Approximation of free-discontinuity problems by elliptic functionals via Γ-convergence

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1 Introduction

A variational formulation of some problems in Computer Vision was given by Mumford and Shah [14], and later elaborated by De Giorgi and Ambrosio [11]. In this framework, problems involving the functional

$$\int_\Omega |\nabla u|^2\, dx + \mathcal{H}^{n-1}(S_u), \tag{1}$$

defined on the space $SBV(\Omega)$ of special functions of bounded variation are studied, where ∇u denotes the approximate gradient of u, and S_u is the set of the discontinuity points of u. In a two-dimensional setting, S_u represents the contours of the object in a picture and u is a smoothing of an imput image. Energies of the same form arise in fracture mechanics for brittle solids, where S_u is interpreted as the crack surface and u as the displacement outside the fractured region ([4]). Problems involving functionals of this form are usually called free-discontinuity problems, after a terminology introduced by De Giorgi (see [11], [5], [7]).

The Ambrosio and Tortorelli approach [6] provides a variational approximation of the Mumford and Shah functional (1) via elliptic functionals to obtain approximate smooth solutions and overcome the numerical problems due to surface detection. The unknown surface S_u is substituted by an additional function variable v which approaches the characteristic of the complement of S_u. The approximating functionals have the form

$$\int_\Omega v^2 |\nabla u|^2\, dx + \int_\Omega \Big(\varepsilon |\nabla v|^2 + \frac{1}{4\varepsilon}(1-v)^2\Big)\, dx, \tag{2}$$

defined on functions u, v such that $u, v \in H^1(\Omega)$ and $0 \le v \le 1$. The interaction of the terms in the second integral provide an approximate interfacial energy.

The adaptation of the Ambrosio and Tortorelli approximation to obtain as limits more complex surface energies does not seem to follow easily from their approach. A double-limit procedure to obtain non-constant energy densities is described in [1]. In this paper we study a variant of the Ambrosio and Tortorelli construction by considering functionals of the form

$$G_\varepsilon(u, v) = \int_\Omega \psi(v) |\nabla u|^2\, dx + \int_\Omega \Big(\varepsilon |\nabla v|^2 + \frac{1}{\varepsilon} W(u - v)\Big)\, dx, \tag{3}$$

where W and ψ are positive function vanishing only at 0. In this case the distance between the functions v and u is increasingly penalized as $\varepsilon \to 0^+$, generating in the limit a functional which depends on the traces $u^\pm$ of u on both sides of S_u. We prove (Theorem 3.1) that G_ε approximate the functional

$$F(u) = \int_\Omega \psi(u)|\nabla u|^2\,dx + \int_{S_u} (\Phi(u^+) + \Phi(u^-))\,d\mathcal{H}^{n-1}, \tag{4}$$

where $\Phi(s) = 2|\int_0^s \sqrt{W(t)}dt|$ is the usual transition energy between 0 and s. In this case, the additional variable v in G_ε approaches u times the characteristic of the complement of S_u. Functionals of the Mumford-Shah type with non-constant surface energy density are obtained by choosing $\psi(z) = 1$ if $z \neq 0$.

2 Notation and preliminaries

We use standard notation for Sobolev and Lebesgue spaces. $\mathcal{L}^n$ will denote the Lebesgue measure in $\mathbf{R}^n$ and $\mathcal{H}^k$ will denote the k-dimensional Hausdorff measure. $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ will be the families of open and Borel sets, respectively. If μ is a Borel measure and E is a Borel set, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) = \mu(A \cap B)$. $[t]^\pm$ denote the positive/negative part of $t \in \mathbf{R}$.

2.1 Γ-convergence

Let (X, d) be a metric space. We say that a sequence $F_j : X \to [-\infty, +\infty]$ Γ-*converges* to $F : X \to [-\infty, +\infty]$ (as $j \to +\infty$) if for all $u \in X$ we have
(i) (*lower limit inequality*) for every sequence (u_j) converging to u

$$F(u) \le \liminf_j F_j(u_j); \tag{5}$$

(ii) (*existence of a recovery sequence*) there exists a sequence (u_j) converging to u such that

$$F(u) \ge \limsup_j F_j(u_j), \tag{6}$$

or, equivalently by (5),

$$F(u) = \lim_j F_j(u_j). \tag{7}$$

The function F is called the Γ-*limit* of (F_j) (with respect to d), and we write $F = \Gamma\text{-}\lim_j F_j$. If (F_ε) is a family of functionals indexed by $\varepsilon > 0$ then we say that F_ε Γ-converges to F as $\varepsilon \to 0^+$ if $F = \Gamma\text{-}\lim_{j\to+\infty} F_{\varepsilon_j}$ for all (ε_j) converging to 0.

The reason for the introduction of this notion is explained by the following fundamental theorem.

Theorem 2.1 *Let* $F = \Gamma\text{-}\lim_j F_j$*, and let a compact set* $K \subset X$ *exist such that* $\inf_X F_j = \inf_K F_j$ *for all* j*. Then*

$$\exists \min_X F = \lim_j \inf_X F_j. \tag{8}$$

Moreover, if (u_j) *is a converging sequence such that* $\lim_j F_j(u_j) = \lim_j \inf_X F_j$ *then its limit is a minimum point for* F*.*

The definition of Γ-convergence can be given pointwise on X. It is convenient to introduce also the notion of Γ-lower and upper limit, as follows: let $F_\varepsilon : X \to [-\infty, +\infty]$ and $u \in X$. We define

$$\Gamma\text{-}\liminf_{\varepsilon\to0^+} F_\varepsilon(u) = \inf\{\liminf_{\varepsilon\to0^+} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\}; \tag{9}$$

$$\Gamma\text{-}\limsup_{\varepsilon\to0^+} F_\varepsilon(u) = \inf\{\limsup_{\varepsilon\to0^+} F_\varepsilon(u_\varepsilon) : \ u_\varepsilon \to u\}. \tag{10}$$

If $\Gamma\text{-}\liminf_{\varepsilon\to0^+} F_\varepsilon(u) = \Gamma\text{-}\limsup_{\varepsilon\to0^+} F_\varepsilon(u)$ then the common value is called the Γ-*limit of* (F_ε) *at* u, and is denoted by $\Gamma\text{-}\lim_{\varepsilon\to0^+} F_\varepsilon(u)$. Note that this definition is in accord with the previous one, and that F_ε Γ-converges to F if and only if $F(u) = \Gamma\text{-}\lim_{\varepsilon\to0^+} F_\varepsilon(u)$ at all points $u \in X$.

We recall that:
(i) if $F = \Gamma\text{-}\lim_j F_j$ and G is a continuous function then $F + G = \Gamma\text{-}\lim_j(F_j + G)$;
(ii) the Γ-lower and upper limits define lower semicontinuous functions.

¿From (i) we get that in the computation of our Γ-limits we can drop all d-continuous terms. Remark (ii) will be used in the proofs combined with approximation arguments.

For an introduction to Γ-convergence we refer to [10]; see also [8] Part II. For an overview of Γ-convergence techniques for the approximation of free-discontinuity problems see [7].

2.2 Functions of bounded variation

Let $u \in L^1(\Omega)$. We say that u is a *function of bounded variation* on Ω if its distributional derivative is a measure; i.e., there exist signed measures μ_i such that

$$\int_\Omega u D_i\phi\, dx = -\int_\Omega \phi\, d\mu_i$$

for all $\phi \in C^1_c(\Omega)$. The vector measure $\mu = (\mu_i)$ will be denoted by Du. The space of all functions of bounded variation on Ω will be denoted by $BV(\Omega)$.

It can be proven that if $u \in BV(\Omega)$ then the complement of the set of Lebesgue points S_u, that will be called the *jump set* of u, is *rectifiable*, i.e. there exists a countable family (Γ_i) of graphs of Lipschitz functions of $(n-1)$ variables such that $\mathcal{H}^{n-1}(S_u \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$. Hence, a *normal* ν_u can be defined $\mathcal{H}^{n-1}$-a.e. on S_u, as well as the *traces* $u^\pm$ of u on both sides of S_u as

$$u^\pm(x) = \lim_{\rho \to 0^+} \fint_{\{y \in B_\rho(x) : \pm\langle y - x, \nu_u(x)\rangle > 0\}} u(y)\, dy\,,$$

where $\fint_B u\, dy = |B|^{-1} \int_B u\, dy$. Note that the notation is similar to that of the positive and negative part of u. In the case $n = 1$, we can always choose $\nu = +1$, so that $u^+(x)$ and $u^-(x)$ coincide with the right-hand side and left-hand side (approximate) limits of u at x, denoted by $u(x+)$ and $u(x-)$, respectively.

If $u \in BV(\Omega)$ we define the three measures $D^a u$, $D^j u$ and $D^c u$ as follows. By the Radon Nikodym Theorem we set $Du = D^a u + D^s u$ where $D^a u << \mathcal{L}^n$ and $D^s u$ is the *singular part* of Du with respect to $\mathcal{L}^n$. $D^a u$ is the *absolutely continuous* part of Du with respect to the Lebesgue measure, $D^j u = Du \llcorner S_u$ is the *jump part* of Du, and $D^c u = D^s u \llcorner (\Omega \setminus S_u)$ is the *Cantor part* of Du. We can then write

$$Du = D^a u + D^j u + D^c u\,.$$

It can be seen that $D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u$, and that the Radon Nikodym derivative of Du with respect of $\mathcal{L}^n$ is the *approximate gradient* ∇u of u (which will be also denoted by u' if $n = 1$).

A function $u \in L^1(\Omega)$ is a *special function of bounded variation* on Ω if $D^c u = 0$, or, equivalently, if its distributional derivative can be written as

$$Du = \nabla u\, \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner S_u\,.$$

The space of special functions of bounded variation on Ω is denoted $SBV(\Omega)$.

For a detailed study of the properties of BV-functions we refer to [5], [12] and [13]. For an introduction to the study of free-discontinuity problems in the BV setting we refer to [5].

2.3 A lower semicontinuity result

We recall a simple lower semicontinuity result for one-dimensional functionals defined on $SBV(a, b)$ (see [2], [3], [7] Chapter 2).

Proposition 2.2 *Let (u_j) be a sequence in$SBV(a, b)$ with*

(i) *(u'_j) is bounded in $L^2(a, b)$;*

(ii) *$\#(S_u)$ is equibounded;*

(iii) *(u_j) is bounded in $L^\infty(a, b)$.*

Then, up to passing to a subsequence, u_j converges in $L^1(a, b)$ to a function $u \in SBV(a, b)$. Furthermore,

(a) *$u'_j \to u'$ weakly in $L^2(a, b)$;*

(b) *for all lower semicontinuous* $\vartheta : \mathbf{R} \times \mathbf{R} \to [0, +\infty)$ *which is also subadditive (i.e.,*

$$\vartheta(r, s) \leq \vartheta(r, t) + \vartheta(t, s)$$

for all $r, s, t \in \mathbf{R}$*) we have*

$$\sum_{x \in S_u} \vartheta(u(x-), u(x+)) \leq \liminf_j \sum_{x \in S_{u_j}} \vartheta(u_j(x-), u_j(x+)).$$

In particular $\#(S_u) \leq \liminf_j \#(S_{u_j})$ *by choosing* $\vartheta = 1$.

3 The main result

Let $W, \psi : \mathbf{R} \to [0, +\infty)$ be two functions vanishing only at $z = 0$, increasing on $[0, +\infty)$ and decreasing on $(-\infty, 0]$, and assume that W is continuous and ψ is lower semicontinuous.

Let Ω be a bounded open subset of $\mathbf{R}^n$. We define the space

$$SBV_*(\Omega) = \{u \in L^1(\Omega) : \ u \vee t, u \wedge (-t) \in SBV(\Omega) \text{ for all } t > 0\}.$$

Note that the approximate gradient ∇u of $u \in SBV_*(\Omega)$ exists for a.e. $x \in \Omega$. Moreover, if $n = 1$ then S_u is at most countable.

Theorem 3.1 *Let* $G_\varepsilon : L^1(\Omega) \times L^1(\Omega) \to [0, +\infty)$ *be defined by*

$$G_\varepsilon(u, v) = \begin{cases} \displaystyle\int_\Omega \Big(\psi(v)|\nabla u|^2 + \frac{1}{\varepsilon} W(u - v) + \varepsilon |\nabla v|^2\Big)\, dx & \text{if } u, v \in H^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Then there exists the $\Gamma\text{-}\lim_{\varepsilon \to 0+} G_\varepsilon(u, v) = G(u, v)$ *with respect to the* $L^1(\Omega) \times L^1(\Omega)$*-convergence, and*

$$G(u, v) = \begin{cases} \displaystyle\int_\Omega \psi(u)|\nabla u|^2\, dx + \int_{S_u} (\Phi(u^+) + \Phi(u^-))\, d\mathcal{H}^{n-1} & \text{if } u \in SBV_*(\Omega) \\ & \text{and } u = v \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\Phi(z) := 2\Big|\int_0^z \sqrt{W(s)} ds\Big| \tag{11}$$

for all $z \in \mathbf{R}$.

The proof of the theorem above will be given in detail only in the case $n = 1$, as a consequence of the propositions in the rest of the section. In this case it suffices to consider $\Omega = (a, b)$ an interval. The case $n \geq 2$ can be easily obtained by slicing and approximation techniques from the study of the 1-dimensional case (for all the details we refer to Chapter 4 in [7]).

Proposition 3.2 *Let F be defined on $L^1(a,b)$ by $F(u) = G(u,u)$. Then F is lower semicontinuous with respect to the L^1-convergence.*

Proof. (a) *lower semicontinuity on non-negative functions*: let $t > 0$, let

$$\phi_t(z) = \begin{cases} \Phi(z) & \text{if } z > t \\ 0 & \text{if } z \leq t, \end{cases}$$

and let $\theta_t(y,z) = \phi_t(y) + \phi_t(z)$. The function θ_t is subadditive and lower semicontinuous.

For all $v \in SBV_*(a,b)$ with $v \geq 0$ let $v^t = v \vee t$; note that

$$\begin{aligned} F(v) &\geq F_t(v^t) := \int_{(a,b)} \psi(v^t)|(v^t)'|^2\,dx + \sum_{S_{v^t}} \theta_t((v^t)^+,(v^t)^-) \\ &\geq \psi(t)\int_{(a,b)} |(v^t)'|^2\,dx + \Phi(t)\,\#(S_{v^t}). \end{aligned}$$

If $u_j \to u$ in $L^1(a,b)$ with $u_j \geq 0$ and $F(u_j) \leq c$, we deduce from the inequality above that $u_j^t \to u^t$ weakly in $SBV(a,b)$. From the lower semicontinuity of F_t we have $u^t \in SBV(a,b)$ and

$$\liminf_j F(u_j) \geq \int_{(a,b)} \psi(u^t)|(u^t)'|^2\,dx + \sum_{S_{u_j^t}} \theta((u^t)^+,(u^t)^-).$$

Taking the supremum for $t > 0$ we get $u \in SBV_*(a,b)$ and

$$\liminf_j F(u_j) \geq F(u).$$

(b) *lower semicontinuity on non-positive functions*: the proof is the same as in Step (a).

(c) *conclusion*: the lower semicontinuity of F follows by noting that $F(u) = F(u \wedge 0) + F(u \vee 0)$. □

We "localize" the functionals G_ε by defining for all A open subset of (a,b) and $u,v \in L^1(a,b)$

$$G_\varepsilon(u,v,A) = \begin{cases} \displaystyle\int_A \Big(\psi(v)|u'|^2 + \frac{1}{\varepsilon}W(u-v) + \varepsilon|\nabla v|^2\Big)\,dx & \text{if } u,v \in H^1(a,b) \\ +\infty & \text{otherwise.} \end{cases}$$

Lemma 3.3 *Let $t > 0$, $x_0 < x_1$ and let $u_0, v_0 \in \mathbf{R}$ and $u,v \in H^1(x_0,x_1)$ satisfy*

$$\begin{cases} u_0, v_0 > t \\ u(x_0) = u_0 \\ v(x_0) = v_0 \\ v(x) > t \text{ for all } x \in (x_0,x_1) \\ v(x_1) = t \end{cases} \qquad \text{or} \qquad \begin{cases} u_0, v_0 < -t \\ u(x_0) = u_0 \\ v(x_0) = v_0 \\ v(x) < -t \text{ for all } x \in (x_0,x_1) \\ v(x_1) = -t. \end{cases}$$

Then for every $\varepsilon > 0$

$$G_\varepsilon(u,v,(x_0,x_1)) \geq \psi(t)\frac{(u_0 - \inf(u \vee 0))^2}{x_1 - x_0} + \Phi\big(\big((\inf(u\vee 0)) \wedge v_0 - t\big) \vee 0\big)$$

in the case $u_0, v_0 > t$*, while*

$$G_\varepsilon(u,v,(x_0,x_1)) \geq \psi(-t)\frac{(u_0 + \sup(u \wedge 0)^2}{x_1 - x_0} + \Phi\big(\big(\sup(u\wedge 0) \vee (-v_0) + t\big) \wedge 0\big)$$

in the case $u_0, v_0 < -t$*. The same estimates hold if the boundary conditions at* x_0 *and at* x_1 *are interchanged.*

Proof. We deal with the case $u_0, v_0 > t$ only, the other case being dealt with in a symmetric way. First, note that

$$\int_{(x_0,x_1)} \psi(v)|u'|^2\,dx \ \geq \psi(t)\frac{(u_0 - \inf(u\vee 0))^2}{x_1}. \tag{12}$$

If $\inf(u\vee 0) \leq t$ then the inequality for $G_\varepsilon(u,v,(x_0,x_1))$ is trivial. If $\inf(u\vee 0) > t$, let

$$\overline{x} = \sup\{x \in [x_0,x_1]:\ v(x) = \inf(u\vee 0)\}$$

($\overline{x} = 0$ if $v < \inf u$ on $[0,x_1]$). Suppose first that $\inf(u\vee 0) \leq v_0$; then

$$\begin{aligned}
\int_{\overline{x}}^{x_1}\Big(\frac{1}{\varepsilon}W(u-v) + \varepsilon|v'|^2\Big)\,dx &\geq \int_{\overline{x}}^{x_1}\Big(\frac{1}{\varepsilon}W((\inf(u\vee 0)) - v) + \varepsilon|v'|^2\Big)\,dx \\
&\geq 2\int_{\overline{x}}^{x_1}\sqrt{W((\inf(u\vee 0)) - v)}|v'|\,dx \\
&= \Phi(\inf(u\vee 0) - t). \qquad (13)
\end{aligned}$$

In the case $\inf(u\vee 0) > v_0$, the same computation carries on with v_0 in place of $\inf(u\vee 0)$. ☐

For all $t > 0$ and $r, s \in \mathbf{R}$ we set

$$\Phi_t(r) = \begin{cases} \Phi(r-t) & \text{if } r > t \\ 0 & \text{if } |r| < t \\ \Phi(r+t) & \text{if } r < -t \end{cases} \tag{14}$$

$$\vartheta_t(r,s) = \Phi_t(r) + \Phi_t(s). \tag{15}$$

Proposition 3.4 *Let* $u \in L^1(a,b)$ *and let* $x \in S_u$*. If* $u_\varepsilon \to u$ *and* $v_\varepsilon \to u$ *in* $L^1(a,b)$ *then*

$$\liminf_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon, v_\varepsilon, (x_0,x_1)) \geq \vartheta_t(u(x-), u(x+))$$

for all $t > 0$ *and for all* $x_0 < x < x_1$.

Proof. We deal with the case $u(x+), u(x-) > t > 0$, the changes in the other cases being clear from the proof and from the statement of Lemma 3.3. It is not restrictive to suppose that we have $\liminf_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon, v_\varepsilon, (x_0, x_1)) < +\infty$.

Let (ε_j) be a sequence of positive numbers converging to 0 such that

$$\lim_j G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) = \liminf_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon, v_\varepsilon).$$

Up to restricting the interval (x_0, x_1), we can suppose that

$$\begin{array}{ll} u_{\varepsilon_j}(x_0) \to u(x_0) > t, & u_{\varepsilon_j}(x_1) \to u(x_1) > t, \\ v_{\varepsilon_j}(x_0) \to u(x_0) > t, & v_{\varepsilon_j}(x_1) \to u(x_1) > t. \end{array} \tag{16}$$

Note that $\lim_j \inf_{(x_0,x_1)} |v_{\varepsilon_j}| = 0$; otherwise, by the properties of ψ, we deduce that a subsequence of (u_{ε_j}) is equibounded in $H^1(x_0, x_1)$ and then $u \in H^1(x_0, x_1)$ contradicting the fact that $x \in S_u$.

Denote

$$x_0^j = \inf\{x \in (x_0, x_1) : \ v_{\varepsilon_j}(x) = t\}, \qquad x_1^j = \sup\{x \in (x_0, x_1) : \ v_{\varepsilon_j}(x) = t\},$$

which are attained for j large enough. By Lemma 3.3, we then have

$$\begin{aligned} G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}, (x_0, x_1)) &\geq G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}, (x_0, x_0^j)) + G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}, (x_1^j, x_1)) \\ &\geq \psi(t)\frac{1}{(x_0^j - x_0)}\Big|u_{\varepsilon_j}(x_0) - \inf_{(x_0,x_0^j)}[u_{\varepsilon_j}]^+\Big|^2 \\ &\quad +\Phi_t\Big((\inf_{(x_0,x_0^j)}[u_{\varepsilon_j}]^+) \wedge v_{\varepsilon_j}(x_0)\Big) \\ &\quad +\psi(t)\frac{1}{(x_1 - x_1^j)}\Big|u_{\varepsilon_j}(x_1) - \inf_{(x_1^j,x_1)}[u_{\varepsilon_j}]^+\Big|^2 \\ &\quad +\Phi_t\Big((\inf_{(x_1^j,x_1)}[u_{\varepsilon_j}]^+) \wedge v_{\varepsilon_j}(x_0)\Big). \end{aligned} \tag{17}$$

With fixed $\eta > 0$, up to restricting the interval (x_0, x_1) further, we can suppose that

$$\limsup_j (|u_{\varepsilon_j}(x_0) - u(x-)| + |v_{\varepsilon_j}(x_0) - u(x-)|) < \eta,$$

$$\limsup_j (|u_{\varepsilon_j}(x_1) - u(x+)| + |v_{\varepsilon_j}(x_1) - u(x-)|) < \eta.$$

Then from (17) we deduce first that

$$\limsup_j |(\inf_{(x_0,x_0^j)}[u_{\varepsilon_j}]^+) - u(x-)| \leq c\frac{1}{\sqrt{\psi(t)}}\sqrt{x_1 - x_0} + \eta,$$

$$\limsup_j |(\inf_{(x_1^j,x_1)}[u_{\varepsilon_j}]^+) - u(x+)| \leq c\frac{1}{\sqrt{\psi(t)}}\sqrt{x_1 - x_0} + \eta,$$

and, consequently, that

$$\liminf_j G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}, (x_0, x_1))$$

$$\geq \vartheta_t\Big(u(x-) - c\frac{1}{\sqrt{\psi(t)}}\sqrt{x_1 - x_0} - \eta, u(x+) - c\frac{1}{\sqrt{\psi(t)}}\sqrt{x_1 - x_0} - \eta\Big).$$

As η and $x_1 - x_0$ can be taken arbitrarily small, we have the thesis. □

Remark 3.5 ¿From the previous proposition we immediately get:

(a) for every open subset Ω' of (a, b) we have

$$\liminf_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon, v_\varepsilon, \Omega') \geq \sum_{S_u\cap\Omega'} \vartheta_t(u(x-), u(x+))$$

if $u_\varepsilon \to u$ and $v_\varepsilon \to u$ in $L^1(a, b)$;

(b) if $\Gamma\text{-}\liminf_{\varepsilon\to 0^+} G_\varepsilon(u, u) < +\infty$ then for all $t > 0$

$$\#(\{x \in S_u : |u(x-)| \vee |u(x+)| > t\}) < +\infty.$$

Proposition 3.6 *We have $G \leq \Gamma\text{-}\liminf_{\varepsilon\to 0^+} G_\varepsilon$.*

Proof. Let $u_\varepsilon \to u$ and $v_\varepsilon \to v$ in $L^1(a, b)$ be such that

$$\liminf_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon, v_\varepsilon) < +\infty.$$

Then we have

$$\liminf_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\int_{(a,b)} W(u_\varepsilon - v_\varepsilon)\, dx < +\infty,$$

which implies $u = v$.

Let (ε_j) be a sequence of positive numbers converging to 0 such that

$$\lim_j G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) = \liminf_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon, v_\varepsilon),$$

and such that $u_{\varepsilon_j} \to u$, $v_{\varepsilon_j} \to u$ a.e. We denote

$$M = \{x \in (a, b) :\ \lim_j u_{\varepsilon_j}(x) = \lim_j v_{\varepsilon_j}(x) = u(x)\}.$$

Let $t > 0$ be fixed. Thanks to Remark 3.5(b) we can suppose that $t \notin \{|u(x+)|, |u(x-)| :\ x \in S_u\}$.

We may select a double-indexed family $\{x_i^N : N \in \mathbf{N},\ i = 0, \ldots, 2^N\}$ of points in $[a, b]$ such that

$$a = x_0^N < x_1^N < \cdots < x_{2^N}^N,$$

$x_i^N \notin S_u$, $x_i^N \in M$ (except for $i = 0, 2^N$), and

$$2^{-N-1} \le x_{i+1}^N - x_i^N \le 2^{-N+1}, \qquad x_{2i}^N = x_i^N$$

for all N and i. We can also suppose that there exist the limits $\lim_j u_{\varepsilon_j}(a) = \lim_j v_{\varepsilon_j}(a) = u(a+)$ and $\lim_j u_{\varepsilon_j}(b) = \lim_j v_{\varepsilon_j}(b) = u(b-)$. This is not restrictive, upon first restricting our analysis to $(a+\eta, b-\eta)$ for some small $\eta > 0$, and then letting η tend to 0.

Fix $N \in \mathbf{N}$, and set

$$J_i = J_i^N = (x_{i-1}^N, x_i^N), \qquad i = 1, \ldots, 2^N,$$

and

$$a_N = \{i \in \{1, \ldots, 2^N\} : \ |u| \le t \text{ a.e. on } J_i\}.$$

For all $i \notin a_N$ we can choose a point $\overline{x}_i \in J_i \cap M$ such that $|u(\overline{x}_i)| > t$. We can therefore suppose that

$$u_{\varepsilon_j}(\overline{x}_i) > t, \ v_{\varepsilon_j}(\overline{x}_i) > t \qquad \text{for all } i \text{ and } j, \tag{18}$$

or

$$u_{\varepsilon_j}(\overline{x}_i) < -t, \ v_{\varepsilon_j}(\overline{x}_i) < -t \qquad \text{for all } i \text{ and } j, \tag{19}$$

in the cases $u(\overline{x}_i) > t$ and $u(\overline{x}_i) < -t$, respectively. Upon extracting a subsequence of (ε_j) we can also assume that for all $i \notin a_N$ only one of the two following possibilities is realized:

(i) for all j we have $|v_{\varepsilon_j}(x)| > t/2$ for all $x \in J_i$;

(ii) for all j there exists $y_i^j \in J_i$ such that $|v_{\varepsilon_j}(y_i^j)| \le t/2$.

We then set

$$b_N = \{i \notin a_N : \ \text{(i) holds}\}, \qquad c_N = \{i \notin a_N : \ \text{(ii) holds}\}.$$

Let $i \in c_N$, and suppose for the sake of simplicity that (18) holds and $\overline{x}_i < y_i^j$. By continuity, we then find $\widehat{x}_i \in (\overline{x}_i, y_i^j)$ such that $v_{\varepsilon_j} > t/2$ on $(\overline{x}_i, \widehat{x}_i)$. ¿From Lemma 3.3, applied with $t/2$ in place of t, we deduce that, in the case that $u(\overline{x}_i) > t$,

$$\begin{aligned} G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}, J_i) &\ge \psi\Big(\frac{t}{2}\Big) 2^{N-1}\Big(t - (\inf_{J_i} u_{\varepsilon_j} \vee \frac{t}{2})\Big)^2 + \Phi_t\Big(\inf_{J_i} u_{\varepsilon_j} \vee \frac{t}{2}\Big) \\ &\ge \psi\Big(\frac{t}{2}\Big)\Big(t - (\inf_{J_i} u_{\varepsilon_j} \vee \frac{t}{2})\Big)^2 + \Phi_t\Big(\inf_{J_i} u_{\varepsilon_j} \vee \frac{t}{2}\Big) \\ &\ge c(t) > 0, \end{aligned}$$

where $c(t)$ is a constant depending only on t. The same conclusion holds in the case $u(\overline{x}_i) < -t$, with the obvious changes coming from Lemma 3.3. Hence, $\#(c_N) \le c$, independent of N.

Let now

$$u_t(s) = [u(s)-t]^+ - [u(s)+t]^- = \begin{cases} u(s)-t & \text{if } u(s) > t \\ 0 & \text{if } |u(s)| \le t \\ u(s)+t & \text{if } u(s) < t. \end{cases}$$

If $i \in a_N$ then we trivially have

$$\liminf_j G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}, J_i) \ge 0 = \int_{J_i} \psi(u)|u_t'|^2\,dx. \tag{20}$$

If $i \in b_N$, we first note that

$$\begin{aligned} \liminf_j G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}, J_i) &\ge \liminf_j \int_{J_i} \psi(v_{\varepsilon_j})|u_{\varepsilon_j}'|^2\,dx \\ &\ge \Big(\psi\Big(\frac{t}{2}\Big) \wedge \psi\Big(-\frac{t}{2}\Big)\Big) \liminf_j \int_{J_i} |u_{\varepsilon_j}'|^2\,dx, \end{aligned}$$

from which we deduce, upon extracting a subsequence, that (u_{ε_j}) converges weakly to u in $H^1(J_i)$, and that we may suppose that $\|u_{\varepsilon_j}\|_{L^\infty(J_i)} \le C < +\infty$. Moreover, letting $\tilde v_{\varepsilon_j} = (v_{\varepsilon_j} \wedge C) \vee (-C)$, after noting that

$$G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}, J_i) \ge G_{\varepsilon_j}(u_{\varepsilon_j}, \tilde v_{\varepsilon_j}, J_i)$$

and that $\tilde v_{\varepsilon_j} \to u$ in $L^1(J_i)$ we obtain

$$\begin{aligned} \liminf_j G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}, J_i) &\ge \liminf_j G_{\varepsilon_j}(u_{\varepsilon_j}, \tilde v_{\varepsilon_j}, J_i) \\ &\ge \liminf_j \int_{J_i} \psi(\tilde v_{\varepsilon_j})|u_{\varepsilon_j}'|^2\,dx \\ &\ge \int_{J_i} \psi(u)|u'|^2\,dx \end{aligned} \tag{21}$$

by the lower semicontinuity of the functional $u \mapsto \int_{J_i} \psi(u)|u'|^2\,dx$ (see e.g. [9]).

If $i \in c_N$, by Remark 3.5(a) we estimate

$$\liminf_j G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}, J_i) \ge \sum_{S_u \cap J_i} \vartheta_t(u(x-), u(x+)). \tag{22}$$

We remark, moreover, that:

(a) if $i \in a_N$ then $u_t = 0$ in J_i, which in particular implies that $S_{u_t} \cap J_i = \emptyset$, so that $\vartheta_t(u(x-), u(x+)) = 0$ for all $x \in S_u \cap J_i$;

(b) if $i \in b_N$ then $u \in H^1(J_i)$ and in particular $S_u \cap J_i = \emptyset$;

hence,

$$\sum_{i \in c_N} \sum_{x \in S_u \cap J_i} \vartheta_t(u(x-), u(x+)) = \sum_{x \in S_u} \vartheta_t(u(x-), u(x+)). \tag{23}$$

We set $K_N = \bigcup_{i\in c_N} J_i$. From (20)–(23) we deduce that

$$\liminf_j G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \geq \int_{(a,b)\setminus K_N} \psi(u)|u_t'|^2\,dx + \sum_{x\in S_u} \vartheta_t(u(x-), u(x+)).$$

Upon noting that

(a) if $i \in a_N$ then $\{2i-1, 2i\} \subset a_{N+1}$;

(b) if $i \in b_N$ then $\{2i-1, 2i\} \subset a_{N+1} \cup b_{N+1}$,

we get that $K_{N+1} \subset K_N$, and from $\#c_N \leq c$ and $|J_i| \leq c2^{-N}$ we deduce that $|K_N| \to 0$. From the Dominated Convergence Theorem we then obtain, letting $N \to +\infty$,

$$\liminf_j G_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \geq \int_{(a,b)} \psi(u)|u_t'|^2\,dx + \sum_{x\in S_u} \vartheta_t(u(x-), u(x+)). \tag{24}$$

Note that by (24) we have that

$$(\psi(t) \wedge \psi(-t)) \int_{(a,b)} |u_t'|^2\,dx < +\infty,$$

and, since

$$(\Phi(t) \wedge \Phi(-t))\#(S_{u_{2t}}) \leq \sum_{S_u} \vartheta_t(u(x-), u(x+)),$$

also that $\#(S_{u_{2t}}) < +\infty$. Hence we deduce that $\#(S_{u_t}) < +\infty$ and $u_t \in H^1((a,b)\setminus S_{u_t})$ for all $t > 0$; in particular, $u \in SBV_*(a,b)$.

Eventually, the thesis of the proposition is obtained by letting $t \to 0^+$ and using the Dominated Convergence Theorem again. □

Proposition 3.7 *We have $G \geq \Gamma\text{-}\limsup_{\varepsilon\to 0^+} G_\varepsilon$.*

Proof. First, let $u \in SBV(a,b)$ with

$$\int_{(a,b)} |u_t'|^2\,dx + \#(S_u) < +\infty.$$

By the local nature of the construction below, we can suppose $S_u = \{0\}$, with $0 \in (a,b)$. We have to construct a family $(u_\varepsilon, v_\varepsilon)$ such that

$$F(u) \geq \limsup_{\varepsilon\to 0^+} G_\varepsilon(u_\varepsilon, v_\varepsilon).$$

We perform the construction only for $x > 0$, the construction for $x < 0$ being symmetric.

With fixed $\eta > 0$ let $T > 0$ and $v_T \in H^1(0,T)$ be such that

$$v_T(0) = 0, \qquad v_T(T) = u(0+),$$

$$\int_0^T \Big(W(u(0+) - v_T) + |v_T'|^2 \Big)\, dx \le \Phi(u(0+)) + \eta,$$

$$\frac{1}{T}\int_0^1 W(\tau u(0+))\, d\tau \le \eta.$$

The existence of such a v_T follows for example from [7] Section 3.2.1. We set, for $\varepsilon > 0$ sufficiently small:

$$u_\varepsilon(x) = \begin{cases} x\frac{T}{\varepsilon}u(0+) & \text{if } 0 \le x < \frac{\varepsilon}{T} \\ u(0+) & \text{if } \frac{\varepsilon}{T} \le x < \varepsilon(T + \frac{1}{T}) \\ u\Big(x - \varepsilon(T + \frac{1}{T})\Big) & \text{if } x \ge \varepsilon(T + \frac{1}{T}), \end{cases}$$

$$v_\varepsilon(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{\varepsilon}{T} \\ v_T\Big(\frac{x}{\varepsilon} - \frac{1}{T}\Big) & \text{if } \frac{\varepsilon}{T} \le x \le \varepsilon(T + \frac{1}{T}) \\ u\Big(x - \varepsilon(T + \frac{1}{T})\Big) & \text{if } x > \varepsilon(T + \frac{1}{T}). \end{cases}$$

It can be immediately verified that $u_\varepsilon \to u$ and $v_\varepsilon \to u$ in $L^1(a,b)$. Moreover,

$$\lim_{\varepsilon \to 0^+} G_\varepsilon(u_\varepsilon, v_\varepsilon, (0,b)) \le \int_{(0,b)} \psi(u)|u'|^2\, dx + \Phi(u(0+)) + 2\eta.$$

¿From the corresponding construction for $x < 0$ we eventually obtain

$$\begin{aligned} \lim_{\varepsilon \to 0^+} G_\varepsilon(u_\varepsilon, v_\varepsilon) &\le \int_{(a,b)} \psi(u)|u'|^2\, dx + \Phi(u(0+)) + \Phi(u(0-)) + 4\eta \\ &= G(u,u) + 4\eta, \end{aligned}$$

which gives the desired inequality by the arbitrariness of $\eta > 0$, after noting that the functions u_ε and v_ε belong to $H^1(a,b)$ thanks to the condition $u_\varepsilon(0) = v_\varepsilon(0) = 0$.

In the general case $u \in SBV_*(a,b)$ with $F(u) < +\infty$, consider the family $(u_t)_{t>0}$ defined by

$$u_t = [u - t]^+ - [u + t]^-.$$

Note that $F(u_t) \le F(u)$, $u_t \in SBV(a,b)$ and

$$\int_{(a,b)} |u'|^2\, dx + \#(S_{u_t}) < +\infty.$$

therefore, by the first part of the proof,

$$\Gamma\text{-}\limsup_{\varepsilon \to 0^+} G_\varepsilon(u_t, u_t) \le F(u_t) \le F(u),$$

and the proof is concluded by letting $t \to 0$, so that $u_t \to u$ in $L^1(a,b)$, and recalling the lower semicontinuity of $v \mapsto \Gamma\text{-}\limsup_{\varepsilon \to 0^+} G_\varepsilon(v,v)$. □

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