A Variational Definition of the Strain Energy for an Elastic String

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1. Introduction

The aim of this paper is to deduce the constitutive equations of an elastic one-dimensional string from the stress-strain relations of nonlinear three-dimensional elasticity, by passing to the limit when the other dimensions go to zero. We will use the variational point of view.

Denote by Σ the reference configuration of the string:

$$\Sigma = \big\{(x_1, x_2, x_3) : 0 \leq x_1 \leq 1,\ x_2 = x_3 = 0\big\},$$

and by Σ_ε the "thick" elastic body

$$\Sigma_\varepsilon = \big\{(x_1, x_2, x_3) : 0 \leq x_1 \leq 1,\ x_2^2 + x_3^2 \leq \varepsilon^2\big\}.$$

We assume the stored strain energy, associated to a displacement field u, to be given by a functional of the form

$$\int_{\Sigma_\varepsilon} f(Du)\, dx,$$

where $f : \mathbb{R}^9 \to [0, +\infty]$ is a suitable function, in general not convex (we refer to [2],[3] for further physical motivations of this model). Assuming also that the exterior loads

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derive from a potential of the form $u^2+g(x)u$, the equilibrium configuration of the body Σ_ε is given by the solution u_ε of the minimum problem

$$\min\left\{\int_{\Sigma_\varepsilon}[f(Du)+u^2+g(x)u]\,dx\right\},$$

where the minimum is taken over all functions u belonging to some Sobolev space $W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)$.

We shall prove that, if $\varepsilon\to 0^+$, then u_ε converges in an appropriate sense to a function u_0 defined on Σ, and this function u_0 turns out to be a solution of the variational (limit) problem

$$\min\left\{\int_{\Sigma}[f_0^{**}(u')+u^2+g(x_1,0,0)u]\,dx_1\right\},$$

where the function f_0^{**} is defined below, and the functional $\int_\Sigma f_0^{**}(u')\,dx_1$ may be taken as a model for the strain energy of the string Σ.

We shall see in Section 4 that for a large class of physically plausible functions f this functional has the property that *the stored energy is positive under tension, but is zero under compression.* This natural phenomenon cannot be seen if we consider for the elastic body Σ_ε a quadratic (or more generally convex) energy density f, as is the case for example in linear elasticity.

Our method is related to Γ-convergence, already used in similar situations (see e.g. [1],[4],[7]); nevertheless, for the reader's convenience, we state and prove all results without using any specific knowledge about Γ-convergence.

We wish to thank Professor Ennio De Giorgi for the stimulating discussions we had on this subject.

2. Notations and statements

In this section we state an abstract theorem that will be proved in Section 3, and from which we shall deduce in Section 4 the result announced in the Introduction. We set

$$\Sigma = \big\{(x_1, x_2, x_3) : 0 \le x_1 \le 1,\ x_2 = x_3 = 0\big\},$$

and for every $\varepsilon > 0$

$$\Sigma_\varepsilon = \big\{(x_1, x_2, x_3) : 0 \le x_1 \le 1,\ x_2^2 + x_3^2 \le \varepsilon^2\big\},$$

$$S_\varepsilon = \big\{(x_2, x_3) : x_2^2 + x_3^2 \le \varepsilon^2\big\}.$$

If $a, b, c \in \mathbb{R}^3$, we denote by $(a|b|c)$ the matrix whose columns are a, b, c. As usual, the symbol φ^{**} denotes the convex envelope of a function φ, that is

$$\varphi^{**} = \max\{\psi \le \varphi : \psi \text{ is convex and lower semicontinuous}\}.$$

Fix any $p > 1$. The assumptions we make on f are the following:

(2.1) *$f : \mathbb{R}^9 \to \mathbb{R} \cup \{+\infty\}$ is continuous;*

(2.2) *if $\det \xi \le 0$ then $f(\xi) = +\infty$;*

(2.3) *for every $\delta > 0$ there exists c_δ such that, if $\det \xi \ge \delta$, then $f(\xi) \le c_\delta\,(1 + |\xi|^p)$;*

(2.4) *$f(\xi) \ge c|\xi|^p - c'$.*

Remark that (2.1) is a natural regularity assumption, (2.2) has the important physical meaning that the energy becomes infinite when the volume locally vanishes; finally, (2.3) and (2.4) are growth assumptions on the energy.

We define on $C^1(\Sigma_\varepsilon; \mathbb{R}^3)$ the functionals

$$F_\varepsilon(u) = \int_{\Sigma_\varepsilon} f(Du)\,dx;$$

also, we define on $\mathbb{R}^3$ the function

$$f_0(z) = \inf\big\{f(z|\alpha|\beta) : \alpha, \beta \in \mathbb{R}^3\big\},$$

and from the assumptions on f we derive some properties of f_0.

By (2.1),(2.4) we have

(2.5) f_0 is continuous,

while (2.3) implies

(2.6) if $|z| \ge \delta$ then $f_0(z) \le c_\delta\,(1 + |z|^p)$,

so that for a suitable c

(2.7) $f_0^{**} \le c\,(1 + |z|^p)$ for all z.

Therefore we may define on $W^{1,p}(\Sigma; \mathbb{R}^3)$ the functional

$$F_0(u) = \int_\Sigma f_0^{**}(u')\,dx_1.$$

Now let $g \in C^0(\Sigma_1;\mathbb{R}^3)$, and set for every $u \in C^1(\Sigma_\varepsilon;\mathbb{R}^3)$

$$G_\varepsilon(u) = \frac{1}{\pi\varepsilon^2}\Big[F_\varepsilon(u) + \int_{\Sigma_\varepsilon} (gu + |u|^p)\,dx\Big],$$

and for every $u \in W^{1,p}(\Sigma;\mathbb{R}^3)$

$$G_0(u) = F_0(u) + \int_\Sigma (gu + |u|^p)\,dx_1.$$

Finally, for every $v \in L^1(\Sigma_\varepsilon;\mathbb{R}^3)$ we set

$$\widetilde{v}(x_1) = \fint_{S_\varepsilon} v(x_1,x_2,x_3)\,dx_2\,dx_3,$$

where the symbol $\fint$ denotes the integral mean.

The main result is

Theorem 2.1 . *Let $(u_\varepsilon)_{\varepsilon>0}$ be such that $u_\varepsilon \in C^1(\Sigma_\varepsilon;\mathbb{R}^3)$ and*

$$\lim_{\varepsilon\to 0}\Big[G_\varepsilon(u_\varepsilon) - \inf\big\{G_\varepsilon(u) : u \in C^1(\Sigma_\varepsilon;\mathbb{R}^3)\big\}\Big] = 0.$$

The sequence $(\widetilde{u_\varepsilon})_{\varepsilon>0}$ is weakly compact in $W^{1,p}(\Sigma;\mathbb{R}^3)$, and, if u_0 is one of its limit points, then

$$G_0(u_0) = \min\big\{G_0(u) : u \in W^{1,p}(\Sigma;\mathbb{R}^3)\big\}.$$

3. Proof of Theorem 2.1

In the sequel we adopt the notations and assumptions of Section 2, and we shall deduce Theorem 2.1 from the following results:

Proposition 3.1 . *The sequence* $(\widetilde{u_\varepsilon})_{\varepsilon>0}$ *is weakly compact in* $W^{1,p}(\Sigma;\mathbb{R}^3)$.

Proposition 3.2 . *For every sequence* $(v_\varepsilon)_{\varepsilon>0}$ *such that*

$$v_\varepsilon \in C^1(\Sigma_\varepsilon;\mathbb{R}^3)\ , \quad \widetilde{v_\varepsilon} \rightharpoonup v \text{ weakly in } W^{1,p}(\Sigma;\mathbb{R}^3)$$

we have

$$F_0(v) \le \liminf_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} F_\varepsilon(v_\varepsilon).$$

Proposition 3.3 . *For every* $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$ *there exists* $v_\varepsilon \in C^1(\Sigma_\varepsilon;\mathbb{R}^3)$ *such that*

$$\widetilde{v_\varepsilon} \rightharpoonup v \text{ weakly in } W^{1,p}(\Sigma;\mathbb{R}^3)\ , \quad F_0(v) = \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} F_\varepsilon(v_\varepsilon).$$

Proof of Proposition 3.1 : Set $m_\varepsilon = \inf\big\{G_\varepsilon(u) : u \in C^1(\Sigma_\varepsilon;\mathbb{R}^3)\big\}$, and remark that, if u^* is the identity function in $\mathbb{R}^3$,

$$-c(g) \le m_\varepsilon \le G_\varepsilon(u^*) \le c(g) < +\infty.$$

Then (2.4) implies

$$\fint_{\Sigma_\varepsilon} \big(|Du_\varepsilon|^p + |u_\varepsilon|^p\big)\,dx \le c + \fint_{\Sigma_\varepsilon} g u_\varepsilon\,dx \le c(g) + \frac{1}{2}\fint_{\Sigma_\varepsilon} |u_\varepsilon|^p\,dx,$$

hence

$$\int_\Sigma \big(|\widetilde{u_\varepsilon}|^p + |D(\widetilde{u_\varepsilon})|^p\big)\,dx_1 = \int_\Sigma \big(|\widetilde{u_\varepsilon}|^p + |\widetilde{D_{x_1}u_\varepsilon}|^p\big)\,dx_1 \le \fint_{\Sigma_\varepsilon} \big(|u_\varepsilon|^p + |Du_\varepsilon|^p\big)\,dx \le c,$$

and the first proposition is proved. ■

Proof of Proposition 3.2 : Assume $\widetilde{v_\varepsilon} \rightharpoonup v$ weakly in $W^{1,p}(\Sigma;\mathbb{R}^3)$; we remark that (2.7) implies that f_0^{**} is everywhere subdifferentiable, and if $\alpha(z) \in \partial f_0^{**}(z)$ then $|\alpha(z)| \le c\,(1+|z|^{p-1})$: therefore, since $v' \in L^p(\Sigma)$ we may select for any $x_1 \in \Sigma$ an $\alpha \in \partial f_0^{**}\big(v'(x_1)\big)$ such that the mapping $x_1 \mapsto \alpha\big(v'(x_1)\big)$ is in $L^{p'}(\Sigma;\mathbb{R}^3)$. Then

$$\frac{1}{\pi\varepsilon^2}\,F_\varepsilon(v_\varepsilon) = \fint_{\Sigma_\varepsilon} f(Dv_\varepsilon)\,dx \ge \fint_{\Sigma_\varepsilon} f_0(D_{x_1}v_\varepsilon)\,dx \ge \fint_{\Sigma_\varepsilon} f_0^{**}(D_{x_1}v_\varepsilon)\,dx$$

$$\ge \fint_{\Sigma_\varepsilon} [f_0^{**}(v') + \langle \alpha, D_{x_1}v_\varepsilon - v'\rangle]\,dx = \int_\Sigma [f_0^{**}(v') + \langle \alpha, (\widetilde{v_\varepsilon} - v)'\rangle]\,dx_1,$$

and the result follows by taking the limit as $\varepsilon \to 0$. ■

Proof of Proposition 3.3 : We set, for every $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$,

$$F^+(v) = \inf\big\{\limsup_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} F_\varepsilon(v_\varepsilon) : v_\varepsilon \in C^1(\Sigma_\varepsilon;\mathbb{R}^3),\ \widetilde{v_\varepsilon} \rightharpoonup v \text{ weakly in } W^{1,p}(\Sigma;\mathbb{R}^3)\big\};$$

then a diagonal argument shows that the infimum is actually a minimum, so that by Proposition 3.2 we need only prove

$$F^+(v) \le \int_\Sigma f_0^{**}(v')\,dx_1.$$

The proof proceeds in several steps.
Step 1 : $F^+(v) \le \int_\Sigma f_0(v')\,dx_1$ for all $v \in C^3(\Sigma;\mathbb{R}^3)$, with $|v'| \neq 0$, $|v''| \neq 0$.
Set for all $\delta > 0$

$$f_0^\delta(z) = \min\big\{f(z|\alpha|\beta) : \det(z|\alpha|\beta) \ge \delta\big\},$$

we remark that f_0^δ is increasing with respect to δ, and by (2.2)

$$f_0 = \lim_{\delta\to 0} f_0^\delta.$$

Take any two C^1 functions φ, ψ such that $\det(v'|\varphi|\psi) > 0$ on Σ. For instance, one may take the normal and binormal to the curve v. We approximate v on Σ with the function

$$w(x) = v(x_1) + x_2\varphi(x_1) + x_3\psi(x_1)$$

on Σ_ε: then $\widetilde{w} = v$, and $Dw = (v' + x_2\varphi' + x_3\psi'|\varphi|\psi)$, so that by the choice of φ, ψ we may assume that for some δ

$$\det Dw \ge \delta > 0 \quad \text{on } \Sigma_\varepsilon \text{ for all } \varepsilon.$$

Then by (2.3) we may apply the dominated convergence theorem to obtain

$$F^+(v) \le \limsup_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} F_\varepsilon(w) = \int_\Sigma f(v'|\varphi|\psi)\,dx_1 \tag{3.1}$$

for every $\varphi, \psi \in C^1(\Sigma;\mathbb{R}^3)$ such that $\det(v'|\varphi|\psi) \ge \delta$. Now, by Lusin's theorem, for every $(\varphi,\psi) \in L^p(\Sigma;\mathbb{R}^3)$ such that $\det(v'|\varphi|\psi) \ge 2\delta$ there exists a sequence $(\varphi_j,\psi_j) \in C^1(\Sigma;\mathbb{R}^3)$ such that

$$\det(v'|\varphi_j|\psi_j) \ge \delta\ , \quad (\varphi_j,\psi_j) \to (\varphi,\psi) \text{ in } L^p(\Sigma;\mathbb{R}^3),$$

hence, by (2.3) and (3.1),

$$\begin{aligned} F^+(v) &\le \inf\left\{\int_\Sigma f(v'|\varphi|\psi)\,dx_1 : \varphi,\psi \in C^1(\Sigma;\mathbb{R}^3),\ \det(v'|\varphi|\psi) \ge \delta\right\} \\ &\le \inf\left\{\int_\Sigma f(v'|\varphi|\psi)\,dx_1 : \varphi,\psi \in L^p(\Sigma;\mathbb{R}^3),\ \det(v'|\varphi|\psi) \ge 2\delta\right\}, \end{aligned}$$

where the last inequality is to be taken a.e.; by the measurable selection lemma ([6], Theorem 1.2 of Chapter VIII) there exists a couple $(\varphi^{2\delta},\psi^{2\delta})$ of measurable functions, which belong to $L^p(\Sigma;\mathbb{R}^3)$ by (2.4), such that

$$f(v'|\varphi^{2\delta}|\psi^{2\delta}) = \min\big\{f(v'|\alpha|\beta) : \det(v'|\alpha|\beta) \geq 2\delta\big\},$$

therefore

$$F^+(v) \leq \int_\Sigma f_0^{2\delta}(v')\,dx_1.$$

Since $f_0^{2\delta}(v') \leq f(v'|\varphi^1|\psi^1) \in L^1(\Sigma)$, we may pass to the limit as $\delta \to 0$, thus concluding the first step.

Step 2 : $F^+(v) \leq \int_\Sigma f_0(v')\,dx_1$ for all $v \in C^1(\Sigma;\mathbb{R}^3)$, with $|v'| \neq 0$.

By (2.4), a standard diagonal argument shows that F^+ is lower semicontinuous in the weak topology of $W^{1,p}(\Sigma;\mathbb{R}^3)$, therefore also in the strong topology of $C^1(\Sigma;\mathbb{R}^3)$, whereas (2.5),(2.6) imply that $\int_\Sigma f_0(v')\,dx_1$ is continuous on the space $\{v \in C^1(\Sigma;\mathbb{R}^3) : |v'| \geq \delta > 0\}$. Step 2 then follows by density.

Before stating and proving step 3, we introduce an auxiliary function. Since f_0 is continuous, infinite at 0 and finite otherwise, we have $f_0 > f_0^{**}$ in $B_R(0)$ for some $R > 0$. It is easy to find a function h which satisfies for some $0 < \varrho < r < R$ the following conditions:

- h is continuous,
- $h = +\infty$ in B_ϱ,
- h is radial in B_r,
- $h = f_0$ outside B_R,
- $h \geq f_0$.

We cut h on B_r and we set

$$\tilde h(z) = \begin{cases} h(z) & \text{if } |z| \geq r \\ h(r) & \text{if } |z| \leq r. \end{cases}$$

Step 3 : $F^+(v) \leq \int_\Sigma \tilde h(v')\,dx_1$ for every $v \in C^1(\Sigma;\mathbb{R}^3)$.

For every $v \in C^1(\Sigma;\mathbb{R}^3)$ we may construct a sequence $(v_n) \subset C^1(\Sigma;\mathbb{R}^3)$ such that:

- $v_n \to v$ uniformly,
- $v_n(x_1) = v(x_1)$ if $|v'(x_1)| \geq r$,
- $|v_n'(x_1)| = r$ if $|v'(x_1)| < r$.

This sequence converges to v weakly in $W^{1,p}(\Sigma;\mathbb{R}^3)$, and, by the semicontinuity of F^+,

$$F^+(v) \leq \liminf_n F^+(v_n) \leq \liminf_n \int_\Sigma f_0(v_n')\,dx_1 \leq \liminf_n \int_\Sigma h(v_n')\,dx_1 = \int_\Sigma \tilde h(v')\,dx_1,$$

thus concluding step 3.

Now we take the weak lower semicontinuous envelope in $W^{1,p}(\Sigma;\mathbb{R}^3)$ of both terms in the inequality

$$F^+(v) \le \int_\Sigma \tilde{h}(v')\,dx_1,$$

which holds in $C^1(\Sigma;\mathbb{R}^3)$. Since $\tilde{h}$ is continuous, and $-c \le \tilde{h}(z) \le c\,(1+|z|^p)$ by (2.6), it is well known (see [5]) that the lower semicontinuous envelope of $\int_\Sigma \tilde{h}(v')\,dx_1$ is $\int_\Sigma \tilde{h}^{**}(v')\,dx_1$, therefore

$$F^+(v) \le \int_\Sigma \tilde{h}^{**}(v')\,dx_1$$

for every $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$. The proof of Krein-Milman's theorem may be easily adapted to show that, since $p > 1$ and $\tilde{h} = f_0$ on the set where $f_0 = f_0^{**}$, we have $\tilde{h}^{**} = f_0^{**}$, hence

$$F^+(v) \le \int_\Sigma f_0^{**}(v')\,dx_1$$

on $W^{1,p}(\Sigma;\mathbb{R}^3)$, thus concluding the proof of Proposition 3.3. ∎

Proof of Theorem 2.1 : By Proposition 3.1 the sequence $(\widetilde{u_\varepsilon})$ is compact; denoting by u_0 one of its limit points, we have, by Proposition 3.2,

$$F_0(u_0) \le \liminf_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} F_\varepsilon(u_\varepsilon).$$

On the other hand, for any $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$ let (v_ε) be the sequence given by Proposition 3.3. We have

$$\begin{aligned}
G_0(u_0) - G_0(v) &= F_0(u_0) - F_0(v) + \int_\Sigma \big(|u_0|^p + gu_0 - |v|^p - gv\big)\,dx_1 \\
&\le \liminf_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} F_\varepsilon(u_\varepsilon) - \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} F_\varepsilon(v_\varepsilon) \\
&\qquad + \lim_{\varepsilon\to 0} \fint_{\Sigma_\varepsilon} \big(|u_\varepsilon|^p + gu_\varepsilon - |v_\varepsilon|^p - gv_\varepsilon\big)\,dx \\
&= \liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon) - \lim_{\varepsilon\to 0} G_\varepsilon(v_\varepsilon) \\
&\le \liminf_{\varepsilon\to 0} \big[G_\varepsilon(u_\varepsilon) - \inf G_\varepsilon\big] = 0,
\end{aligned}$$

i.e., u_0 is a minimum point for G_0. ∎

4. Some examples

In this Section we consider some particular cases of Theorem 2.1 arising in nonlinear elasticity. In this case the function $f(\xi)$ must satisfy the following additional properties:

(4.1) f *is objective* (i.e., $f(Q\xi) = f(\xi)$ for every proper orthogonal matrix Q),

(4.2) $f(I) = \min f = 0$,

and we can prove the

Theorem 4.1 . *If* (4.1),(4.2) *are satisfied, then* $f_0(z)$ *depends only on* $|z|$, *and*

$$f_0^{**}(z) = 0 \quad \text{whenever } |z| \le 1.$$

PROOF : By (4.1), for every $z \in \mathbb{R}^3$ and every proper orthogonal matrix Q we have

$$\begin{aligned} f_0(Qz) &= \inf \left\{ f(Qz|\alpha|\beta) : \alpha, \beta \in \mathbb{R}^3 \right\} \\ &= \inf \left\{ f(Qz|Qa|Qb) : a, b \in \mathbb{R}^3 \right\} \\ &= \inf \left\{ f(z|a|b) : a, b \in \mathbb{R}^3 \right\} = f_0(z), \end{aligned}$$

so that $f_0(z)$ depends only on $|z|$.

The second assertion follows since f_0^{**} is convex and

$$0 = \min f \le f_0^{**}(z) \le f_0(z) \le f(I) = 0$$

for every $z \in \mathbb{R}^3$ with $|z| = 1$. ∎

To justify the result that the energy is zero under compression (so that for instance to a constant deformation sending the whole string into a single point corresponds zero energy) we remark that, due to the division by the *area* of the section in G_ε, our functionals will approximate *strings*, which disregard torsion and bending: it is easy to shape a very thin rope into a very small coil with very little effort, which means that a one-dimensional string can conceivably be "coiled up" into a single point with no effort; remark that the sequence (v_n) in step 3 does exactly that.

We recall that the strain energy density has commonly the form

$$f(\xi) = |\xi|^p + \beta(\det \xi) \tag{4.3}$$

for suitable choices of p and β (see for instance [2],[8]). As an example, we consider the case

$$f(\xi) = |\xi|^2 + 6(\det \xi)^{-1/3} - 9,$$

defined only if $\det \xi > 0$. Easy computations show that

$$f_0(z) = |z|^2 + 8|z|^{-1/4} - 9,$$

so that

$$f_0^{**}(z) = \begin{cases} f_0(z) & \text{if } |z| \ge 1 \\ 0 & \text{if } |z| < 1. \end{cases}$$

We remark that if in (4.3) we have $p > 3$ and $\beta : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is any convex and continuous function, with $\beta(0) = +\infty$, then the integrand (4.3) satisfies the assumptions of Theorem 2.1; moreover, by Ball's semicontinuity results [2], for every $\varepsilon > 0$ the three-dimensional elasticity problem

$$\min\left\{ \fint_{\Sigma_\varepsilon} [f(Du) + |u|^p + g(x)u]\, dx : u \in W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^3) \right\}$$

has a solution u_ε, and we may apply Theorem 2.1 to conclude that $\widetilde{u_\varepsilon}$ (or a subsequence) converges weakly in $W^{1,p}(\Sigma; \mathbb{R}^3)$ to a solution of the "string" problem

$$\min\left\{ \int_{\Sigma} [f_0^{**}(u') + |u|^p + g(x_1, 0, 0)u]\, dx_1 : u \in W^{1,p}(\Sigma; \mathbb{R}^3) \right\}.$$

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