ON THE LIMITS OF PERIODIC RIEMANNIAN METRICS[†]

By

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I. Introduction

In this paper we study the asymptotic behaviour as $\varepsilon \to 0^+$ of the infima of the functionals

$$F_\varepsilon(u) = \int_0^1 f\left(\frac{u(t)}{\varepsilon}, u'(t)\right) dt$$

on the space $W(z_0, z_1) = \{u \in W^{1,p}(0,1;\mathbf{R}^n) : u(0) = z_0,\ u(1) = z_1\}$, where $p > 1$, $z_0, z_1 \in \mathbf{R}^n$ and $f(s,z)$ is a Borel function which is convex in z, periodic in s and satisfying

$$\lambda |z|^p \leqq f(s,z) \leqq \Lambda(1 + |z|^p) \qquad (0 < \lambda \leqq \Lambda)$$

for every $(s,z) \in \mathbf{R}^n \times \mathbf{R}^n$.

Our main result (Theorem III.1) may be stated as follows:

There exists a convex function $\varphi : \mathbf{R}^n \to \mathbf{R}$ satisfying

$$\lambda |z|^p \leqq \varphi(z) \leqq \Lambda(1 + |z|^p) \qquad \text{for every } z \in \mathbf{R}^n,$$

such that for every $z_0, z_1 \in \mathbf{R}^n$ and for every bounded continuous function $g : \mathbf{R}^n \to \mathbf{R}$

$$\lim_{\varepsilon \to 0^+} \inf \left\{ F_\varepsilon(u) + \int_0^1 g(u)dt : u \in W(z_0, z_1) \right\}$$

$$= \min \left\{ \int_0^1 [\varphi(u') + g(u)]dt : u \in W(z_0, z_1) \right\}.$$

If $f(s,z)$ is p-homogeneous with respect to z, then φ is p-homogeneous. Moreover if $(u_\varepsilon)_{\varepsilon > 0} \subset W(z_0, z_1)$ is such that

$$\lim_{\varepsilon \to 0^+} \left[F_\varepsilon(u_\varepsilon) + \int_0^1 g(u_\varepsilon)dt - \inf \left\{ F_\varepsilon(u) + \int_0^1 g(u)dt : u \in W(z_0, z_1) \right\} \right] = 0$$

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then from $u_\varepsilon \to u$ *in* $L^p(0,1;\mathbf{R}^n)$ *it follows that*

$$\int_0^1 [\varphi(u')+g(u)]dt = \min\left\{\int_0^1 [\varphi(u')+g(u)]dt : u \in W(z_0, z_1)\right\}.$$

To prove this result we make use of the techniques of Γ-convergence, a concept first introduced by E. De Giorgi and T. Franzoni [4] in 1975 and later developed by several authors (for further references see [3]).

In Section IV we produce an example showing that in the case $f(s,z)=a(s)|z|^2$ the function φ need not be a quadratic form. Note that the energy integrals $\int_0^1 a(u/\varepsilon)|u'|^2 dt$ are associated with Riemannian metrics, while $\int_0^1 \varphi(u')dt$ is not (although we may say that it is associated with a Finsler metric). Therefore the space of Riemannian metrics is not closed in the space of Finsler metrics, with respect to the Γ-convergence of the energy integrals.

II. Preliminary lemmas

We give hereafter the definitions and main results of the Γ-convergence theory, which will be used in the proofs of the theorems in Section III. The general statements and the proofs of what follows may be found in [1], [4].

Let X denote a topological space, and for every $h \in \mathbf{N}$ take a function $F_h : X \to [-\infty, +\infty]$.

Definition II.1. For all $x \in X$ we set

$$\Gamma^-(X)\liminf_h F_h(x) = \sup_{U\in\mathcal{T}(x)} \liminf_h \inf_{y\in U} F_h(y), \tag{2.1}$$

$$\Gamma^-(X)\limsup_h F_h(x) = \sup_{U\in\mathcal{T}(x)} \limsup_h \inf_{y\in U} F_h(y), \tag{2.2}$$

where $\mathcal{T}(x)$ is the family of the neighbourhoods of x in X. If the two Γ-limits (2.1), (2.2) coincide at the point x, their common value will be indicated by $\Gamma^-(X)\lim_h F_h(x)$.

Proposition II.2. *The* Γ*-limits* (2.1), (2.2), *regarded as functions of x, are lower semicontinuous with respect to the topology of X.*

Proposition II.3. *If* $G : X \to \mathbf{R}$ *is continuous, then for every* $x \in X$

$$\Gamma^-(X)\liminf_h (G+F_h)(x) = G(x) + \Gamma^-(X)\liminf_h F_h(x),$$

$$\Gamma^-(X)\limsup_h (G+F_h)(x) = G(x) + \Gamma^-(X)\limsup_h F_h(x).$$

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$c \in \mathbf{R}$ there exists a compact set $K \subseteq X$ such that, for all h, $\{x \in X : F_h(x) \leqq c\} \subseteq K$. Assume also that for all $x \in X$ there exists

$$\Gamma^-(X) \lim_h F_h(x) = F(x).$$

Then we have

$$\lim_h \inf_{x \in X} F_h(x) = \min_{x \in X} F(x).$$

If in addition (x_h) is a sequence such that

$$\lim_h F_h(x_h) = \lim_h \inf_{x \in X} F_h(x),$$

then by $x_h \to x_0$ in X it follows that $F(x_0) = \min_{x \in X} F(x)$.

Proposition II.5. *Assume X is a metric space; then for every $x \in X$*

$$\Gamma^-(X) \liminf_h F_h(x) = \min \left\{ \liminf_h F_h(x_h) : x_h \to x \text{ in } X \right\},$$

$$\Gamma^-(X) \limsup_h F_h(x) = \min \left\{ \limsup_h F_h(x_h) : x_h \to x \text{ in } X \right\}.$$

Proposition II.6. *Assume X is a separable metric space; then from every sequence (F_h) of functions from X into $[-\infty, +\infty]$ we may select a subsequence (F_{h_k}) such that for every $x \in X$ there exists $\Gamma^-(X) \lim_k F_{h_k}(x)$.*

Remark. If instead of a sequence $(F_h)_{h \in \mathbf{N}}$, with $h \to +\infty$, we deal with a family $(F_\varepsilon)_{\varepsilon > 0}$, with $\varepsilon \to 0^+$, of functions from X into $[-\infty, +\infty]$, we may define $\Gamma^-(X) \liminf_\varepsilon F_\varepsilon(x)$ and $\Gamma^-(X) \limsup_\varepsilon F_\varepsilon(x)$ by modifying (2.1), (2.2) in the natural way. Propositions II.2, II.3 and II.4 still hold for $(F_\varepsilon)_{\varepsilon > 0}$, with the obvious changes in the statements.

Proposition II.7. *Assume X is a metric space, and let $x \in X$. The following conditions are equivalent:*

$$\Gamma^-(X) \lim_\varepsilon F_\varepsilon(x) = L;$$

from every sequence $\varepsilon_h \to 0^+$ we may select a subsequence (ε_{h_k}) such that $\Gamma^-(X) \lim_k F_{\varepsilon_{h_k}}(x) = L$.

III. Results

Let $n \geqq 1$ be an integer, $p > 1$ a real number. We will denote by I the interval $(0, 1)$, and by $\mathscr{A}$ the family of the open sets of I. If $A \in \mathscr{A}$, the symbols $L^p(A)$,

$W^{1,p}(A)$ will always stand for $L^p(A;\mathbf{R}^n)$, $W^{1,p}(A;\mathbf{R}^n)$ respectively. If $A, B \in \mathscr{A}$, by $A \subset\subset B$ we mean that $\bar{A} \subset B$. Finally, we will denote by Y the cube $[0,1)^n$.

Let $f: \mathbf{R}^n \times \mathbf{R}^n \to [0, +\infty[$ be a Borel function satisfying:

$$\text{(3.1)} \quad \lambda |z|^p \leqq f(s,z) \leqq \Lambda(1+|z|^p) \qquad \text{for all } (s,z) \in \mathbf{R}^n \times \mathbf{R}^n \quad (0<\lambda \leqq \Lambda);$$

$$\text{the function } z \mapsto f(s,z) \text{ is convex on } \mathbf{R}^n, \text{ for every } s \in \mathbf{R}^n;$$

$$\text{the function } s \mapsto f(s,z) \text{ is } Y\text{-periodic, for every } z \in \mathbf{R}^n.$$

For all $\varepsilon > 0$, $A \in \mathscr{A}$ and $u \in W^{1,p}(A)$ we define

$$F_\varepsilon(u,A) = \int_A f\left(\frac{u}{\varepsilon}, u'\right) dt;$$

moreover if $v \in W^{1,p}(A)$ we set

$$\Phi_v(u,A) = \begin{cases} 0 & \text{if } u - v \in W_0^{1,p}(A), \\ +\infty & \text{otherwise.} \end{cases}$$

Our main result is the following.

Theorem III.1. *Let f satisfy the hypotheses above. Then there exists a convex function $\varphi: \mathbf{R}^n \to [0, +\infty[$ satisfying*

$$\lambda |z|^p \leqq \varphi(z) \leqq \Lambda(1+|z|^p) \qquad \text{for all } z \in \mathbf{R}^n$$

such that for every $A \in \mathscr{A}$ and $u, u_0 \in W^{1,p}(A)$

$$\Gamma^-(L^p(A)) \lim_\varepsilon F_\varepsilon(u,A) = \int_A \varphi(u')dt,$$

$$\text{(3.2)} \qquad \Gamma^-(L^p(A)) \lim_\varepsilon [F_\varepsilon(u,A) + \Phi_{u_0}(u,A)] = \int_A \varphi(u')dt + \Phi_{u_0}(u,A).$$

$$\text{(3.2)} \qquad \Gamma^-(L^p(A)) \lim_\varepsilon [F_\varepsilon(u,A) + \Phi_{u_0}(u,A)] = \int_A \varphi(u')dt + \Phi_{u_0}(u,A).$$

Moreover for every $z \in \mathbf{R}^n$ the following representation formula for the function φ holds:

$$\varphi(z) = \lim_\varepsilon \inf\{F_\varepsilon(u,I) : u \in W^{1,p}(I),\ u(0)=0,\ u(1)=z\}$$

$$= \lim_\varepsilon \inf\{F_\varepsilon(u,I) : u \in W^{1,p}(I),\ u(1)-u(0)=z\}.$$

We remark that from Proposition II.3 and Theorem III.1 it follows that for every bounded continuous function $g: \mathbf{R}^n \to \mathbf{R}$, for every $A \in \mathscr{A}$ and $u, u_0 \in W^{1,p}(A)$

$$\Gamma^{-}(L^{p}(A))\lim_{\varepsilon}\left[F_{\varepsilon}(u,A)+\int_{A}g(u)dt\right]=\int_{A}[\varphi(u')+g(u)]dt,$$

$$\Gamma^{-}(L^{p}(A))\lim_{\varepsilon}\left[F_{\varepsilon}(u,A)+\int_{A}g(u)dt+\Phi_{u_0}(u,A)\right]$$

$$=\int_{A}[\varphi(u')+g(u)]dt+\Phi_{u_0}(u,A).$$

We shall prove Theorem III.1 after some propositions.

Fix a sequence $\varepsilon_h\to 0^+$; for all $A\in\mathscr{A}$ and $u\in W^{1,p}(A)$ we define

$$\bar{F}(u,A)=\Gamma^{-}(L^{p}(A))\limsup_{h}F_{\varepsilon_h}(u,A).$$

Proposition III.2. *Let $A,B,C\in\mathscr{A}$, with $C\subset\subset A\cup B$. For every $u\in W^{1,p}(A\cup B)$*

$$\bar{F}(u,C)\leqq\bar{F}(u,A)+\bar{F}(u,B).$$

Proof. Let K be a compact subset of A such that $\bar{C}\setminus B\subset\mathring{K}$, and put $\delta=\operatorname{dist}(K,\partial A)$. Fix an integer $\nu\geqq 1$ and define for $i=1,\ldots,\nu$

$$A_i=\left\{t\in I:\operatorname{dist}(t,K)<i\frac{\delta}{\nu}\right\}.$$

Set $A_0=\mathring{K}$, and let $\varphi_i\in C_0^{\infty}(A_i)$ be such that

$$0\leqq\varphi_i\leqq 1,$$

$$\varphi_i=1\quad\text{on } A_{i-1},$$

$$|\varphi_i'|\leqq 2\frac{\nu}{\delta}.$$

We denote hereafter bv the same letter c all the positive constants which do not

We denote hereafter by the same letter c all the positive constants which do not depend on K, i, ν and δ.

By Proposition II.5 there exist two sequences (u_h), (v_h) such that $u_h\to u$ in $L^p(A)$, $v_h\to u$ in $L^p(B)$ and

$$\bar{F}(u,A)=\limsup_{h}F_{\varepsilon_h}(u_h,A),\tag{3.3}$$

$$\bar{F}(u,B)=\limsup_{h}F_{\varepsilon_h}(v_h,B).\tag{3.4}$$

Set

$$w_{i,h}=\varphi_i u_h+(1-\varphi_i)v_h;$$

we have

$$F_{\varepsilon_h}(w_{i,h}, C) \leqq F_{\varepsilon_h}(u_h, C \cap A_{i-1}) + F_{\varepsilon_h}(v_h, C \setminus \bar{A}_i)$$

$$+ \Lambda \int_{C \cap (A_i \setminus A_{i-1})} [1 + |\varphi_i'(u_h - v_h) + \varphi_i u_h' + (1 - \varphi_i) v_h'|^p] dt$$

$$\leqq F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B)$$

$$+ c\left[\left(\frac{\nu}{\delta}\right)^p \int_C |u_h - v_h|^p dt + \int_{C \cap (A_i \setminus A_{i-1})} (1 + |u_h'|^p + |v_h'|^p) dt\right].$$

For every h, there exists an index $i_h \leqq \nu$ such that

$$\int_{C \cap (A_{i_h} \setminus A_{i_h - 1})} (1 + |u_h'|^p + |v_h'|^p) dt \leqq \frac{1}{\nu} \int_{C \cap A \cap B} (1 + |u_h'|^p + |v_h'|^p) dt$$

$$\leqq \frac{c}{\nu} [1 + F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B)],$$

so that

$$F_{\varepsilon_h}(w_{i_h,h}, C) \leqq \left(1 + \frac{c}{\nu}\right) [F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B)] + \frac{c}{\nu} + c\left(\frac{\nu}{\delta}\right)^p \int_C |u_h - v_h|^p dt. \tag{3.5}$$

It is easy to see that the sequence $(w_{i_h,h})$ converges to u in $L^p(C)$. Letting $h \to +\infty$ in (3.5), by (3.3) and (3.4) we obtain

$$\bar{F}(u, C) \leqq \limsup_h F_{\varepsilon_h}(w_{i_h,h}, C)$$

$$\leqq \left(1 + \frac{c}{\nu}\right) [F(u, A) + F(u, B)] + \frac{c}{\nu},$$

and the proof is completed since ν was arbitrary. ∎

A slight modification of the proof above yields the following result.

Proposition III.3. *For every $A, B \in \mathscr{A}$ with $B \subset\subset A$, for every compact subset K of B and every $u \in W^{1,p}(A)$ we have $\bar{F}(u, A) \leqq \bar{F}(u, B) + \bar{F}(u, A \setminus K)$, so that $\bar{F}(u, A) = \sup\{\bar{F}(u, B) : B \subset\subset A\}$.*

Proposition III.4. *We can select from (ε_h) a subsequence (ε_{h_k}) such that for all $A \in \mathscr{A}$ and $u \in W^{1,p}(A)$ there exists*

$$F(u, A) = \Gamma^-(L^p(A)) \lim_k F_{\varepsilon_{h_k}}(u, A). \tag{3.6}$$

In addition for all $u \in W^{1,p}(I)$ the set function $A \mapsto F(u, A)$ is the restriction to $\mathscr{A}$ of a regular Borel measure.

Proof. Choose a countable base $\mathscr{U}$ for the open sets of I, closed under finite unions. By Proposition II.6 we may construct (by a diagonal process) a subsequence (ε_{h_k}) such that for all $B \in \mathscr{U}$ there exists

$$G(u, B) = \Gamma^-(L^p(B)) \lim_k F_{\varepsilon_{h_k}}(u, B)$$

for all $u \in W^{1,p}(B)$. Set for all $A \in \mathscr{A}$ and $u \in W^{1,p}(A)$

$$F(u, A) = \sup\{G(u, B) : B \in \mathscr{U}, B \subset\subset A\}.$$

Applying Proposition III.2 and Proposition III.3 to the sequence (ε_{h_k}) we obtain that the set function $B \mapsto G(u, B)$ is subadditive on $\mathscr{U}$. Moreover it is clear, by the definition of G, that for all $A, B \in \mathscr{U}$ with $A \cap B = \varnothing$ and for all $u \in W^{1,p}(A \cup B)$

$$G(u, A \cup B) \geqq G(u, A) + G(u, B).$$

Then the set function $A \mapsto F(u, A)$ is subadditive and superadditive on $\mathscr{A}$, and it is also regular from the inside. By proposition (5.5) and theorem (5.6) of [5], for all $u \in W^{1,p}(I)$ the set function $A \mapsto F(u, A)$ is the restriction to $\mathscr{A}$ of a regular Borel measure.

We still have to prove (3.6). Put for all $A \in \mathscr{A}$ and $u \in W^{1,p}(A)$

$$\underline{H}(u, A) = \Gamma^-(L^p(A)) \liminf_k F_{\varepsilon_{h_k}}(u, A),$$

$$\bar{H}(u, A) = \Gamma^-(L^p(A)) \limsup_k F_{\varepsilon_{h_k}}(u, A).$$

Since $G = \underline{H} = \bar{H}$ on $\mathscr{U}$, by Proposition III.3

$$\bar{H}(u, A) = \sup\{\bar{H}(u, B) : B \subset\subset A\} = \sup\{G(u, B) : B \in \mathscr{U}, B \subset\subset A\} \leqq \underline{H}(u, A),$$
$$H(u, A) = \sup\{H(u, B) : B \subset\subset A\} = \sup\{G(u, B) : B \in \mathscr{U}, B \subset\subset A\} \leqq \underline{H}(u, A),$$

so that $\underline{H}(u, A) = \bar{H}(u, A) = F(u, A)$. ■

Proposition III.5. *Let $A \in \mathscr{A}$, and $u \in W^{1,p}(A)$. If F is the functional defined in* (3.6), *then*

$$F(u, A) \geqq \inf\left\{\limsup_k F_{\varepsilon_{h_k}}(v_k, A) : v_k - u \in W_0^{1,p}(A),\ v_k \to u \text{ in } L^p(A)\right\}.$$

Proof. Let K be any compact subset of A and let δ, A_i, φ_i, (u_h) be as in the proof of Proposition III.2. Set

$$w_{i,k} = \varphi_i u_k + (1 - \varphi_i) u;$$

we have

$$F_{\varepsilon_{h_k}}(w_{i,k}, A) \leqq F_{\varepsilon_{h_k}}(u_k, A) + \Lambda \int_{A\setminus K} (1+|u'|^p)dt$$

$$+\Lambda \int_{A_i\setminus A_{i-1}} [1+|\varphi_i'(u_k-u)+\varphi_i u_k' + (1-\varphi_i)u'|^p]dt.$$

As in Proposition III.2, for a suitable (i_k) we obtain

$$\limsup_k F_{\varepsilon_{h_k}}(w_{i_k,k}, A) \leqq \left(1+\frac{c}{\nu}\right)\left[F(u,A)+\Lambda \int_{A\setminus K} (1+|u'|^p)dt + \frac{c}{\nu}\right],$$

and the result follows since ν and K were arbitrary. ∎

Proposition III.6. *Let F be as in* (3.6). *For all $A \in \mathscr{A}$ and $u, u_0 \in W^{1,p}(A)$ we have*

$$(3.7) \qquad F(u,A)+\Phi_{u_0}(u,A) = \Gamma^-(L^p(A)) \lim_k [F_{\varepsilon_{h_k}}(u,A)+\Phi_{u_0}(u,A)].$$

Proof. If $u-u_0 \notin W_0^{1,p}(A)$, then the left-hand side in (3.7) is $+\infty$; let $u_k \to u$ in $L^p(A)$, and suppose that

$$\liminf_k [F_{\varepsilon_{h_k}}(u_k,A)+\Phi_{u_0}(u_k,A)] < +\infty.$$

Then $S = \{k \in \mathbf{N} : \Phi_{u_0}(u_k, A) < +\infty\}$ is infinite and by the coercivity (3.1) of f a subsequence of $\{u_k : k \in S\}$ converges weakly to u in $W^{1,p}(A)$, but then necessarily $u-u_0 \in W_0^{1,p}(A)$, which is a contradiction.

In the case $u-u_0 \in W^{1,p}(A)$, we have $\Phi_{u_0}(\cdot, A) \equiv \Phi_u(\cdot, A)$, and by Propositions III.5 and II.5

$$F(u,A) \geqq \inf\left\{\limsup_k [F_{\varepsilon_{h_k}}(v_k,A)+\Phi_u(v_k,A)] : v_k \to u \text{ in } L^p(A)\right\}$$

$$= \inf\left\{\limsup_k [F_{\varepsilon_{h_k}}(v_k,A)+\Phi_{u_0}(v_k,A)] : v_k \to u \text{ in } L^p(A)\right\}$$

$$= \Gamma^-(L^p(A)) \limsup_k [F_{\varepsilon_{h_k}}(u,A)+\Phi_{u_0}(u,A)]$$

$$\geqq \Gamma^-(L^p(A)) \liminf_k [F_{\varepsilon_{h_k}}(u,A)+\Phi_{u_0}(u,A)]$$

$$\geqq \Gamma^-(L^p(A)) \liminf_k F_{\varepsilon_{h_k}}(u,A)$$

$$= F(u,A) = F(u,A)+\Phi_{u_0}(u,A).$$ ∎

Proposition III.7. *Let F be as in* (3.6). *There exists a convex function*

$\varphi : \mathbf{R}^n \to \mathbf{R}$, which is p-homogeneous if $f(s, z)$ is p-homogeneous in z, such that for every $A \in \mathscr{A}$ and $u \in W^{1,p}(A)$

$$F(u, A) = \int_A \varphi(u')dt.$$

Proof. Fix $A \in \mathscr{A}$, $u \in W^{1,p}(A)$, $a \in \mathbf{R}^n$. Then

$$F(u + a, A) = F(u, A). \tag{3.8}$$

To see this, take $u_k \to u$ in $L^p(A)$ such that

$$F(u, A) = \lim_k F_{\varepsilon_{h_k}}(u_k, A);$$

it is easy to find a sequence $a_k \to a$ in $\mathbf{R}^n$ such that $(\varepsilon_{h_k})^{-1} a_k \in \mathbf{Z}^n$. The sequence $(u_k + a_k)$ tends to $u + a$ in $L^p(A)$, therefore

$$F(u + a, A) \leqq \liminf_k \int_A f((\varepsilon_{h_k})^{-1}(u_k + a_k), u'_k)dt$$

$$= \lim_k \int_A f((\varepsilon_{h_k})^{-1} u_k, u'_k)dt = F(u, A).$$

The opposite inequality may be proved in the same way, thus obtaining (3.8).

Define for every $A \in \mathscr{A}$ and $v \in L^p(A)$

$$L(v, A) = F(w, A),$$

where $w \in W^{1,p}(A)$ is any function such that $w' = v$ a.e. on A. By (3.8) the functional $v \mapsto L(v, A)$ is well defined; moreover, by Proposition II.2, it is lower semicontinuous on $L^p(A)$, and the set function $A \mapsto L(v, A)$ may be extended to a measure (defined on the Borel sets $\mathscr{B}$ of I), which we denote by $\tilde{L}(v, \cdot)$. We prove that the functional $\tilde{L}$ is local on $\mathscr{B}$, i.e. that $\tilde{L}(v_1, B) = \tilde{L}(v_2, B)$ whenever $v_1 = v_2$
that the functional $\tilde{L}$ is local on $\mathscr{B}$, i.e. that $\tilde{L}(v_1, B) = \tilde{L}(v_2, B)$ whenever $v_1 = v_2$
a.e. on $B \in \mathscr{B}$.

Let $v_1, v_2 \in L^p(I)$ with $v_1 = v_2$ a.e. on $B \in \mathscr{B}$: it is not restrictive to assume that $v_1 = v_2$ everywhere on B, and that $v_1 \leqq v_2$ on I. By Lusin's theorem, for any $\varepsilon > 0$ there exists $A_\varepsilon \in \mathscr{A}$, with $\text{meas}(A_\varepsilon) < \varepsilon$, such that v_1 and v_2 are continuous on $I \setminus A_\varepsilon$. Then the set

$$B_\varepsilon = A_\varepsilon \cup \{t \in I : v_2(t) < v_1(t) + \varepsilon\}$$

is open, and $B \subseteq B_\varepsilon$. Define

$$v_\varepsilon = \begin{cases} v_2 & \text{on } B_\varepsilon \\ v_1 + \varepsilon & \text{otherwise} \end{cases}$$

so that $v_\varepsilon \to v_1$ in $L^p(I)$ as $\varepsilon \to 0^+$. Take $A \in \mathscr{A}$ and K compact such that $K \subseteq B \subseteq A$. Then, since $L(\cdot, A)$ is lower semicontinuous on $L^p(A)$, we have

$$\begin{aligned}
\tilde{L}(v_1, B) &\leqq L(v_1, A)\\
&\leqq \liminf_\varepsilon L(v_\varepsilon, A)\\
&\leqq \liminf_\varepsilon\,[L(v_\varepsilon, A \cap B_\varepsilon) + L(v_\varepsilon, A \setminus K)]\\
&\leqq L(v_2, A) + \Lambda \lim_\varepsilon \int_{A\setminus K} (1 + |v_\varepsilon|^p)dt\\
&= L(v_2, A) + \Lambda \int_{A\setminus K} (1 + |v_1|^p)dt
\end{aligned}$$

whence $\tilde{L}(v_1, B) \leqq \tilde{L}(v_2, B)$ since A and K are arbitrary. Taking $w_\varepsilon = v_1$ on B_ε and $w_\varepsilon = v_2 - \varepsilon$ otherwise, one proves the opposite inequality, thus obtaining the locality of $\tilde{L}$.

By theorem 1.4 of [2] there exists a function $\varphi(t, z)$ convex in $z \in \mathbf{R}^n$ and such that for every $B \in \mathscr{B}$ and $v \in L^p(I)$

$$\tilde{L}(v, B) = \int_B \varphi(t, v(t))dt.$$

It is easy to see that since f is independent of t, the same is true for φ. Then for every $A \in \mathscr{A}$ and $u \in W^{1,p}(A)$

$$F(u, A) = L(u', A) = \tilde{L}(u', A) = \int_A \varphi(u')dt.$$

Using Proposition II.5 one may prove, after some simple calculations, that if $f(s, z)$ is p-homogeneous in z, then also φ is p-homogeneous. ∎

Define for every $z \in \mathbf{R}^n$ and $T > 0$

$$M'_T(z) = \inf\left\{\frac{1}{T}\int_0^T f(u, u')dt : u \in W^{1,p}(0, T),\ u(T) - u(0) = Tz\right\},$$

$$M''_T(z) = \inf\left\{\frac{1}{T}\int_0^T f(u, u')dt : u \in W^{1,p}(0, T),\ u(0) = 0,\ u(T) = Tz\right\}.$$

Clearly, $M'_T(z) \leqq M''_T(z)$.

Proposition III.8. *For every* $z \in \mathbf{R}^n$ *there exists* $M''(z) = \lim_{T\to\infty} M''_T(z)$.

We shall divide the proof in several steps.

Step 1. *For every* $z \in \mathbf{Q}^n$ *there exists* $S > 0$ *such that the limit* $\lim_{k\to\infty} M''_{2^kS}(z)$ *exists.*

To see this, let $z = (p_1/q_1, \dots, p_n/q_n)$ with p_i, q_i integers, $q_i > 0$, and put $S = q_1 \cdot \cdots \cdot q_n$. Take $k \in \mathbf{N}$ and $\varepsilon > 0$, and let $u \in W^{1,p}(0, 2^kS)$ be such that $u(0) = 0$, $u(2^kS) = 2^kSz$ and

$$\int_0^{2^kS} f(u, u')dt \leqq 2^kS(M''_{2^kS}(z) + \varepsilon).$$

Define

$$v(t) = \begin{cases} u(t) & \text{if } 0 \leqq t \leqq 2^kS, \\ u(t - 2^kS) + 2^kSz & \text{if } 2^kS \leqq t \leqq 2^{k+1}S. \end{cases}$$

Remarking that $Sz \in \mathbf{Z}^n$, we get

$$2^{k+1}SM''_{2^{k+1}S}(z) \leqq \int_0^{2^{k+1}S} f(v, v')dt$$

$$= 2\int_0^{2^kS} f(u, u')dt \leqq 2^{k+1}S(M''_{2^kS}(z) + \varepsilon).$$

Since ε was arbitrary, $M''_{2^{k+1}S}(z) \leqq M''_{2^kS}(z)$ and the sequence decreases to a limit which we call $M''(z)$.

Step 2. *For all* $z \in \mathbf{Q}^n$ *we have* $\lim_{T\to\infty} M''_T(z) = M''(z)$.

To prove this, fix $z \in \mathbf{Q}^n$ and take S as in Step 1. For all $T > 0$ we may write

To prove this, fix $z \in \mathbf{Q}^n$ and take S as in Step 1. For all $T > 0$ we may write

$$T = \sum_{k=0}^{+\infty} \alpha_k 2^kS + a \qquad (0 \leqq a < S)$$

for suitable coefficients $\alpha_k \in \{0, 1\}$, all vanishing for k large enough. Set $T_0 = 0$, $T_k = \sum_{i=0}^{k-1} \alpha_i 2^iS$; fix $\varepsilon > 0$ and for all $k \in \mathbf{N}$ let $u_k \in W^{1,p}(0, 2^kS)$ be such that $u_k(0) = 0$, $u_k(2^kS) = 2^kSz$ and

$$\int_0^{2^kS} f(u_k, u'_k)dt \leqq 2^kS(M''_{2^kS}(z) + \varepsilon).$$

Put

$$v(t) = \begin{cases} u_k(t - T_k) + T_k z & \text{if } T_k \leqq t \leqq T_{k+1}, \\ tz & \text{if } T - a \leqq t \leqq T. \end{cases}$$

By simple changes of variables we obtain

$$TM''_T(z) \leqq \int_0^T f(v, v')dt = \sum_{k=0}^{+\infty} \alpha_k \int_0^{2^k S} f(u_k, u_k')dt + \int_{T-a}^{T} f(tz, z)dt$$

$$\leqq \sum_{k=0}^{+\infty} \alpha_k 2^k S(M''_{2^k S}(z) + \varepsilon) + \int_{T-S}^{T} f(tz, z)dt.$$

By Step 1, for a suitable k_ε we have $M''_{2^k S}(z) < M''(z) + \varepsilon$ for every $k > k_\varepsilon$, so that

$$TM''_T(z) \leqq \sum_{k=0}^{k_\varepsilon} 2^k SM''_{2^k S}(z) + 2^{k_\varepsilon+1} S\varepsilon + T(M''(z) + 2\varepsilon) + \int_{T-S}^{T} f(tz, z)dt.$$

Dividing both sides by T and letting $T \to +\infty$ we obtain, since ε was arbitrary,

(3.9) $$\limsup_{T \to +\infty} M''_T(z) \leqq M''(z).$$

We reason by contradiction: suppose that

(3.10) $$\liminf_{T \to +\infty} M''_T(z) < M''(z).$$

Every $T > 0$ may be written as $T = m(T)S - b(T)$, with $m(T) \in \mathbf{N}$ and $0 \leqq b(T) < S$. Fix $\varepsilon > 0$ and choose $u \in W^{1,p}(0, T)$ such that $u(0) = 0$, $u(T) = Tz$ and $\int_0^T f(u, u')dt \leqq T(M''_T(z) + \varepsilon)$. Set

$$v(t) = \begin{cases} u(t) & \text{if } 0 \leqq t \leqq T, \\ tz & \text{if } T \leqq t \leqq T + b(T); \end{cases}$$

tz if $T \leqq t \leqq T + b(T)$;

reasoning as above we have

$$M''_{m(T)S}(z) \leqq \frac{T}{T + b(T)} M''_T(z) + \frac{1}{T + b(T)} \int_T^{T+S} f(tz, z)dt.$$

If (3.10) holds, there exists T_0 such that $M''_{m(T_0)S}(z) < M''(z)$; since $m(T_0) \in \mathbf{N}$, the same argument used in Step 1 yields that the sequence $(M''_{2^k m(T_0)S}(z))$ decreases to a limit $L < M''(z)$. On the other hand, we may prove an inequality analogous to (3.9) with $m(T_0)S$ in place of S, so that

$$\limsup_{T \to +\infty} M''_T(z) \leqq \lim_k M''_{2^k m(T_0)S}(z) = L < M''(z).$$

But

$$M''(z) \leqq \limsup_{T\to+\infty} M''_T(z),$$

therefore (3.10) is impossible.

Step 3. *For every $z \in \mathbf{R}^n$ there exists* $\lim_{T\to+\infty} M''_T(z)$.

Fix $z, w \in \mathbf{R}^n$, $T>0$, $\frac{1}{2}<k<1$ and $\varepsilon>0$, and let $u \in W^{1,p}(0,T)$ be such that $u(0)=0$, $u(T)=Tw$ and $\int_0^T f(u,u')dt \leqq T(M''_T(w)+\varepsilon)$. Set

$$v(t)=\begin{cases} u(t/k) & \text{if } 0\leqq t\leqq kT,\\ Tw+\dfrac{t-kT}{1-k}(z-w) & \text{if } kT\leqq t\leqq T.\end{cases}$$

Since $f(s,\cdot)$ is convex and controlled by (3.1), for every $(s,z)\in\mathbf{R}^n\times\mathbf{R}^n$ and $1<\alpha<2$ we have

$$|f(s,\alpha z)-f(s,z)|\leqq c(\alpha-1)(1+|z|^p)$$

with c independent of α, s, z. Then, denoting by the same letter c any positive constant which does not depend on z, w, T, k, ε, we have

$$\begin{aligned} TM''_T(z) &\leqq \int_0^T f(v,v')dt\\ &\leqq \int_0^{kT} f\left(u(t/k),\frac{1}{k}u'(t/k)\right)dt+(1-k)T\Lambda\left(1+\frac{|z-w|^p}{(1-k)^p}\right)\\ &= k\int_0^T [f(u,u'/k)-f(u,u')]dt+k\int_0^T f(u,u')dt\\ &\quad +(1-k)T\Lambda\left(1+\frac{|z-w|^p}{(1-k)^p}\right)\\ &\leqq c(1-k)\int_0^T (1+|u'|^p)dt+kT(M''_T(w)+\varepsilon)\\ &\quad +(1-k)T\Lambda\left(1+\frac{|z-w|^p}{(1-k)^p}\right)\\ &\leqq c(1-k)T\left[1+M''_T(w)+\frac{|z-w|^p}{(1-k)^p}\right]+kT(M''_T(w)+\varepsilon). \end{aligned}$$

Dividing both sides by T and letting $\varepsilon\to 0^+$ we have

$$M''_T(z) \leqq c(1-k)\left[1 + M''_T(w) + \frac{|z-w|^p}{(1-k)^p}\right] + kM''_T(w). \tag{3.11}$$

Now take $x \in \mathbf{R}^n$ and $y \in \mathbf{Q}^n$. Letting $T \to +\infty$ in (3.11), with $z = x$ and $w = y$, we obtain

$$\limsup_{T\to+\infty} M''_T(x) \leqq c(1-k)\left[1 + M''(y) + \frac{|x-y|^p}{(1-k)^p} + kM''(y)\right];$$

letting $y \to x$ and then $k \to 1$ yields

$$\limsup_{T\to+\infty} M''_T(x) \leqq \liminf_{\substack{y\to x\\ y\in\mathbf{Q}^n}} M''(y). \tag{3.12}$$

We use again (3.11), with $z = y$ and $w = x$. As above we obtain

$$\limsup_{\substack{y\to x\\ y\in\mathbf{Q}^n}} M''(y) \leqq \liminf_{T\to+\infty} M''_T(x),$$

which together with (3.12) completes the proof of Proposition III.8. ∎

Let (ε_h) be any sequence such that $\varepsilon_h \to 0^+$ and that (F_{ε_h}) is Γ-converging to some limit F. By Proposition III.7 the limit may be written as

$$F(u, A) = \int_A \varphi(u')dt$$

for a suitable convex function φ. We want to identify φ in terms of M' and M''.

Proposition III.9. *For every $z \in \mathbf{R}^n$, $\varphi(z) \geqq M''(z)$.*

Proof. Fix $z \in \mathbf{R}^n$; by Propositions III.6 and II.5 there exists a sequence $(u_h) \subset W^{1,p}(I)$ converging in $L^p(I)$ to the function $u(t) = zt$ and satisfying $u_h(0) = 0$, $u_h(1) = z$, $\lim_h F_{\varepsilon_h}(u_h) = \varphi(z)$. Then we have

0, $u_h(1) = z$, $\lim_h F_{\varepsilon_h}(u_h) = \varphi(z)$. Then we have

$$M''(z) = \lim_h M''_{1/\varepsilon_h}(z) \leqq \lim_h F_{\varepsilon_h}(u_h) = \varphi(z). \qquad ∎$$

Proposition III.10. *For every $z \in \mathbf{R}^n$, $\varphi(z) \leqq \liminf_h M'_{1/\varepsilon_h}(z)$.*

Proof. Take $z \in \mathbf{R}^n$; we may assume that the sequence $M'_{1/\varepsilon_h}(z)$ converges to some real number. By definition of $M'_T(z)$ there exists a sequence $(u_h) \subset W^{1,p}(I)$ such that

$$F_{\varepsilon_h}(u_h) \leqq M'_{1/\varepsilon_h}(z) + 1/h, \qquad u_h(1) - u_h(0) = z. \tag{3.13}$$

Since f is periodic with respect to s, it is not restrictive to assume that $|u_h(0)| \leqq \varepsilon_h$. By (3.13) the sequence (u_h) is bounded in $W^{1,p}(I)$; so we may select a subsequence, still denoted by (u_h), which converges weakly in $W^{1,p}(I)$ to a function v such that $\int_0^1 v'dt = z$. Then Jensen's inequality yields

$$\varphi(z) \leqq \int_0^1 \varphi(v')dt \leqq \lim_h F_{\varepsilon_h}(u_h) = \lim_h M'_{1/\varepsilon_h}(z). \quad \blacksquare$$

Proof of Theorem III.1. Propositions III.9 and III.10 imply that for any sequence (ε_h) such that (F_{ε_h}) is Γ-convergent, the corresponding function φ satisfies $\varphi(z) = M''(z) = \lim_h M'_{1/\varepsilon_h}(z)$ for every $z \in \mathbf{R}^n$. Thus, for every $z \in \mathbf{R}^n$ the limit $\lim_{T\to+\infty} M'_T(z) = M'(z)$ exists and

$$\varphi(z) = M'(z) = M''(z).$$

Let $\varepsilon_h \to 0^+$; by Proposition III.4 we may select a subsequence (ε_{h_k}) such that the functionals $F_{\varepsilon_{h_k}}(u, A)$ are Γ-converging to a limit $F(u, A)$ which by Proposition III.7 we may write as $\int_A \varphi(u')dt$. By the argument above

$$F(u, A) = \int_A M''(u')dt$$

for every $A \in \mathscr{A}$ and $u \in W^{1,p}(A)$. Then by Proposition II.7

$$\Gamma^-(L^p(A)) \lim_\varepsilon F_\varepsilon(u, A) = \int_A \varphi(u')dt.$$

To prove (3.2) apply Proposition III.6. $\blacksquare$

IV. An example

Let $n = 2$, and take two real constants α and β, with $0 < \alpha \leqq \beta$. Define on $[0, 1) \times [0, 1)$ the following function:

$$a(x) = \begin{cases} \beta & \text{if } x \in [(0, \frac{1}{2}) \times (\frac{1}{2}, 1)] \cup [(\frac{1}{2}, 1) \times (0, \frac{1}{2})] \\ \alpha & \text{otherwise} \end{cases}$$

and extend it by periodicity to a function defined on $\mathbf{R}^2$ which we still denote by a. For every $\varepsilon > 0$ set

$$W_\varepsilon = \{x \in \mathbf{R}^2 : a(x/\varepsilon) = \alpha\}, \qquad B_\varepsilon = \{x \in \mathbf{R}^2 : a(x/\varepsilon) = \beta\}$$

(for every $\varepsilon > 0$ we may think of $\mathbf{R}^2$ as of a chessboard composed by the "white" squares, W_ε, and the "black" squares, B_ε).

Consider the functionals

$$F_\varepsilon(u) = \int_0^1 a(u/\varepsilon)|u'|^2 dt,$$

defined on $W^{1,2}(I)$. By Theorem III.1 there exists a convex function $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that

$$\Gamma^-(L^2(I)) \lim_{\varepsilon} F_\varepsilon(u) = \int_0^1 \varphi(u')dt$$

for every $u \in W^{1,2}(I)$. In this section we shall prove:

Theorem IV.1. *If $\alpha \neq \beta$, the function $z \mapsto \varphi(z)$ is not a quadratic form.*

Proof. Fix $z \in \mathbf{Z}^2$, $z \neq (0,0)$, and set $u_0(t) = tz$. For every $h \in \mathbf{N}$ the functional $F_{1/h} + \Phi_{u_0}$ is coercive and lower semicontinuous in the weak topology of $W^{1,2}(I)$; denote by u_h one of its minimum points. By the convexity of φ and by Proposition II.4

$$\varphi(z) = \min\left\{\int_0^1 \varphi(u')dt : u - u_0 \in W_0^{1,2}(I)\right\} = \lim_h F_{1/h}(u_h). \tag{4.1}$$

We divide the remaining part of the proof into several steps.

Step 1. *For every h the function u_h is piecewise affine.*

Indeed, let Q be one of the white squares of $W_{1/h}$ such that the set $T_Q = \{t : u(t) \in Q\}$ is not empty, and set $t_1 = \min T_Q$, $t_2 = \max T_Q$. If $t_1 < t_2$ set

$$v(t) = \begin{cases} u_h(t) & \text{if } t \leqq t_1 \text{ or } t \geqq t_2 \\ \dfrac{u_h(t_2) - u_h(t_1)}{t_2 - t_1}(t - t_1) + u(t_1) & \text{if } t_1 \leqq t \leqq t_2 \end{cases}$$

and we have

$$\int_{t_1}^{t_2} \alpha |v'|^2 dt \leqq \int_{t_1}^{t_2} a(hu_h)|u_h'|^2 dt,$$

where the equality holds only if $u_h \equiv v$ in $[t_1, t_2]$.

This argument shows that u_h is affine on each of the white squares it crosses (there are only finitely many such squares, because $|u_h|$ is bounded). The set $B_{1/h}(u_h) = \{t \in (0,1) : u_h(t) \in B_{1/h}\}$ is then composed by finitely many open intervals: on each of them u_h is affine, since it solves the Euler equation $\beta u_h'' = 0$.

Step 2. *For every h the velocity $|u_h'(t)|$ is constant on the set $W_{1/h}(u_h) = \{t : u_h(t) \in W_{1/h}\}$ and on the set $B_{1/h}(u_h)$; the two constants need not be equal.*

Let $[t_1, s_1]$ and $[t_2, s_2]$ be any two intervals contained in $W_{1/h}(u_h)$, and in each of which u_h is affine. We may assume $t_1 < s_1 \leqq t_2 < s_2$. Set

$$c = \frac{|u_h(s_1) - u_h(t_1)| + |u_h(s_2) - u_h(t_2)|}{(s_1 - t_1) + (s_2 - t_2)},$$

$$s_1' = t_1 + \frac{|u_h(s_1) - u_h(t_1)|}{c}, \qquad t_2' = s_1' + (t_2 - s_1),$$

and define

$$v(t) = \begin{cases} u_h(t) & \text{if } t \leqq t_1 \text{ or } t \geqq s_2, \\ u_h(t_1) + \dfrac{u_h(s_1) - u_h(t_1)}{s_1' - t_1}(t - t_1) & \text{if } t_1 \leqq t \leqq s_1', \\ u_h(t - s_1' + s_1) & \text{if } s_1' \leqq t \leqq t_2', \\ u_h(t_2) + \dfrac{u_h(s_2) - u_h(t_2)}{s_2 - t_2'}(t - t_2') & \text{if } t_2' \leqq t \leqq s_2. \end{cases}$$

Then $F_{1/h}(v) \leqq F_{1/h}(u_h)$, and the equality holds if and only if $u_h = v$, that is, if $|u_h'|$ takes the same constant value on both $[t_1, s_1]$ and $[t_2, s_2]$. A similar argument may be employed for the intervals of $B_{1/h}(u_h)$.

For every $u \in W^{1,2}(I)$ and every $h \in \mathbf{N}$ set

$$L_h^W(u) = \int_{W_{1/h}(u)} |u'(t)|\,dt, \qquad L_h^B(u) = \int_{B_{1/h}(u)} |u'(t)|\,dt.$$

Step 3. *For every h we have $F_{1/h}(u_h) = [\sqrt{\alpha} L_h^W(u_h) + \sqrt{\beta} L_h^B(u_h)]^2$.*

By Step 2, if $L_h^W(u_h) = 0$ or $L_h^B(u_h) = 0$ the result is trivial. In the general case call c_h^W and c_h^B the two speeds of u_h, so that

$$|u_h'| = \begin{cases} c_h^W & \text{on } W_{1/h}(u_h), \\ c_h^B & \text{on } B_{1/h}(u_h). \end{cases}$$

Then $F_{1/h}(u_h) = \alpha c_h^W L_h^W(u_h) + \beta c_h^B L_h^B(u_h)$. If v is a different parametrization of the

curve u_h such that for some constants c^W and c^B
Then $F_{1/h}(u_h) = \alpha c_h^W L_h^W(u_h) + \beta c_h^B L_h^B(u_h)$. If v is a different parametrization of the

curve u_h, such that for some constants c^W and c^B

$$|v'| = \begin{cases} c^W & \text{on } W_{1/h}(v), \\ c^B & \text{on } B_{1/h}(v), \end{cases}$$

then we have $F_{1/h}(v) = \alpha c^W L_h^W(v) + \beta c^B L_h^B(v)$ and $L_h^W(v)/c^W + L_h^B(v)/c^B = 1$. But $L_h^W(v) = L_h^W(u_h)$ and $L_h^B(v) = L_h^B(u_h)$, therefore by $F_{1/h}(u_h) \leqq F_{1/h}(v)$ it follows that

$$F_{1/h}(u_h) = \min\{\alpha x L_h^W(u_h) + \beta y L_h^B(u_h) : x, y > 0,\ L_h^W(u_h)/x + L_h^B(u_h)/y = 1\}$$

$$= [\sqrt{\alpha} L_h^W(u_h) + \sqrt{\beta} L_h^B(u_h)]^2.$$

Step 4. *For every h there exists v_h such that $v_h - u_0 \in W_0^{1,2}(I)$ and*

$$v_h(t) \in W_{1/h} \qquad \textit{for all } t, \tag{4.2}$$

$$[F_{1/h}(v_h)]^{1/2} \leqq [F_{1/h}(u_h)]^{1/2} + (2\sqrt{\alpha} - \sqrt{\beta}) L_h^B(u_h). \tag{4.3}$$

If (t_1, t_2) is one of the intervals composing $B_{1/h}(u_h)$, the points $u_h(t_1)$, $u_h(t_2)$ belong to the boundary of a "black square" Q. Let $w_Q : (t_1, t_2) \to \mathbf{R}^2$ be a piecewise affine function joining $u_h(t_1)$ with $u_h(t_2)$, and such that $w_Q(t)$ belongs to the boundary of Q for all t. It is possible to choose w_Q so that it satisfies

$$\int_{t_1}^{t_2} |w_Q'(t)|\,dt \leqq 2 \int_{t_1}^{t_2} |u_h'(t)|\,dt. \tag{4.4}$$

Call w_h the piecewise affine function obtained from u_h by substituting it with the corresponding w_Q in each of the intervals of $B_{1/h}(u_h)$. An argument similar to the proof of Step 2 shows that it is possible to choose a new parametrization of w_h in order to obtain a curve v_h such that $|v_h'|$ is constant. By (4.4) it follows that

$$L_h^W(v_h) = L_h^W(w_h) \leqq L_h^W(u_h) + 2L_h^B(u_h),$$

but $F_{1/h}(v_h) = \alpha [L_h^W(v_h)]^2$ and the result follows by Step 3.

We now conclude the proof of Theorem IV.1. Since $\varphi(0,1) = \varphi(\sqrt{2}/2, \sqrt{2}/2) = \varphi(1,0) = \alpha$, if φ is a quadratic form then necessarily

$$\varphi(z) = \alpha |z|^2 \qquad \text{for all } z \in \mathbf{R}^2. \tag{4.5}$$

By Step 3 it follows that

$$\begin{aligned}
[F_{1/h}(u_h)]^{1/2} &= \sqrt{\alpha}[L_h^W(u_h) + L_h^B(u_h)] + (\sqrt{\beta} - \sqrt{\alpha}) L_h^B(u_h) \\
&= \sqrt{\alpha} \int_0^1 |u_h'|\,dt + (\sqrt{\beta} - \sqrt{\alpha}) L_h^B(u_h) \\
&\geqq \sqrt{\alpha}|z| + (\sqrt{\beta} - \sqrt{\alpha}) L_h^B(u_h).
\end{aligned}$$

Letting $h \to +\infty$, since $\alpha < \beta$ by (4.1) and (4.5) we have

$$\lim_h L_h^B(u_h) = 0. \tag{4.6}$$

Let (v_h) be as in Step 4; then by (4.3), (4.6) and (4.1)

$$\begin{aligned}
|z| &\leqq \liminf_h L_h^W(v_h) \leqq \limsup_h L_h^W(v_h) = \frac{1}{\sqrt{\alpha}} \limsup_h [F_{1/h}(v_h)]^{1/2} \\
&\leqq \frac{1}{\sqrt{\alpha}} \lim_h [[F_{1/h}(u_h)]^{1/2} + (2\sqrt{\alpha} - \sqrt{\beta}) L_h^B(u_h)] = |z|
\end{aligned}$$

by (4.3), (4.6) and (4.1), therefore

$$\lim_h L_h^W(v_h) = |z|. \tag{4.7}$$

If (z_1, z_2) are the coordinates of z, it is easy to see that (4.2) implies $L_h^W(v_h) \geqq (\sqrt{2}-1)\min(|z_1|,|z_2|) + \max(|z_1|,|z_2|)$ for all h, and by (4.7)

$$\sqrt{z_1^2 + z_2^2} \geqq (\sqrt{2}-1)\min(|z_1|,|z_2|) + \max(|z_1|,|z_2|).$$

But this is false unless $z_1 = 0$ or $z_2 = 0$ or $|z_1| = |z_2|$. ∎

Remark. It is easy to see that if β/α is large enough, then the sequence (u_h) of the minimum points of $(F_{1/h})$ satisfies $u_h(t) \in W_{1/h}$ for all t, so that

$$\varphi(z) = [(\sqrt{2}-1)\min(|z_1|,|z_2|) + \max(|z_1|,|z_2|)]^2 \cdot \alpha$$

(for $\beta/\alpha \geqq 4$ this follows by Step 4).

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