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Thin inclusions in linear elasticity: a variational approach

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I. Introduction

The inclusion of a very thin layer of very rigid material into a given elastic body has been widely considered, and in the classic literature (see for instance [9]) we find that when the Lamé coefficients of the material in the layer grow as $1/\varepsilon^3$, where ε is the thickness of the layer, the problem becomes the inclusion of a plate (governed by an elliptic fourth-order equation) into the original elastic body.

More recently, the method of formal asymptotic expansion (see for instance [2], [3], [4], [10]) has been used for a more rigorous study of many problems of this type (see also [8]).

In this paper we study the thin inclusion problem from a different point of view: in fact, we are interested in the "variational behaviour" of the approximating energies, and the limit problem is identified by its energy functional. More precisely, we consider energies of the form

$$F_\varepsilon^\lambda(u) = G(u, \Omega) + \frac{1}{\varepsilon^\lambda} \int_{\Sigma_\varepsilon} f(x, e(u))\, dx, \tag{1. 1}$$

where $G(u, \Omega)$ is the stored energy of the surrounding body Ω, Σ_ε is the layer, $e(u) = (Du + {}^tDu)/2$ is the usual linearized strain tensor, and $f(x, \cdot)$ is a convex function such that

$$|z|^p \leqq f(x, z) \leqq c(1 + |z|^p),$$

with $p > 1$. In this way, the approximating problem (for example with Neumann boundary conditions) may be written in the form

$$\min \left\{ F_\varepsilon^\lambda(u) + \alpha \int_\Omega |u|^p\, dx - \langle L, u \rangle : u \in W^{1,p}(\Omega; \mathbb{R}^n) \right\}, \tag{1. 2}$$

where $\alpha > 0$, and where L is a given load. In order to study the asymptotic behaviour of the solutions u_ε of (1. 2) we apply the Γ-convergence theory to the energies

F_ε^λ defined in (1. 1). Indeed, it is well known (see Theorem II. 1) that the Γ-convergence of the energies implies the convergence of minimum points and minimum values. In our case, the expression of the limit energy obviously depends on λ; the first critical exponent is $\lambda = 1$, and we obtain (see Theorem II. 4) the limit

$$F^1(u) = G(u, \Omega) + \int_\Sigma f_0(\sigma, e_\tau(u))\, d\sigma,$$

where Σ is the inclusion (with normal vector ν), $e_\tau(u)$ is the tangential strain, defined in Section II, and the function f_0 is given by

$$f_0(\sigma, z) = \min_{\xi \in \mathbb{R}^n} f(\sigma, z + \xi \otimes \nu(\sigma)).$$

In this first case, the energy density in Σ depends only on the first derivatives of the displacement u, so that no plate or shell phenomena occur.

The second critical exponent is $\lambda = p+1$; in this case the limit energy takes the form (see Theorem II. 5)

$$F^{p+1}(u) = \begin{cases} G(u, \Omega) + \dfrac{2^{-p}}{p+1} \int_\Sigma f_0(\sigma, \nu\delta\delta u)\, d\sigma, & \text{if } e_\tau(u) = 0 \text{ on } \Sigma, \\ +\infty & \text{, otherwise,} \end{cases}$$

where δ is the tangential derivative operator

$$\delta u = Du - (Du\, \nu) \otimes \nu.$$

In this second case the energy density in Σ depends on the second derivatives of u, so that the inclusion behaves like a shell.

In Section II we give the notation and we state our main results; Sections III and IV are devoted to the proof of Theorems II. 4 and II. 5, while in Section V we consider some explicit examples in two dimensions.

II. Notation and statement of results

In the sequel we denote by Σ a smooth, compact $(n-1)$-dimensional manifold of $\mathbb{R}^n$. For the sake of simplicity, we make the following assumption, which may be dropped by a localization argument:

there exists a single parametrization $\Phi\colon\ \omega \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$ *of* Σ,

where ω is a regular open set. Then we denote by ν the unit normal vector to Σ. Another assumption we make on Σ is the following: set $T^\alpha = \partial\Phi / \partial\xi_\alpha$; then

$\{T^1, \ldots, T^{n-1}\}$ *is an orthogonal set of tangent vectors to* Σ.

We remark that this assumption is always satisfied when $n=2$ or $n=3$ (the physical case): see [6]; we also set $\tau^\alpha = T^\alpha / \| T^\alpha \|$.

Let $h\colon \Sigma \to (0, \infty)$ be a smooth function (the regularity assumptions on Σ and h might be considerably weakened), and for all $\varepsilon > 0$ set

$$\Sigma_\varepsilon = \{\sigma + t\nu(\sigma)\colon \ \sigma \in \Sigma, \ |t| < \varepsilon h(\sigma)\}.$$

The mapping $(\sigma, t) \mapsto \sigma + t\nu(\sigma)$ is invertible on Σ_ε if ε is sufficiently small, which we shall suppose henceforth, therefore the mappings $\sigma(x)$ and $N(x) = \nu(\sigma(x))$ are well defined on Σ_ε.

We want to study the inclusion of the thin plate Σ in a domain Ω which is a regular open subset of $\mathbb{R}^n$ whose closure contains Σ and is not tangent to Σ. Again for the sake of simplicity we will assume that Ω is the interior of Σ_{ε_0}, for some small $\varepsilon_0 > 0$.

We now introduce the energy we will use. For every square matrix A, the symbol A^* denotes the symmetric part of A. Take $p > 1$ and let $f\colon \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$ satisfy:

(2.1) *the function* $z \mapsto f(x, z)$ *is convex*;

(2.2) $f(x, z) = f(x, z^*)$;

(2.3) *there exists a continuous function* $\omega\colon [0, +\infty) \to [0, +\infty)$ *which is increasing and vanishing at the origin, such that*

$$|f(x, z) - f(y, z)| \leqq \omega(|x - y|)(1 + |z|^p);$$

(2.4) $|z^*|^p \leqq f(x, z) \leqq c(1 + |z^*|^p)$.

For every $\varepsilon > 0$, $\lambda > 0$ and $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ we set

$$F_\varepsilon^\lambda(u) = \int_{\Omega \setminus \Sigma_\varepsilon} f(x, e(u))\, dx + \frac{1}{\varepsilon^\lambda} \int_{\Sigma_\varepsilon} f(x, e(u))\, dx,$$

and we denote again by F_ε^λ the functional defined on $L^p(\Omega; \mathbb{R}^n)$ as

$$\begin{cases} F_\varepsilon^\lambda(u), & \text{if } \ u \in W^{1,p}(\Omega; \mathbb{R}^n), \\ +\infty, & \text{otherwise}. \end{cases}$$

We want to characterize the Γ-limit of F_ε^λ in the topology $L^p(\Omega)$, depending on the values of the parameter λ. Indeed, it is well known that the Γ-convergence of a sequence of functionals is strictly related to the convergence of their minimum points and minimum values: more precisely, let X be a metric space, let $(F_\varepsilon)_{\varepsilon > 0}$ be mappings from X into $\overline{\mathbb{R}}$, and let $x \in X$. We set

$$\Gamma^-(X) \liminf_{\varepsilon \to 0} F_\varepsilon(x) = \inf\{\liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon)\colon \ x_\varepsilon \to x \text{ in } X\},$$

$$\Gamma^-(X) \limsup_{\varepsilon \to 0} F_\varepsilon(x) = \inf\{\limsup_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon)\colon \ x_\varepsilon \to x \text{ in } X\}.$$

If these two Γ-limits are the same at x, their common value will be denoted by

$$\Gamma^{-}(X) \lim_{\varepsilon \to 0} F_\varepsilon(x).$$

Theorem II. 1 (see [5], Theorems 2. 3 and 2. 6). *Assume that*

(i) *the family* (F_ε) *is equicoercive, i.e., for every* $c > 0$ *there is a compact subset* K_c *of* X *such that* $\{x \in X\colon F_\varepsilon(x) \leqq c\} \subseteqq K_c$ *for every* $\varepsilon > 0$;

(ii) *for every* $x \in X$, *there exists* $F(x) = \Gamma^{-}(X) \lim_{\varepsilon \to 0} F_\varepsilon(x)$.

Then we have:

(a) F *has a minimum on* X *and* $\min_X F = \lim_{\varepsilon \to 0} (\inf_X F_\varepsilon)$;

(b) *if* x_ε *is a minimum point for* F_ε *and* $x_\varepsilon \to x$ *in* X, *then* x *is a minimum point for* F;

(c) *if* $C\colon X \to \mathbb{R}$ *is continuous, then* $\Gamma^{-}(X) \lim_{\varepsilon \to 0} (C + F_\varepsilon) = C + F$, *and therefore* (a), (b) *apply also to* $C + F_\varepsilon$.

For every $u \in W^{1,p}(\Sigma_\varepsilon)$ we set

$$D_\nu u = \langle Du, \nu \rangle \nu,$$
$$\delta u = Du - D_\nu u.$$

We say that $u \in W^{m,p}(\Sigma)$ if $u \circ \Phi \in W^{m,p}(\omega)$. If $u \in W^{1,p}(\Sigma)$ then u is the trace of a function $\tilde{u} \in W^{1+\frac{1}{p},p}(\Omega)$, so that $\delta \tilde{u} \in W^{\frac{1}{p},p}(\Omega)$ and we may define δu as the trace of $\delta \tilde{u}$ on Σ, which belongs to $L^p(\Sigma)$. This definition is independent of $\tilde{u}$, because if $\tilde{u} = 0$ on Σ then $\delta \tilde{u} = 0$ on Σ. An easy computation shows that the following properties hold for the operator δ.

Proposition II. 2. *Let* f, g *be smooth functions defined on* Σ, *and assume* f *has compact support in* Σ. *If we set*

$$\delta_i^{-1} f = -\delta_i f + f \nu_i \delta_j \nu_j$$

then

$$\int_\Sigma f \delta_i g \, d\sigma = \int_\Sigma g \delta_i^{-1} f \, d\sigma.$$

Proposition II. 3. *Let* u *be a smooth function defined in a neighbourhood of* Σ. *Then*

$$\delta_i \left(\int_0^{h(\sigma)} u(\sigma + t\nu(\sigma)) \, dt \right) = u(\sigma + h(\sigma)\, \nu(\sigma)) \, \delta_i h(\sigma) + \int_0^{h(\sigma)} (\delta_i u + t \delta u \delta_i \nu) \, dt.$$

For $u \in W^{1,p}(\Omega; \mathbb{R}^n)$, or $u \in W^{1,p}(\Sigma; \mathbb{R}^n)$, we define

$$e_\tau(u) = [(I - \nu \otimes \nu) \, \delta u]^*.$$

It is readily verified that

$$\begin{aligned} e_\tau(u) &= (I - v \otimes v)\,(\delta u)^*\,(I - v \otimes v) \\ &= (\langle \delta u, \tau^\alpha \otimes \tau^\beta \rangle\, \tau^\alpha \otimes \tau^\beta)^* \\ &= (\langle \delta(u - \langle u, v \rangle v) + \langle u, v \rangle\, \delta v, \tau^\alpha \otimes \tau^\beta \rangle\, \tau^\alpha \otimes \tau^\beta)^*, \end{aligned}$$

which shows that $e_\tau(u) \in L^p(\Sigma)$ if

$$\langle u, v \rangle \in L^p(\Sigma), \quad u - \langle u, v \rangle v \in W^{1,p}(\Sigma).$$

Finally we set

$$W_1(\Sigma) = \{u \in W^{1,p}(\Omega) : u - \langle u, v \rangle v \in W^{1,p}(\Sigma)\},$$

$$W_{p+1}(\Sigma) = \{u \in W_1(\Sigma) : e_\tau(u) = 0,\ \langle u, v \rangle \in W^{2,p}(\Sigma)\},$$

$$f_0(\sigma, z) = \min_{\xi \in \mathbb{R}^n} f(\sigma, z + \xi \otimes v(\sigma)),$$

$$G(u) = \int_\Omega f(x, e(u))\, dx \quad \text{for all} \quad u \in W^{1,p}(\Omega; \mathbb{R}^n),$$

$$F^1(u) = \begin{cases} G(u) + 2 \int_\Sigma h(\sigma)\, f_0(\sigma, e_\tau(u))\, d\sigma, & \text{if} \quad u \in W_1, \\ +\infty & \text{, otherwise,} \end{cases}$$

(2. 5)
$$F^{p+1}(u) = \begin{cases} G(u) + \dfrac{2}{p+1} \int_\Sigma h^{p+1}(\sigma)\, f_0(\sigma, v\, \delta\, \delta u)\, d\sigma, & \text{if} \quad u \in W_{p+1}, \\ +\infty & \text{, otherwise.} \end{cases}$$

We may now state our main Γ-convergence results.

Theorem II. 4. *Let f satisfy* (2. 1), …, (2. 4); *then*

$$\Gamma^-(L^p(\Omega)) \lim_{\varepsilon \to 0} F_\varepsilon^1 = F^1.$$

Theorem II. 5. *Let f satisfy* (2. 1), …, (2. 4) *and assume also that*

$$f(x, \cdot) \quad \textit{is } p\textit{-homogeneous.}$$

Then

$$\Gamma^-(L^p(\Omega)) \lim_{\varepsilon \to 0} F_\varepsilon^{p+1} = F^{p+1}.$$

Remark II. 6. A similar problem arises when we consider functionals of the form

$$F_\varepsilon^\lambda(u) = \int_{\Omega \setminus \Sigma_\varepsilon} f(x, Du)\, dx + \frac{1}{\varepsilon^\lambda} \int_{\Sigma_\varepsilon} f(x, Du)\, dx,$$

where $u : \Omega \to \mathbb{R}$ is a scalar function and f satisfies (2. 1), …, (2. 4) with z instead of z^*.

In this case the same argument employed in the proof of Theorem II. 4 shows that the only critical exponent is $\lambda = 1$, and we get the Γ-limit

$$F^1(u) = \int_\Omega f(x, Du)\, dx + \int_\Sigma 2\, h(\sigma)\, f_0(\sigma, \delta u)\, d\sigma,$$

where

$$f_0(\sigma, z) = \min_{t \in \mathbb{R}} f(\sigma, z + t\, v(\sigma)).$$

III. Proof of Theorem II. 4

To prove Theorem II. 4 we must verify the following inequalities for every $u \in W^{1,p}(\Omega; \mathbb{R}^n)$:

$$(3.\ 1) \qquad F^1(u) \leqq F^1_-(u),$$

$$(3.\ 2) \qquad F^1_+(u) \leqq F^1(u).$$

Set for all $h \in \mathbb{N}$

$$f_h(x, z) = \inf\{f(x, w) + h|w - z^*|^p : w \in \mathbb{R}^{n^2}\};$$

then for a suitable sequence (ω_h), vanishing as $h \to \infty$, we have:

(i) the function $z \mapsto f_h(x, z)$ is convex and of class C^1;

(ii) f_h depends only on x and z^*;

(iii) f_h satisfies (2. 3) uniformly with respect to h;

(iv) $c' |z^*|^p \leqq f_h(x, z) \leqq c(1 + |z^*|^p)$;

(v) $|f(x, z) - f_h(x, z)| \leqq \omega_h(1 + |z^*|^p)$.

Define $F^1_{\varepsilon,h}$ and F^1_h as F^1_ε and F^1, but with f_h instead of f: the properties above immediately imply

$$F^1_{\varepsilon,h} \leqq F^1_\varepsilon \leqq (1 + \omega_h)\, F^1_{\varepsilon,h},$$

$$F^1_h \leqq F^1 \leqq (1 + \omega_h)\, F^1_h.$$

It is then enough to prove (3. 1), (3. 2) for F^1_h; therefore we shall suppose henceforth that $f(x, z)$ is of class C^1 in z.

Proof of (3. 1). We begin with the following lemma:

Lemma III. 1. *If* $F^1_-(u) < +\infty$ *then* $u \in W_1$.

Proof. For $u \in L^p(\Omega)$, let $u_\varepsilon \to u$ in $L^p(\Omega)$ satisfy

$$(3.\ 3) \qquad \liminf_\varepsilon F^1_\varepsilon(u_\varepsilon) < +\infty.$$

It is not restrictive to assume that the sequence $(F_\varepsilon^1(u_\varepsilon))$ actually converges, so that by (2. 4)

$$\int_\Omega |e(u_\varepsilon)|^p\,dx + \frac{1}{\varepsilon}\int_{\Sigma_\varepsilon} |e(u_\varepsilon)|^p\,dx \leqq C. \tag{3. 4}$$

Korn's inequality then implies

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in} \quad W^{1,p}(\Omega), \tag{3. 5}$$

so that $u \in W^{1,p}(\Omega)$, and in particular $u \in L^p(\Sigma)$. Since $|e_\tau(u_\varepsilon)|^p \leqq |e(u_\varepsilon)|^p$, Hölder inequality and (3. 4) yield

$$\int_\Sigma \left| \frac{1}{\varepsilon} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} e_\tau(u_\varepsilon)\,dt \right|^p d\sigma \leqq C. \tag{3. 6}$$

We set

$$v_\varepsilon(\sigma) = \frac{1}{\varepsilon} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} e_\tau(u_\varepsilon)\,dt.$$

For every $g \in C_0^1(\Sigma)$, integrating by parts on Σ we have

$$\int_\Sigma g \frac{1}{\varepsilon} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} (I - \nu \otimes \nu)\,\delta u_\varepsilon dt\,d\sigma = \int_\Sigma g(I - \nu \otimes \nu) \frac{1}{\varepsilon} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} \delta u_\varepsilon\,dt\,d\sigma$$

$$= \omega_\varepsilon + \int_\Sigma g(I - \nu \otimes \nu) \frac{1}{\varepsilon} \left\{ \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} \delta u_\varepsilon (I + t\,\delta \nu)\,dt + \varepsilon(u_\varepsilon^+ + u_\varepsilon^-)\,\delta h \right\} d\sigma$$

$$- \int_\Sigma g(I - \nu \otimes \nu)\,(u_\varepsilon^+ + u_\varepsilon^-)\,\delta h\,d\sigma,$$

where $u_\varepsilon^\pm(\sigma) = u_\varepsilon(\sigma \pm \varepsilon h(\sigma)\,\nu)$. Recalling Propositions II. 2, II. 3, and taking the limit as $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon \to 0} \int_\Sigma g \frac{1}{\varepsilon} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} (I - \nu \otimes \nu)\,\delta u_\varepsilon\,dt\,d\sigma = \int_\Sigma g(I - \nu \otimes \nu) \cdot 2h\,\delta u\,d\sigma,$$

therefore

$$\lim_{\varepsilon \to 0} \int_\Sigma v_\varepsilon\,g\,d\sigma = \int_\Sigma 2h\,e_\tau(u)\,g\,d\sigma,$$

which together with (3. 6) implies that $e_\tau(u) \in L^p(\Sigma)$, and

$$v_\varepsilon \rightharpoonup 2h e_\tau(u) \quad \text{weakly in} \quad L^p(\Sigma). \tag{3. 7}$$

Now,

$$e_\tau(u) = \langle (\delta u)^*, \tau^\alpha \otimes \tau^\beta \rangle \, \tau^\alpha \otimes \tau^\beta,$$

so that for every α, β

$$(3.8) \qquad (\delta_j u^i + \delta_i u^j) T_i^\alpha T_j^\beta = \| T^\alpha \| \, \| T^\beta \| \, \langle (\delta u)^*, \tau^\alpha \otimes \tau^\beta \rangle \in L^p(\Sigma).$$

Define $u_\alpha = \langle u, T^\alpha \rangle$ and $U_\alpha = u_\alpha \circ \Phi$: then (3. 8) becomes

$$2\, e(U) = \frac{\partial U_\alpha}{\partial \xi_\beta} + \frac{\partial U_\beta}{\partial \xi_\alpha} = T_j^\beta \delta_j u_\alpha + T_i^\alpha \delta_i u_\beta$$
$$= (\delta_j u^i + \delta_i u^j) \, T_i^\alpha T_j^\beta + u^i (T_j^\alpha \delta_j T_i^\beta + T_j^\beta \delta_j T_i^\alpha) \in L^p(\Sigma);$$

Korn's inequality implies that $U \in W^{1,p}(\omega)$, and therefore $\langle u, \tau^\alpha \rangle \in W^{1,p}(\Sigma)$ for all α, i.e., $u \in W_1$. ■

Now (3. 1) is easy: since for every v

$$e(v) = e_\tau(v) + \{[2\, e(v) v - \langle e(v) v, v \rangle v] \otimes v\}^*,$$

we have

$$(3.9) \quad f(\sigma, e(v)) \geqq \min_{\xi \in \mathbb{R}^n} f(\sigma, e(v) + \xi \otimes v) = \min_{\xi \in \mathbb{R}^n} f(\sigma, e_\tau(v) + \xi \otimes v) = f_0(\sigma, e_\tau(v)).$$

Let $u_\varepsilon \to u$ in L^p satisfy (3. 3): by (3. 5) we have

$$(3.10) \qquad G(u) \leqq \liminf_\varepsilon G(u_\varepsilon),$$

and by (3. 9) and the convexity of f_0 we deduce that

$$\frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} f(\sigma, e(u_\varepsilon)) \, dx \geqq \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} f_0(\sigma, e_\tau(u_\varepsilon)) \, dx$$
$$\geqq \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} f_0(\sigma, e_\tau(u(\sigma))) \, dx + \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} \langle f_0'(\sigma, e_\tau(u(\sigma))), e_\tau(u_\varepsilon) - e_\tau(u(\sigma)) \rangle \, dx.$$

Therefore by (3. 7) and (2. 3)

$$(3.11) \qquad \liminf_\varepsilon \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} f(x, e(u_\varepsilon)) \, dx \geqq \int_\Sigma 2\, h f_0(\sigma, e_\tau(u)) \, d\sigma,$$

which together with (3. 10) proves (3. 1). ■

Proof of (3. 2). By Lemma III. 3 we may assume that $u \in W_1$, and therefore by the semicontinuity of F_+^1 and by the density of $C^1(\bar{\Omega})$ in W_1 we may confine ourselves to prove (3. 2) when $u \in C^1(\bar{\Omega})$.

Let θ be a smooth function satisfying

$$\theta(t) = 1 \text{ for } |t| \leqq 1, \quad \theta(t) = 0 \text{ for } |t| \geqq 2, \quad |\theta'(t)| \leqq 2,$$

and set

$$\theta_\varepsilon(x) = \theta\left(\frac{t}{\varepsilon h(\sigma)}\right);$$

clearly, $\theta_\varepsilon = 0$ far from Σ. We define

$$v_\varepsilon(x) = [u(\sigma) + t\varphi(\sigma)]\,\theta_\varepsilon(x) + u(x)\,[1 - \theta_\varepsilon(x)], \tag{3. 12}$$

where φ is any smooth function from Σ into $\mathbb{R}^n$ with compact support in Σ. Then

$$\begin{aligned} F_\varepsilon^1(v_\varepsilon) &= \int_{\Omega\setminus\Sigma_{2\varepsilon}} f(x, e(u))\,dx + \int_{\Sigma_{2\varepsilon}\setminus\Sigma_\varepsilon} f(x, e(u))\,dx \\ &\quad + \frac{1}{\varepsilon}\int_{\Sigma_\varepsilon} f(x, D(u(\sigma)) + \varphi(\sigma)\otimes\nu(\sigma) + tD(\varphi(\sigma)))\,dx \\ &= S_1 + S_2 + S_3. \end{aligned} \tag{3. 13}$$

Denoting by ω_ε any quantity which vanishes with ε, we have

$$S_1 = G(u) + \omega_\varepsilon; \tag{3. 14}$$

$$\begin{aligned} |S_2| &\leqq c\varepsilon + c\int_{\Sigma_{2\varepsilon}\setminus\Sigma_\varepsilon} |Dv_\varepsilon|^p\,dx \\ &\leqq \omega_\varepsilon + c\int_{\Sigma_{2\varepsilon}\setminus\Sigma_\varepsilon}\left[\frac{1}{\varepsilon^p}|u(x) - u(\sigma) - t\varphi(\sigma)|^p + |D(u(\sigma) + t\varphi(\sigma))|^p + |Du|^p\right]dx \\ &\leqq \omega_\varepsilon + \frac{c}{\varepsilon^p}\int_{\Sigma_{2\varepsilon}\setminus\Sigma_\varepsilon} |u(x) - u(\sigma)|^p\,dx \leqq \omega_\varepsilon + \int_{\Sigma_{2\varepsilon}} |Du|^p\,dx \leqq \omega_\varepsilon. \end{aligned} \tag{3. 15}$$

As for S_3, we remark that the quantity

$$\alpha_\varepsilon = \frac{1}{\varepsilon}\int_{\Sigma_\varepsilon} [f(x, D(u(\sigma)) + \varphi\otimes\nu + tD(\varphi(\sigma))) - f(x, \delta u + \varphi\otimes\nu)]\,dx$$

vanishes as $\varepsilon \to 0$: indeed,

$$D(u(\sigma)) = \delta u - t\,\delta u\,DN,$$

and the convexity of $f(x, \cdot)$ and (2. 3) imply

$$|f(x, z) - f(x, w)| \leqq c\,|z - w|\,(1 + |z|^{p-1} + |w|^{p-1}),$$

so that

$$|\alpha_\varepsilon| \leqq c \int_{\Sigma_\varepsilon} |\delta\varphi + \delta u| \, dx \leqq \omega_\varepsilon.$$

Therefore

$$S_3 = \omega_\varepsilon + \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} f(x, \delta u + \varphi \otimes \nu) \, dx. \tag{3. 16}$$

By (3. 14), (3. 15) and (3. 16) we obtain

$$F_\varepsilon^1(u_\varepsilon) = \omega_\varepsilon + G(u) + \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} f(x, \delta u + \varphi \otimes \nu) \, dx$$

and, taking the limit as $\varepsilon \to 0$, we have

$$F_+^1(u) \leqq \limsup_\varepsilon F_\varepsilon^1(u_\varepsilon) = G(u) + \int_\Sigma 2\, h(\sigma)\, f(\sigma, \delta u + \varphi \otimes \nu) \, dx.$$

Since φ is arbitrary,

$$F_+^1(u) \leqq G(u) + \inf_\varphi \int_\Sigma 2\, h(\sigma)\, f(\sigma, \delta u + \varphi \otimes \nu) \, dx, \tag{3. 17}$$

where the infimum is to be taken on all smooth functions φ, or equivalently over all $\varphi \in L^1(\Sigma)$. By the measurable selection lemma (see [7], Theorem 1. 2 of Chapter VIII) there exists a function $\varphi \in L^1(\Sigma)$ such that

$$f(\sigma, \delta u(\sigma) + \varphi(\sigma) \otimes \nu(\sigma)) = \min_{\xi \in \mathbb{R}^n} f(\sigma, \delta u(\sigma) + \xi \otimes \nu(\sigma)) \quad \text{for all } \sigma.$$

Then (3. 17) reduces to

$$F_+^1(u) \leqq G(u) + \int_\Sigma 2\, h(\sigma) \min_{\xi \in \mathbb{R}^n} f(\sigma, \delta u + \xi \otimes \nu) \, d\sigma = F^1(u). \quad \blacksquare$$

IV. Proof of Theorem II. 5

To prove Theorem II. 5 we must verify the following inequalities for every $u \in W^{1,p}(\Omega; \mathbb{R}^n)$:

$$F^{p+1}(u) \leqq F_-^{p+1}(u), \tag{4. 1}$$

$$F_+^{p+1}(u) \leqq F^{p+1}(u). \tag{4. 2}$$

As in Section III, we may assume that $f(x, z)$ is of class C^1 in z.

Proof of (4. 2). We prove first the analogue of Lemma III. 1.

Lemma IV. 1. *If* $u \in L^p(\Omega)$ *and* $F_-^{p+1}(u) < +\infty$, *then* $u \in W_{p+1}$.

Proof. Let $u \in L^p(\Omega)$ and let $u_\varepsilon \in W^{1,p}(\Omega)$ satisfy

$$u_\varepsilon \to u \text{ in } L^p(\Omega), \qquad \liminf_\varepsilon F_\varepsilon^{p+1}(u_\varepsilon) < +\infty.$$

Then

$$\int_\Omega |e(u_\varepsilon)|^p\,dx + \frac{1}{\varepsilon^{p+1}} \int_{\Sigma_\varepsilon} |e(u_\varepsilon)|^p\,dx \leqq C,$$

and Korn's inequality implies $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, hence $\langle u, v\rangle \in L^p(\Sigma)$. Moreover, by Theorem II. 4 we have $f_0(e_\tau(u)) = 0$ on Σ, i.e., $\langle u, \tau^\alpha\rangle \in W^{1,p}(\Sigma)$ and $e_\tau(u) = 0$. As in Lemma III. 1 the functions

$$V_\varepsilon(\sigma) = \frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} |t|^{p-2}\, t\, e_\tau(u_\varepsilon)\,dt \tag{4. 3}$$

are bounded in $L^p(\Sigma)$, and we identify their weak limit in L^p through their limit in the sense of distributions. Recalling the definition of e_τ, for every $\theta \in C_0^\infty(\Sigma)$ we compute

$$\lim_\varepsilon \int_\Sigma \theta \frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} |t|^{p-2}\, t[(I - v\otimes v)\,\delta u_\varepsilon]^*\,dt\,d\sigma = \lim_\varepsilon A_\varepsilon^*,$$

where

$$\begin{aligned}
A_\varepsilon &= \int_\Sigma \theta(I - v\otimes v) \frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} \delta(|t|^{p-2}\, t u_\varepsilon)\,dt\,d\sigma \\
&= \int_\Sigma \theta(I - v\otimes v) \frac{1}{\varepsilon^{p+1}} \Bigg\{\delta \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} |t|^{p-2}\, t u_\varepsilon\,dt \\
&\quad - \left[h^{p-1}\,\varepsilon^p\,\delta h \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} D u_\varepsilon v\,dt + \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} |t|^p\,\delta u_\varepsilon\,\delta v\,dt\right]\Bigg\}\,d\sigma \\
&= \int_\Sigma \delta^{-1}(\theta(I - v\otimes v)) \frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} |t|^{p-2}\, t u_\varepsilon\,dt\,d\sigma \\
&\quad - \int_\Sigma \theta(I - v\otimes v) \left[\frac{h^{p-1}\,\delta h}{\varepsilon} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} D u_\varepsilon v\,dt + \frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} |t|^p\,\delta u_\varepsilon\,\delta v\,dt\right] d\sigma \\
&= I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon.
\end{aligned}$$

An integration by parts with respect to the variable t yields

$$I_1^\varepsilon = \int_\Sigma \delta^{-1}(\theta(I - v\otimes v)) \left(\frac{h^p}{\varepsilon p} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} D u_\varepsilon v\,dt - \frac{1}{p\varepsilon^{p+1}} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} |t|^p D u_\varepsilon\, v\,dt\right) d\sigma. \tag{4. 4}$$

Now we recall that

$$\frac{1}{\varepsilon}\int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} e(u_\varepsilon)\,dt, \qquad \frac{1}{\varepsilon^{p+1}}\int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} |t|^p\, e(u_\varepsilon)\,dt, \qquad \frac{1}{\varepsilon}\int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} v\, Du_\varepsilon\, v\otimes v\, dt$$

go to zero in $L^p(\Sigma)$, and so (4. 4) becomes

$$\begin{aligned}
I_1^\varepsilon &= \omega_\varepsilon + \int_\Sigma \delta^{-1}(\theta(I-v\otimes v))\left(-\frac{h^p}{p\varepsilon}\int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} v\,\delta u_\varepsilon\, dt + \frac{1}{p\varepsilon^{p+1}}\int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} |t|^p\, v\,\delta u_\varepsilon\, dt\right) d\sigma \\
&= \omega_\varepsilon + \int_\Sigma \delta^{-1}(\theta(I-v\otimes v))\, v\left[-\frac{h^p}{p\varepsilon}\,\delta\left(\int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} u_\varepsilon\, dt\right) + \frac{1}{p\varepsilon^{p+1}}\,\delta\left(\int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} |t|^p u_\varepsilon\, dt\right)\right] d\sigma \\
&= \omega_\varepsilon + \int_\Sigma \delta^{-1}\left(\delta^{-1}(\theta(I-v\otimes v))\, v\,\frac{h^p}{p}\right)\cdot\left(-\frac{1}{\varepsilon}\int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} u_\varepsilon\, dt\right) d\sigma \\
&\quad + \frac{1}{p}\int_\Sigma \delta^{-1}(\delta^{-1}(\theta(I-v\otimes v))\, v)\cdot\left(\frac{1}{\varepsilon^{p+1}}\int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} |t|^p\, u_\varepsilon\, dt\right) d\sigma \\
&= \omega_\varepsilon - \int_\Sigma \delta^{-1}\left(\delta^{-1}(\theta(I-v\otimes v))\, v\frac{h^p}{p}\right) 2hu\, d\sigma + \frac{2}{p(p+1)}\int_\Sigma \delta^{-1}(\delta^{-1}(\theta(I-v\otimes v))\, v)\, h^{p+1} u\, ds \\
&= \omega_\varepsilon - \frac{2}{p+1}\int_\Sigma \delta^{-1}(\theta(I-v\otimes v))\, v\, h^{p+1}\,\delta u\, d\sigma.
\end{aligned}$$

Similar computations show that

$$I_2^\varepsilon = \omega_\varepsilon + 2\int_\Sigma \theta(I-v\otimes v)\,(v\,\delta u\otimes\delta h)\, d\sigma,$$

$$I_3^\varepsilon = \omega_\varepsilon - \frac{2}{p+1}\int_\Sigma \theta(I-v\otimes v)\,\delta u\,\delta v\, d\sigma,$$

so that

$$A_\varepsilon = \omega_\varepsilon + \int_\Sigma \theta(I-v\otimes v)\left[-\frac{2}{p+1}\, h^{p+1}(\delta(v\,\delta u) + \delta u\,\delta v)\right] d\sigma.$$

This implies that the matrices V_ε defined in (4. 3) converge in the sense of distributions, and therefore weakly in $L^p(\Sigma)$, to

$$-\frac{2}{p+1}\, h^{p+1}\,[(I-v\otimes v)\,(\delta(v\,\delta u) + \delta u\,\delta v)]^*.$$

Now, $\delta(v\,\delta u) + \delta u\,\delta v = v\delta\,\delta u + 2(\delta u)^*\,\delta v$, and $\delta v = (I-v\otimes v)\,\delta v$, so that

$$[(I-v\otimes v)\,(\delta(v\,\delta u) + \delta u\,\delta v)]^* = [(I-v\otimes v)\,(\delta u)^*\,(I-v\otimes v)\,\delta v]^* = [e_\tau(u)\,\delta v]^* = 0.$$

Then we have

$$V_\varepsilon \rightharpoonup -\frac{2}{p+1} h^{p+1} [(I - \nu \otimes \nu)(\nu\, \delta \delta u)]^*$$

weakly in $L^p(\Sigma)$, and in particular

$$\langle \nu\, \delta \delta u, \tau^\alpha \otimes \tau^\beta + \tau^\beta \otimes \tau^\alpha \rangle \in L^p(\Sigma) \quad \text{for all } \alpha, \beta. \tag{4. 5}$$

Define $u_\nu = \langle u, \nu \rangle$ and $U_\nu = u_\nu \circ \Phi$; recalling the definition of u_α and that $u_\nu \in L^p(\Sigma)$, $u_\alpha \in W^{1,p}(\Sigma)$, and substituting u with $u_\alpha \tau^\alpha + u_\nu \nu$ in (4. 5), we obtain

$$\langle \delta \delta u_\nu, \tau^\alpha \otimes \tau^\beta + \tau^\beta \otimes \tau^\alpha \rangle \in L^p(\Sigma) \quad \text{for all } \alpha, \beta.$$

But $\delta f = \langle \delta f, \tau^\gamma \rangle \tau^\gamma$, hence the formula above may be reduced to

$$\begin{aligned} &\langle T^\alpha, \delta(\langle T^\beta, \delta u_\nu \rangle)\rangle + \langle T^\beta, \delta(\langle T^\alpha, \delta u_\nu \rangle)\rangle \\ &\qquad + \langle T^\gamma, \delta u_\nu \rangle \left\langle \delta\left(\frac{T^\gamma}{\|T^\gamma\|^2}\right), T^\alpha \otimes T^\beta + T^\beta \otimes T^\alpha \right\rangle \in L^p(\Sigma). \end{aligned}$$

We may then write

$$D^2_{\alpha\beta} U_\nu + \langle D U_\nu, g_{\alpha\beta} \rangle \in L^p(\omega) \quad \text{for all } \alpha, \beta, \tag{4. 6}$$

where the vector $g_{\alpha\beta}$ is defined by

$$g^\gamma_{\alpha\beta} = \left\langle T^\alpha, D_\beta\left(\frac{T^\gamma}{\|T^\gamma\|^2}\right)\right\rangle + \left\langle T^\beta, D_\alpha\left(\frac{T^\gamma}{\|T^\gamma\|^2}\right)\right\rangle.$$

In particular,

$$\Delta U_\nu + \langle D U_\nu, g_{\alpha\alpha} \rangle \in L^p(\omega),$$

and by Theorem (6. 1) of [1], $U_\nu \in L^p(\omega)$, we obtain $U_\nu \in W^{2,p}_{\mathrm{loc}}(\omega)$. But on every ball $B \Subset \omega$

$$\|D U_\nu\|_p \leqq \varepsilon \|D^2 U_\nu\|_p + \frac{c}{\varepsilon} \|U_\nu\|_p,$$

with c independent of the particular B, therefore (4. 6) implies

$$\|D^2 U_\nu\|_{L^p(B)} \leqq \|D^2 U_\nu + \langle D U_\nu, g_{\alpha\beta}\rangle\|_{L^p(\omega)} + c\, \|D U_\nu\|_{L^p(B)} \leqq c + \varepsilon\, \|D^2 U_\nu\|_{L^p(B)},$$

and finally, since B is arbitrary, $U_\nu \in W^{2,p}(\omega)$, which concludes the proof. ■

We may now prove (4. 1); set $A = v(\delta\delta u)^*$: then for every $u \in W_{p+1}$ and $u_\varepsilon \to u$ in $L^p(\Omega)$ such that $\liminf F_\varepsilon^{p+1}(u_\varepsilon) < +\infty$, we have

$$\begin{aligned}\liminf_\varepsilon F_\varepsilon^{p+1}(u_\varepsilon) &\geqq G(u) + \liminf_\varepsilon \frac{1}{\varepsilon^{p+1}} \int_{\Sigma_\varepsilon} f(x, e(u_\varepsilon))\, dx \\ &= G(u) + \liminf_\varepsilon \frac{1}{\varepsilon^{p+1}} \int_{\Sigma_\varepsilon} f(\sigma, e(u_\varepsilon))\, dx \\ &= G(u) + \liminf_\varepsilon \frac{1}{\varepsilon^{p+1}} \int_{\Sigma_\varepsilon} f_0(\sigma, e_\tau(u_\varepsilon))\, dx,\end{aligned}$$

where we used the continuity of f with respect to x. Then, by the convexity of f_0,

$$\begin{aligned}\liminf_\varepsilon F_\varepsilon^{p+1}(u_\varepsilon) &\geqq G(u) + \liminf_\varepsilon \frac{1}{\varepsilon^{p+1}} \int_\Sigma \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} f_0(\sigma, -tA)\, dt\, d\sigma \\ &\quad + \liminf_\varepsilon \frac{1}{\varepsilon^{p+1}} \int_\Sigma \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} \langle f_0'(\sigma, -tA), e_\tau(u_\varepsilon) + tA \rangle\, dt\, d\sigma \\ &= F^{p+1}(u) + \liminf_\varepsilon \int_\Sigma \left\langle f_0'(\sigma, -A), \frac{1}{\varepsilon^{p+1}} \int_{-\varepsilon h(\sigma)}^{\varepsilon h(\sigma)} (|t|^{p-2} t\, e_\tau(u_\varepsilon) + |t|^p A)\, dt \right\rangle d\sigma \\ &= F^{p+1}(u) + \liminf_\varepsilon \int_\Sigma \left\langle f_0'(\sigma, -A), \left(V_\varepsilon(\sigma) + \frac{2}{p+1} h^{p+1} A \right) \right\rangle d\sigma = F^{p+1}(u),\end{aligned}$$

since $V_\varepsilon \rightharpoonup -\dfrac{2}{p+1} h^{p+1} A$ weakly in $L^p(\Sigma)$. ■

Proof of (4. 2). Fix $u \in W_{p+1}$ and set $\varphi = -v\,\delta u$: then

$$(\delta u + \varphi \otimes v)^* = 0. \tag{4. 7}$$

In addition

$$\varphi = -v\,\delta(u_\alpha \tau^\alpha + u_v v) = -u_\alpha v\,\delta\tau^\alpha - \delta u_v,$$

so that $\varphi \in W^{1,p}(\Sigma)$.

Choose θ_ε as in the proof of (3. 2), and let η be an arbitrary smooth function with compact support in Σ. Set

$$v_\varepsilon(x) = \left(u(\sigma) + t\varphi(\sigma) + \frac{t^2}{2} \eta(\sigma) \right) \theta_\varepsilon(x) + u(x)\,(1 - \theta_\varepsilon(x));$$

then

$$\begin{aligned}F_\varepsilon^{p+1} &= \int_{\Omega \setminus \Sigma_{2\varepsilon}} f(x, e(u))\, dx + \int_{\Sigma_{2\varepsilon} \setminus \Sigma_\varepsilon} f(x, e(v_\varepsilon))\, dx \\ &\quad + \frac{1}{\varepsilon^{p+1}} \int_{\Sigma_\varepsilon} f\left(x, D(u(\sigma)) + tD(\varphi(\sigma)) + \frac{t^2}{2} D(\eta(\sigma)) + \varphi \otimes v + t\eta \otimes v \right) dx \\ &= S_1 + S_2 + S_3.\end{aligned}$$

Following again the proof of (3. 2) we get

$$S_1 = G(u) + \omega_\varepsilon,$$
$$|S_2| \leqq \omega_\varepsilon$$

and

$$S_3 = \omega_\varepsilon + \frac{1}{\varepsilon^{p+1}} \int_{\Sigma_\varepsilon} f(x, \delta u + \varphi \otimes v + t(-\delta u\, \delta v + \delta \varphi + \eta \otimes v))\, dx$$

$$= \omega_\varepsilon + \frac{1}{\varepsilon^{p+1}} \int_{\Sigma_\varepsilon} |t|^p f(\sigma, -\delta u\, \delta v + \delta \varphi + \eta \otimes v)\, dx$$

by (4. 7). Finally,

$$S_3 = \omega_\varepsilon + \frac{1}{p+1} \int_\Sigma 2\, h^{p+1} f(\sigma, -\delta u\, \delta v + \delta \varphi + \eta \otimes v)\, d\sigma.$$

Since η is arbitrary, again we have

$$F_+^{p+1}(u) \leqq G(u) + \frac{1}{p+1} \int_\Sigma 2\, h^{p+1} \min_{\xi \in \mathbb{R}^n} f(\sigma, -\delta u\, \delta v + \delta \varphi + \xi \otimes v)\, d\sigma. \tag{4. 8}$$

Our choice of φ implies

$$-\delta u\, \delta v + \delta \varphi = -v\, \delta \delta u - 2(\delta u)^*\, \delta v,$$

therefore, recalling that $e_\tau(u) = 0$ and that $v \delta v = \delta v v = 0$, the argument of f may be reduced to

$$-v \otimes (v\, \delta u\, \delta v - \xi) - v\, \delta \delta u,$$

so that (4. 8) gives

$$F_+^{p+1}(u) \leqq G(u) + \frac{1}{p+1} \int_\Sigma 2\, h^{p+1} f_0(\sigma, v\, \delta \delta u)\, d\sigma,$$

which concludes the proof. ■

V. Examples

In this Section we want to write explicitly the limit energy given by (2. 5) in some particular cases, with $n = 2$ or $n = 3$, when the energy density has the usual form

$$f(A) = \frac{\lambda}{2} (\operatorname{tr} A)^2 + \mu |A|^2$$

(linear elasticity, isotropic material).

Example V. 1. *The one-dimensional case.* We represent the limit energy for a beam embedded in $\mathbb{R}^2$ (the case of beams in $\mathbb{R}^3$ does not fall within our setting). Let $\gamma(s)$ be a parametrization of the beam, with $|\gamma'|=1$; if we denote by $c=\langle\tau, v'\rangle$ the curvature and by U_τ, U_v the tangential and normal components of the displacement, then the limit energy of the beam turns out to be

$$\frac{\mu(\lambda+\mu)}{6(\lambda+2\mu)}\int_\Sigma (2h)^3\,|U_v''+c^2U_v-c'U_\tau|^2\,ds$$

(where the prime denotes derivation in s), with the constraint

$$U_\tau'+cU_v=0$$

deriving from the condition $e_\tau=0$.

We apply this formula to find the deformation of a semicircular beam

$$\{x^2+y^2=1,\quad y\geqq 0\},$$

with constant thickness h. Then the deformation energy is

$$K\int_0^\pi |U_v''+U_v|^2\,ds,$$

with the constraint

$$U_\tau'+U_v=0;$$

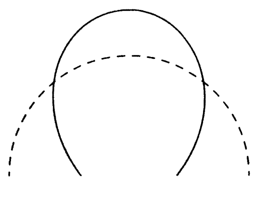
we consider the simply supported case

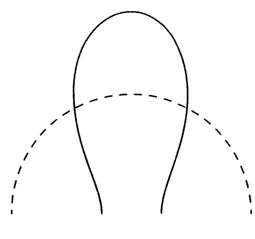
$$U_\tau(0)=U_\tau(\pi)=0,\qquad U_v(0)=U_v(\pi)=-k$$

and the clamped case, in which we add the conditions

$$U_v'(0)=U_v'(\pi)=0.$$

Solving the Euler-Lagrange equation one easily obtains the expression of the deformation: for $k=0.6$ we then have the pictures





which represent the simply supported and the clamped case respectively.

Now we pass to the case of shells in $\mathbb{R}^3$; for any given 2-dimensional manifold Σ, let $M(\sigma)$ be the 2×3 matrix whose rows are the orthonormal tangent vectors τ_1 and τ_2.

Then, if we set $A_0 = M A^t M$, an easy computation shows that

$$(5.1) \qquad f_0(\sigma, A) = \frac{2\mu(\lambda+\mu)}{\lambda+2\mu} (\operatorname{tr} A_0)^2 - 2\mu \det A_0,$$

and we must substitute this expression in (2. 5).

Example V. 2. *The flat case.* Assume Σ is a portion of the plane $\{z=0\}$; then, denoting by w the vertical displacement, and substituting the Lamé coefficients λ, μ with their expression in terms of the Young modulus E and the Poisson coefficient σ, we have for the limit energy of the plate the well known formula

$$\int\!\!\int_{\Sigma} (2h)^3 \frac{E}{24(1-\sigma^2)} [\Delta w - 2(1-\sigma) \det D^2 w] \, dx \, dy,$$

with the constraint that the horizontal displacement is a rigid motion in the plane.

Example V. 3. *The cylindrical case.* Assume Σ is a portion of the cylinder with radius R and axis $\{x=y=0\}$; denote by ϱ, θ, z the cylindrical coordinates, and by U_ϱ, U_θ and U_z the components of the displacement. Then, to obtain the limit energy of the shell it is enough to substitute in (5. 1)

$$A_0 = \begin{pmatrix} D_{zz} U_\varrho & [D_{z\theta} U_\varrho - D_z U_\theta]/R \\ [D_{z\theta} U_\varrho - D_z U_\theta]/R & [U_\varrho + D_{\theta\theta} U_\varrho]/R^2 \end{pmatrix}$$

with the constraints

$$D_z U_z = 0, \qquad U_\varrho + D_\theta U_\theta = 0, \qquad R D_z U_\theta + D_\theta U_z = 0.$$

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