsilon}} \varrho\,(R)\,(1 + |D^2u_\varepsilon|^p)\, dx \\ &\geqq \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, D^2u_\varepsilon)\, dx - \varepsilon \operatorname{meas}(\Sigma_\varepsilon)\, \varrho(0)\,(2 + R^p) - c\varrho(R).\end{aligned}$$

Since R is arbitrary, we have only to prove that

$$\liminf_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, D^2u_\varepsilon)\, dx \geqq LK_p \int_{\partial\Omega} |u|^p\, h^{1-2p}\gamma(\sigma, \nu \otimes \nu)\, dH^{n-1}. \tag{3.10}$$

It is easy to construct by convolutions a sequence (γ_k) of functions from $\mathbb{R}^n \times \mathbb{R}^{n\times n}$ into $[0, +\infty[$ such that

(i) for every $x \in \mathbb{R}^n$ the function $\gamma_k(x, \cdot)$ is convex and p-homogeneous;
(ii) γ_k is of class C^∞ on $\mathbb{R}^n \times (\mathbb{R}^{n\times n} \setminus \{0\})$;
(iii) for every $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^{n\times n}$

$$|z|^p \leqq \gamma_k(x, z) \leqq c\,|z|^p;$$

(iv) for every $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^{n\times n}$

$$|\gamma_k(x, z) - \gamma(x, z)| \leqq \frac{1}{k}\,|z|^p.$$

By (3.9), (3.11) (iv) we obtain

$$\begin{aligned}&\lim_{k\to\infty} \left[\liminf_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \gamma_k(x, D^2u_\varepsilon)\, dx - LK_p \int_{\partial\Omega} |u|^p\, h^{1-2p}\gamma_k(\sigma, \nu \otimes \nu)\, dH^{n-1}\right] \\ &\quad = \liminf_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, D^2u_\varepsilon)\, dx - LK_p \int_{\partial\Omega} |u|^p\, h^{1-2p}\gamma(\sigma, \nu \otimes \nu)\, dH^{n-1},\end{aligned}$$

and so in (3.10) we may assume that γ is of class C^∞ on $\mathbb{R}^n \times (\mathbb{R}^{n\times n} \setminus \{0\})$. For simplicity we set

$$\Phi_\varepsilon''(x) = \frac{1}{r_\varepsilon^2 h^2(x)}\, \Phi''\left(\frac{\delta(x)}{r_\varepsilon h(x)}\right),$$

$$\gamma_{ij}'(x) = \frac{\partial\gamma}{\partial z_{ij}}\,(x, \nu(x) \otimes \nu(x));$$

moreover, we adopt the usual summation convention over repeated indices. Because γ is convex and p-homogeneous,

$$\gamma(x, D^2u_\varepsilon) \geqq \gamma(x, u\Phi_\varepsilon''\, \nu \otimes \nu) + (D_{ij}^2 u_\varepsilon - u\Phi_\varepsilon'' \nu_i \nu_j) \frac{\partial \gamma}{\partial z_{ij}}(x, u\Phi_\varepsilon''\, \nu \otimes \nu)$$
$$= |u\Phi_\varepsilon''|^p\, \gamma(x, \nu \otimes \nu) + (D_{ij}^2 u_\varepsilon - u\Phi_\varepsilon'' \nu_i \nu_j)\, |u\Phi_\varepsilon''|^{p-2}\, u\Phi_\varepsilon''\gamma_{ij}'$$

and by (3.4)

$$(3.12) \qquad \gamma(x, D^2u_\varepsilon) \geqq |u\Phi_\varepsilon''|^p\, \gamma(x, \nu \otimes \nu) + K_p(D_{ij}^2 u_\varepsilon - u\Phi_\varepsilon'' \nu_i \nu_j) \frac{|u|^{p-2}u}{h^{2p-2} r_\varepsilon^{2p-2}} \left(\frac{\delta}{hr_\varepsilon} - \frac{1}{2}\right) \gamma_{ij}'.$$

Applying Lemma [III.1] yields

$$(3.13) \qquad \lim_{\varepsilon \to 0} \varepsilon \int_{\Sigma_\varepsilon} |u\Phi_\varepsilon''|^p\, \gamma(x, \nu \otimes \nu)\, dx = LK_p \int_{\partial\Omega} |u|^p\, h^{1-2p} \gamma(\sigma, \nu \otimes \nu)\, dH^{n-1}.$$

Inequalities (3.5) imply that

$$|u\Phi_\varepsilon''\nu \otimes \nu - D^2(u\varphi_\varepsilon)| \leqq \frac{c}{r_\varepsilon}(|u| + |Du| + |D^2u|),$$

so that

$$(3.14) \qquad \lim_{\varepsilon \to 0} \varepsilon \int_{\Sigma_\varepsilon} |u\Phi_\varepsilon''\nu \otimes \nu - D^2(u\varphi_\varepsilon)|\, |u|^{p-1}\, r_\varepsilon^{2-2p} dx = 0.$$

By (3.12), (3.13), (3.14) to conclude the proof of (3.10) we have only to show that

$$\lim_{\varepsilon \to 0} r_\varepsilon \int_{\Sigma_\varepsilon} D_{ij}^2(u_\varepsilon - u\varphi_\varepsilon) \frac{|u|^{p-2}\, u}{h^{2p-2}} \left(\frac{\delta}{hr_\varepsilon} - \frac{1}{2}\right) \gamma_{ij}'\, dx = 0.$$

Integrating by parts, and recalling that $u_\varepsilon - u\varphi_\varepsilon \in W_0^{2,p}(\Omega_\varepsilon)$, we make the integral above become

$$r_\varepsilon \int_{\Sigma_\varepsilon} (u_\varepsilon - u\varphi_\varepsilon)\, D_{ij}^2 \left[\frac{|u|^{p-2}\, u}{h^{2p-2}} \left(\frac{\delta}{hr_\varepsilon} - \frac{1}{2}\right) \gamma_{ij}'\right] dx$$
$$+ \frac{r_\varepsilon}{2} \int_{\partial\Omega} D_i(u_\varepsilon - u) \frac{|u|^{p-2}\, u}{h^{2p-2}} \gamma_{ij}'\, \nu_j\, dH^{n-1}$$
$$+ r_\varepsilon \int_{\partial\Omega} (u_\varepsilon - u)\, D_j \left[\frac{|u|^{p-2}\, u}{h^{2p-2}} \left(\frac{\delta}{hr_\varepsilon} - \frac{1}{2}\right) \gamma_{ij}'\right] \nu_i\, dH^{n-1}.$$

It is easy to see that the boundary integrals vanish as $\varepsilon \to 0$ since $u_\varepsilon \to u$ in $W^{1,p}(\partial\Omega)$. Moreover

$$\left| D_{ij}^2 \left[\frac{|u|^{p-2}\, u}{h^{2p-2}} \left(\frac{\delta}{hr_\varepsilon} - \frac{1}{2}\right) \gamma_{ij}'\right]\right| \leqq \frac{c}{r_\varepsilon}(|u|^{p-1} + |Du|^{p-1} + |D^2u|^{p-1}),$$

and so also the first integral vanishes as $\varepsilon \to 0$. □

Remark [III.2]. If $\varepsilon \gg r_\varepsilon^{2p-1}$ (i.e. $L = +\infty$), then, for any $L > 0$ we have for small enough ε

$$F_\varepsilon(u) \geqq G(u) + L r_\varepsilon^{2p-1} \int_{\Sigma_\varepsilon} f(x, D^2 u)\, dx$$

for every $u \in W^{2,p}(\Omega_\varepsilon)$; hence by the foregoing discussion

$$F^-(u) \geqq G(u) + L K_p \int_{\partial\Omega} |u|^p h^{1-2p} \gamma(\sigma, \nu \otimes \nu)\, dH^{n-1}.$$

Since L is arbitrary, this implies that $F^-(u) = +\infty$ if $u \notin W_0^{1,p}(\Omega)$. □

Case $\varepsilon \sim r_\varepsilon^{p-1}$, $F^+ \leqq G''_M$.
By the definition of G''_M, we have to prove the inequality $F^+(u) \leqq G''_M(u)$ only for $u \in W^{2,p}(\mathbb{R}^n) \cap W_0^{1,p}(\Omega)$. Choose such a function u, and let $v_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ be such that

$$v_\varepsilon \to u \quad \text{strongly in } W^{2,p}(\mathbb{R}^n), \qquad \|v_\varepsilon\|_{C^3(\mathbb{R}^n)} \leqq r_\varepsilon^{-\frac{1}{2}}.$$

Let Ψ_ε be the solution of the minimum problem

$$\min\left\{\int_\Omega |D^2\psi|^p\, dx : (D\psi, \nu) = 0 \quad \text{and} \quad \psi = -v_\varepsilon - \frac{r_\varepsilon h}{2}(Dv_\varepsilon, \nu) \quad \text{on} \quad \partial\Omega\right\}.$$

Since $u = 0$ on $\partial\Omega$, $\Psi_\varepsilon \to 0$ strongly in $W^{2,p}(\Omega)$. Set

$$\vartheta_\varepsilon(x) = -\frac{\langle Dv_\varepsilon(\sigma(x)), \nu(x)\rangle}{r_\varepsilon h(\sigma(x))} \quad \text{in } \Sigma_\varepsilon,$$

$$w_\varepsilon(x) = \begin{cases} v_\varepsilon(x) + \Psi_\varepsilon(x) & \text{in } \Omega, \\ \dfrac{[\delta(x) - r_\varepsilon h(\sigma(x))]^2}{2}\vartheta_\varepsilon(x) & \text{in } \Sigma_\varepsilon; \end{cases}$$

then $w_\varepsilon \in W_0^{2,p}(\Omega_\varepsilon)$, $w_\varepsilon \to u \cdot 1_\Omega$ in $L^p(\mathbb{R}^n)$ and $w_\varepsilon \to u$ in $W^{2,p}(\Omega)$. Moreover

$$\|D^2 w_\varepsilon - \vartheta_\varepsilon \nu \otimes \nu\|_{L^p(\Sigma_\varepsilon)} \leqq c. \tag{3.15}$$

If we use (2.5), the argument employed in the proof of (3.6), (3.7) yields for every $t \in (0, 1)$

$$\limsup_{\varepsilon\to 0} F_\varepsilon(w_\varepsilon) \leqq G(u) + t^{1-p} \limsup_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, \vartheta_\varepsilon \nu \otimes \nu)\, dx.$$

By the homogeneity of γ, it is enough to apply Lemma [III.1] and to let $t \to 1$ to obtain $F^+ \leqq G''_M$. □

Case $\varepsilon \sim r_\varepsilon^{p-1}$, $G''_M \leqq F^-$.
Let $u \in W^{2,p}(\mathbb{R}^n) \cap W_0^{1,p}(\Omega)$ and take $u_\varepsilon \in W_0^{2,p}(\Omega_\varepsilon)$ such that $u_\varepsilon \to u \cdot 1_\Omega$ in $L^p(\mathbb{R}^n)$. As we did in the part $G'_L \leqq F^-$, we may assume that (3.9) holds and that $u_\varepsilon \to u$ weakly in $W^{2,p}(\Omega)$, and it suffices to prove that

$$\liminf_{\varepsilon\to 0} \varepsilon \int_{\Sigma_\varepsilon} \gamma(x, D^2 u_\varepsilon)\, dx \geqq M \int_{\partial\Omega} |\langle Du, \nu\rangle|^p h^{1-p} \gamma(\sigma, \nu \otimes \nu)\, dH^{n-1}, \tag{3.16}$$

with γ of class C^∞ on $\mathbb{R}^n \times (\mathbb{R}^{n\times n} \setminus \{0\})$.

If we define γ'_{ij} as when $G'_L \leqq F^-$ and ϑ_ε, w_ε as when $F^+ \leqq G''_M$, then we have on Σ_ε

$$\gamma(x, D^2u_\varepsilon) \geqq \gamma(x, \vartheta_\varepsilon \nu \otimes \nu) + (D^2_{ij}u_\varepsilon - \vartheta_\varepsilon \nu_i \nu_j) \frac{\partial \gamma}{\partial z_{ij}}(x, \vartheta_\varepsilon \nu \otimes \nu) \tag{3.17}$$

$$= |\vartheta_\varepsilon|^p \gamma(x, \nu \otimes \nu) + D^2_{ij}(u_\varepsilon - w_\varepsilon) |\vartheta_\varepsilon|^{p-2} \vartheta_\varepsilon \gamma'_{ij}$$

$$+ (D^2_{ij}w_\varepsilon - \vartheta_\varepsilon \nu_i \nu_j) |\vartheta_\varepsilon|^{p-2} \vartheta_\varepsilon \gamma'_{ij}.$$

By (3.15)

$$\varepsilon \int_{\Sigma_\varepsilon} |D^2_{ij}w_\varepsilon - \vartheta_\varepsilon \nu_i \nu_j| \, |\vartheta_\varepsilon|^{p-1} |\gamma'_{ij}| \, dx \leqq c \left(\int_{\Sigma_\varepsilon} |Dv_\varepsilon|^p \, dx \right)^{\frac{p-1}{p}}, \tag{3.18}$$

which vanishes as $\varepsilon \to 0$. Moreover by Lemma [III.1]

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Sigma_\varepsilon} |\vartheta_\varepsilon|^p \gamma(x, \nu \otimes \nu) \, dx = M \int_{\partial\Omega} |\langle Du, \nu \rangle|^p h^{1-p} \gamma(\sigma, \nu \otimes \nu) \, dH^{n-1}. \tag{3.19}$$

An integration by parts yields

$$\varepsilon \int_{\Sigma_\varepsilon} D^2_{ij}(u_\varepsilon - w_\varepsilon) |\vartheta_\varepsilon|^{p-2} \vartheta_\varepsilon \gamma'_{ij} \, dx \tag{3.20}$$

$$= -\varepsilon \int_{\Sigma_\varepsilon} D_i(u_\varepsilon - w_\varepsilon) D_j(|\vartheta_\varepsilon|^{p-2} \vartheta_\varepsilon \gamma'_{ij}) \, dx$$

$$- \varepsilon \int_{\partial\Omega} D_i(u_\varepsilon - w_\varepsilon) |\vartheta_\varepsilon|^{p-2} \vartheta_\varepsilon \gamma'_{ij} \nu_j \, dH^{n-1}.$$

The last term vanishes as $\varepsilon \to 0$: indeed $\varepsilon |\vartheta_\varepsilon|^{p-1} \leqq c |Dv_\varepsilon|^{p-1}$ and $u_\varepsilon - w_\varepsilon \to 0$ weakly in $W^{2,p}(\Omega)$, so that $D(u_\varepsilon - w_\varepsilon) \to 0$ strongly in $L^p(\partial\Omega)$. As for the first term we have

$$\left| \varepsilon \int_{\Sigma_\varepsilon} D_i(u_\varepsilon - w_\varepsilon) D_j(|\vartheta_\varepsilon|^{p-2} \vartheta_\varepsilon \gamma'_{ij}) \, dx \right| \tag{3.21}$$

$$\leqq c \int_{\Sigma_\varepsilon} |D(u_\varepsilon - w_\varepsilon)| \, (|Dv_\varepsilon|^{p-1} + |Dv_\varepsilon|^{p-2} |D^2 v_\varepsilon|) \, dx$$

$$\leqq c \left[\int_{\Sigma_\varepsilon} |D(u_\varepsilon - w_\varepsilon)|^p \, dx \right]^{1/p}$$

since (v_ε) is bounded in $W^{2,p}(\mathbb{R}^n)$. For every $x \in \Sigma_\varepsilon$

$$|D(u_\varepsilon - w_\varepsilon)(x)|^p \leqq \left[\int_0^{r_\varepsilon h(x)} |D^2(u_\varepsilon - w_\varepsilon)(\sigma(x) + t\nu(x))| \, dt \right]^p$$

$$\leqq c r_\varepsilon^{p-1} \int_0^{r_\varepsilon h(x)} |D^2(u_\varepsilon - w_\varepsilon)(\sigma(x) + t\nu(x))|^p \, dt,$$

so that, by (3.9) and (3.15),

$$\int_{\Sigma_\varepsilon} |D(u_\varepsilon - w_\varepsilon)|^p \, dx \leqq c r_\varepsilon^p \int_{\Sigma_\varepsilon} |D^2(u_\varepsilon - w_\varepsilon)|^p \, dx$$

$$\leqq c r_\varepsilon \varepsilon \int_{\Sigma_\varepsilon} (|D^2 u_\varepsilon|^p + |D^2 w_\varepsilon|^p) \, dx \leqq c r_\varepsilon;$$

together with (3.20), (3.21) this implies

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Sigma_\varepsilon} D_{ij}^2(u_\varepsilon - w_\varepsilon) \, |\vartheta_\varepsilon|^{p-2} \, \vartheta_\varepsilon \gamma'_{ij} \, dx = 0,$$

and (3.16) follows by (3.17), (3.18), (3.19). □

Case $\varepsilon \ll r_\varepsilon^{2p-1}$. The inequality $F^- \geqq G'_0 = G$ is trivial. On the other hand, for every $L > 0$ we have for all sufficiently small ε

$$F_\varepsilon(u) \leqq G(u) + L r_\varepsilon^{2p-1} \int_{\Sigma_\varepsilon} f(x, D^2 u) \, dx,$$

whence $F^+ \leqq G'_L$ for all $L > 0$, and $F^+ \leqq G'_0$ follows as $L \to 0$. □

Case $r_\varepsilon^{2p-1} \ll \varepsilon \ll r_\varepsilon^{p-1}$. We already proved in Remark [III.2] that $F^- \geqq G''_0$, and the inequality $F^+ \leqq G''_0$ is proved as in the case above. □

Case $\varepsilon \gg r_\varepsilon^{p-1}$.
Since $F^+ \leqq G''_\infty$ is trivial, we need only prove the inequality $F^- \geqq G''_\infty$, which is derived from the case $\varepsilon \sim r_\varepsilon^{p-1}$ by an argument like to the one we used in Remark [III.2]. □

We pass now to the last assertion of Theorem [II.3]. Let $\lim_{\varepsilon \to 0} \varepsilon / r_\varepsilon^{2p-1} > 0$ and $g \in L^q(\mathbb{R}^n)$. By Theorem [II.1] the functionals

$$\tilde{F}_\varepsilon(u) = F_\varepsilon(u) + \int_{\Omega_\varepsilon} g u \, dx$$

are $\Gamma^-(L^p(\mathbb{R}^n))$-convergent to $F(u) + \int_\Omega g u \, dx$; hence they satisfy condition (ii) of Theorem [II.2]. To conclude the proof of Theorem [II.3] we only have to show that the functionals $\tilde{F}_\varepsilon$ satisfy also condition (i), that is

$$\tilde{F}_\varepsilon(u_\varepsilon) \leqq c \quad \text{for all } \varepsilon \Rightarrow (u_\varepsilon) \text{ is relatively compact in } L^p(\mathbb{R}^n).$$

By (2.2), (2.6) we may assume that

$$\int_\Omega |D^2 u_\varepsilon|^p \, dx + \varepsilon \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^p \, dx + \int_{\Omega_\varepsilon} g u_\varepsilon \, dx \leqq c.$$

Take any $\eta > 0$; for a suitable constant C_η

$$(3.22) \qquad \int_\Omega |D^2 u_\varepsilon|^p \, dx + \varepsilon \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^p \, dx \leqq C_\eta + \eta \int_{\Omega_\varepsilon} |u_\varepsilon|^p \, dx.$$

For every $\sigma \in \partial\Omega$ and $t \in [0, r_\varepsilon h(\sigma)]$

$$|u_\varepsilon(\sigma + t\nu(\sigma))|^p = \left| \int_t^{r_\varepsilon h(\sigma)} (s - t) \langle D^2 u_\varepsilon(\sigma + s\nu(\sigma)), \nu \otimes \nu \rangle \, ds \right|^p$$

$$\leqq c r_\varepsilon^{2p-1} \int_0^{r_\varepsilon h(\sigma)} |D^2 u_\varepsilon(\sigma + s\nu(\sigma))|^p \, ds,$$

so that

$$\int_{\partial\Omega} |u_\varepsilon|^p \, dH^{n-1} \leqq c r_\varepsilon^{2p-1} \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^p \, dx, \tag{3.23}$$

$$\int_{\Sigma_\varepsilon} |u_\varepsilon|^p \, dx \leqq c r_\varepsilon^{2p} \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^p \, dx. \tag{3.24}$$

On the other hand for every $v \in W^{2,p}(\Omega)$

$$\int_\Omega |v|^p \, dx \leqq c \left[\int_\Omega |D^2 v|^p \, dx + \int_{\partial\Omega} |v|^p \, dH^{n-1} \right],$$

with c depending only on Ω and p; hence by (3.23) and since $\lim_{\varepsilon \to 0} \varepsilon / r_\varepsilon^{2p-1} > 0$

$$\int_\Omega |u_\varepsilon|^p \, dx \leqq c \left(\int_\Omega |D^2 u_\varepsilon|^p \, dx + \varepsilon \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^p \, dx \right). \tag{3.25}$$

If η is properly chosen, inequalities (3.22), (3.24), (3.25) yield

$$\int_\Omega |D^2 u_\varepsilon|^p \, dx + \varepsilon \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^p \, dx \leqq c;$$

then $\int_{\Sigma_\varepsilon} |u_\varepsilon|^p \, dx \to 0$ by (3.24) and $\|u_\varepsilon\|_{W^{2,p}(\Omega)} \leqq c$ by (3.25). □

IV. Remarks

In this section we give some extensions of Theorem [II.3], and we show how this applies to the mechanical problem mentioned in the introduction.

Remark [IV.1]. The function $h : \partial\Omega \to]0, +\infty[$ may be assumed to be only continuous: in this case it is enough to approximate h uniformly by smooth functions h_j; applying Theorem [II.3] to h_j and passing to the limit in j gives the result for h.

Remark [IV.2]. Theorem [II.3] can be extended by a slight modification of the proof to the case

$$F_\varepsilon(u) = G_\varepsilon(u) + \varepsilon \int_{\Sigma_\varepsilon} f(x, D^2 u) \, dx,$$

where the functionals G_ε satisfy

(4.1) $\liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon) \geqq G(u)$ if $u_\varepsilon \to u$ weakly in $W^{2,p}(\Omega)$.

(4.2) $\lim_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon) = G(u)$ if $u_\varepsilon \to u$ strongly in $W^{2,p}(\Omega)$.

(4.3) $G_\varepsilon(u) \geqq \int_\Omega |D^2u|^p \, dx$ for every $\varepsilon > 0$ and $u \in W^{2,p}(\Omega)$.

The Γ-limits G'_L and G''_M are the functionals defined in Section II.

Remark [IV.3]. It is clear that if one of the two limits $\lim_{\varepsilon\to 0} \varepsilon/r_\varepsilon^{2p-1}$ and $\lim_{\varepsilon\to 0} \varepsilon/r_\varepsilon^{p-1}$ fails to exist, then the functionals F_ε do not $\Gamma^-(L^p(\mathbb{R}^n))$-converge.

We are now able to study the convergence of the solutions u_ε of (1.2): it is enough to prove that the functionals F_ε defined in (1.1) satisfy the requirements of Remark [IV.2].

We recall that the Young modulus E_ε of the material in Σ_ε vanishes as $\varepsilon \to 0$, and the Poisson coefficient σ_ε converges to some $\sigma_0 > -1$. Since for every $u \in H_0^2(\Omega_\varepsilon)$

$$\int_{\Omega_\varepsilon} \det D^2u \, dx = 0,$$

we may write

$$F_\varepsilon(u) = G_\varepsilon(u) + \frac{E_\varepsilon}{1-\sigma_\varepsilon^2} \int_{\Sigma_\varepsilon} f(D^2u) \, dx,$$

where

$$G_\varepsilon(u) = \int_\Omega \left[\frac{E}{1-\sigma^2} |\Delta u|^2 - 2 \left(\frac{E}{1+\sigma} + \frac{(\sigma_\varepsilon - \sigma_0) E_\varepsilon}{1-\sigma_\varepsilon^2} \right) \det D^2u \right] dx,$$

$$f(D^2u) = |\Delta u|^2 - 2(1-\sigma_0) \det D^2u.$$

Since the Poisson coefficients are numbers less than $\frac{1}{2}$ (and greater than 0 for all known materials: see [6]), the function f satisfies hypotheses (2.1), (2.2), (2.3); moreover it is easy to see that G_ε satisfies hypotheses (4.1), (4.2), (4.3). Then Remark [IV.2] applies, and so the conclusion described in the introduction is attained.

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Scuola Normale Superiore
Pisa, Italy

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