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AN EXTENSION THEOREM FROM CONNECTED SETS, AND HOMOGENIZATION IN GENERAL PERIODIC DOMAINS

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1. INTRODUCTION

The homogenization problem for an elliptic equation in a periodic domain consists of the study of the behaviour of the solutions when the period goes to zero. Given a bounded open set $\Omega \subseteq \mathbf{R}^n$ with Lipschitz boundary and a function $g \in L^2(\Omega)$, in this paper we take an arbitrary periodic open subset E of $\mathbf{R}^n$ with Lipschitz boundary and, denoting the ε-homothetic set by $E_\varepsilon = \varepsilon E$, we consider the solutions u_ε to the Neumann boundary value problems

$$\begin{cases} -\Delta u_\varepsilon + u_\varepsilon = g & \text{in } \Omega \cap E_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial n_\varepsilon} = 0 & \text{in } \partial(\Omega \cap E_\varepsilon), \end{cases} \tag{1.1}$$

where n_ε is the outward unit normal to $\partial(\Omega \cap E_\varepsilon)$. Under the only additional assumption that E is connected, we prove that there exists an extension $\tilde{u}_\varepsilon$ of u_ε to the whole of Ω, such that $(\tilde{u}_\varepsilon)$ converges to the solution u to the problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} \alpha_{ij} D_i D_j u + u = g & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.2}$$

where (α_{ij}) is the positive definite symmetric matrix defined by

$$\sum_{i,j=1}^{n} \alpha_{ij}\xi_i\xi_j = \inf\left\{\frac{1}{|Q \cap E|}\int_{Q\cap E} |Du(x) + \xi|^2 \, dx : u \text{ is } Q\text{-periodic, } u \in \mathcal{C}^1(\mathbf{R}^n)\right\},$$

for every $\xi \in \mathbf{R}^n$, where Q is the periodicity cell for E. This result is contained in a more general form in theorem 3.1.

Many problems similar to (1.1) may be found in the existing literature. The typical hypothesis in the first papers on this subject is that the complement of the periodic set E is disconnected. More precisely, E is obtained by removing from the periodicity cell Q a set B with smooth boundary such that $\operatorname{dist}(B, \partial Q) > 0$, and repeating this structure by periodicity (see [1, 4, 6, 7, 11, 13, 15]). While this hypothesis is satisfactory in dimension 2, it is not fulfilled in many interesting examples in dimension $n \geq 3$, where both E and $\mathbf{R}^n \setminus E$ may be connected. As an example we may consider the three-dimensional grid

$$E = \{x \in \mathbf{R}^3 \colon \mathrm{d}(x, C) < \tfrac{1}{4}\},$$

where C is the set of the edges of the unit cube Q extended by periodicity. Some particular structures of this kind have been considered in [8, 9, 16].

The main difficulty in this setting is the extension of u_ε, which is defined only on $\Omega \cap E_\varepsilon$, to the whole of Ω. This extension property, which is easy when the complement of E is disconnected, has been taken by Khruslov [14] as the definition of *strong connectivity* of $\Omega \cap E_\varepsilon$ and used as the main assumption in the proof of the convergence of the solutions u_ε to (1.1). More generally, a uniform local extension property has been introduced in [10]. A similar extension problem has been considered in [3] for a more general class of periodic sets E whose complement is not necessarily disconnected.

The main new feature of our paper is the proof of an extension result (see theorem 2.1) under the only assumption that the periodic set E is connected. This result allows us to prove the convergence of the extended functions $\tilde{u}_\varepsilon$. In order to show that the limit function u is a solution to (1.2), we write (1.1) in an equivalent form, as a minimum problem, and we use Γ-convergence methods to determine the limit equation (1.2). These results are proven in Section 3 for more general minimum problems defined by convex integral functionals.

2. CONSTRUCTION OF THE EXTENSION OPERATORS

The aim of this section is to prove the existence of suitable extension operators. They will be the fundamental tool to prove the compactness of the solutions to the homogenization problems considered in Section 3.

Let $p \in \mathbf{R}^n$, with $1 \leq p < +\infty$, and let Q be the open unit cube of $\mathbf{R}^n$, centred at the origin, i.e. $Q =]-\frac{1}{2}, \frac{1}{2}[^n$. We say that a set $E \subseteq \mathbf{R}^n$ is periodic if $E + e_i = E$ for every $i = 1, 2, \dots, n$, where (e_i) is the canonical basis of $\mathbf{R}^n$. Moreover, we say that an open set $E \subseteq \mathbf{R}^n$ has Lipschitz boundary at $x \in \partial E$ if ∂E is locally the graph of a Lipschitz function, in the sense that there exist a coordinate system $(y_1, \dots, y_n)$, a Lipschitz function Φ of $n - 1$ variables, and an open rectangle U_x in the y-coordinates, centred at x, such that $E \cap U_x = \{y \colon y_n < \Phi(y_1, \dots, y_{n-1})\}$ and that ∂E splits U_x into two connected sets, $E \cap U_x$ and $U_x \setminus \bar{E}$. If this property holds for every $x \in \partial E$ with the same Lipschitz constant, we say that E has Lipschitz boundary. We say also that E has the cone property if there exists a finite open cone C such that each point $x \in E$ is the vertex of a finite cone C_x contained in E and congruent to C. It is clear that an open set with Lipschitz boundary has the cone property.

In the sequel, if $A \subset \mathbf{R}^n$ is any open set and $\lambda > 0$, we shall use the following symbols:

A_λ or λA the λ-homothetic set $\{\lambda x \colon x \in A\}$,
$A(\lambda)$ the retracted subset $\{x \in A \colon \mathrm{dist}(x, \partial A) > \lambda\}$.

The main result of this section is the following theorem.

THEOREM 2.1. Let E be a periodic, connected, open subset of $\mathbf{R}^n$, with Lipschitz boundary. Given a bounded open set $\Omega \subset \mathbf{R}^n$, and a real number $\varepsilon > 0$, there exist a linear and continuous extension operator $T_\varepsilon \colon W^{1,p}(\Omega \cap E_\varepsilon) \to W^{1,p}_{\mathrm{loc}}(\Omega)$, and three constants $k_0, k_1, k_2 > 0$, such that

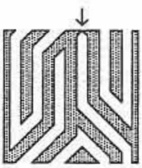
$$T_\varepsilon u = u \qquad \text{a.e. in } \Omega \cap E_\varepsilon, \tag{2.1}$$

$$\int_{\Omega(\varepsilon k_0)} |T_\varepsilon u|^p \, \mathrm{d}x \le k_1 \int_{\Omega \cap E_\varepsilon} |u|^p \, \mathrm{d}x, \tag{2.2}$$

$$\int_{\Omega(\varepsilon k_0)} |D(T_\varepsilon u)|^p \, \mathrm{d}x \le k_2 \int_{\Omega \cap E_\varepsilon} |Du|^p \, \mathrm{d}x, \tag{2.3}$$

for every $u \in W^{1,p}(\Omega \cap E_\varepsilon)$. The constants k_0, k_1, k_2 depend on E, n, p, but are independent of ε and Ω.

In our general hypotheses, it is not possible to construct a family of extension operators T_ε: $W^{1,p}(\Omega \cap E_\varepsilon) \to W^{1,p}(\Omega)$ satisfying (2.1), ..., (2.3) with $\Omega(\varepsilon k_0)$ replaced by Ω, since we do not have any control on the behaviour of E_ε near $\partial\Omega$. In particular, even with $\Omega = Q$, it may happen that $\Omega \cap E_\varepsilon$ is disconnected for every $0 < \varepsilon < 1$ (see Fig. 1), and this prevents (2.3) to hold with Ω in place of $\Omega(\varepsilon k_0)$.



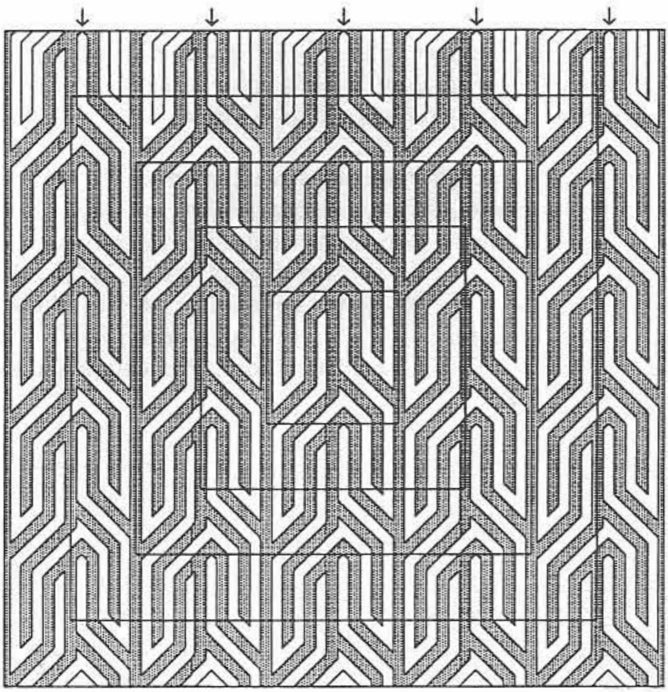


Fig. 1. The smaller picture on the left represents the intersection $E \cap Q$ of a periodic open subset E of $\mathbf{R}^2$ with the unit square. Note that although $E \cap Q$ is not connected and does not have Lipschitz boundary in a neighbourhood of the point marked with an arrow, the periodic set E is connected and has a Lipschitz boundary, as one can see in the larger picture on the right. The smallest square in that picture represents Q, while the other squares represent $2Q$, $3Q$, $4Q$, $5Q$. The shaded region is the connected component A of $E \cap 5Q$ which contains $E \cap 2Q$. One can see easily that, although E is connected, the sets $E \cap kQ$ are not connected for any $k \in \mathbf{N}$. This cannot be avoided by a suitable choice of affine coordinates: in fact $E \cap P$ is not connected for any parallelogram P containing a periodicity cell. Note that $k = 5$ is the smallest integer such that $E \cap 2Q$ is contained in one connected component of $E \cap kQ$. Finally, note that A does not have a Lipschitz boundary in a neighbourhood of the points marked with an arrow.

For what concerns its application to the homogenization problems considered in Section 3, the crucial point of theorem 2.1 is the estimate (2.3) for the gradients. The general proof of this result is quite technical and will be given after some preliminary lemmas. In order to give an idea of the proof, let us consider the particular case where the set $E \cap 2Q$ is connected and has Lipschitz boundary. These properties yield the existence of an extension operator $\tau: W^{1,p}(E \cap 2Q) \to W^{1,p}(2Q)$ (see [7, lemma 3]) that has separate estimates for the gradients. To construct a global extension operator we first consider the family $\tau^\alpha: W^{1,p}(E \cap (\alpha + 2Q)) \to W^{1,p}(\alpha + 2Q)$ of extension operators obtained by translating τ by an integer vector $\alpha \in \mathbf{Z}^n$. The next step consists in glueing together the extension operators $(\tau^\alpha)_\alpha$ by means of a periodic partition of unity, and in proving that an estimate between the gradients still holds. This will be carried out in lemma 2.7. Since the assumption that $E \cap 2Q$ is connected and has Lipschitz boundary is not always satisfied (see Fig. 1), we have proven lemma 2.7 under the general hypotheses of theorem 2.1. In this case the role of the operator τ is played by a different extension operator, that still has separate estimates for the gradients, whose existence is proven in lemma 2.6. The fact that the set $E \cap 2Q$ may be disconnected is taken into account in lemmas 2.2 and 2.3, while the difficulties due to the lack of regularity of its boundary are overcome by means of lemmas 2.4 and 2.5.

LEMMA 2.2. Let A, ω, ω' be open subsets of $\mathbf{R}^n$. Assume that ω, ω' are bounded, with $\omega \subset\subset \omega'$ and that A has Lipschitz boundary at each point of $\partial A \cap \bar{\omega}$. Then the number of connected components of $A \cap \omega'$ that intersect $A \cap \omega$ is finite.

Proof. Since A has Lipschitz boundary at each point of $\partial A \cap \bar{\omega}$, for every $x \in \partial A \cap \bar{\omega}$ there exists an open neighbourhood U_x of x, contained in ω', such that $U_x \cap A$ is connected. For every $x \in A \cap \bar{\omega}$, let U_x be any open ball centred at x, contained in $A \cap \omega'$. Since $\bar{A} \cap \bar{\omega}$ is compact, there exist $x_1, \dots, x_N \in \bar{A}$ such that $\bar{A} \cap \bar{\omega} \subseteq \bigcup_{i=1}^N U_{x_i}$. In particular, $A \cap \omega \subseteq \bigcup_{i=1}^N (U_{x_i} \cap A)$. Now, if C is a connected component of $A \cap \omega'$ that intersects $A \cap \omega$, then C intersects $U_{x_{i_0}} \cap A$ for some $i_0 = 1, \dots, N$. But, since $U_{x_{i_0}} \cap A$ is connected, then $U_{x_{i_0}} \cap A \subseteq C$, and this implies that the number of connected components of $A \cap \omega'$ which intersect $A \cap \omega$ is at most N. ■

LEMMA 2.3. Let E be a connected open subset of $\mathbf{R}^n$, with Lipschitz boundary. Then, there exists $k \in \mathbf{N}$, $k \geq 3$ such that $2Q \cap E$ is contained in a single connected component of $kQ \cap E$ (see Fig. 1).

Proof. By applying lemma 2.2 with $A = E$, $\omega = 2Q$, $\omega' = 3Q$, we can assume that $2Q \cap E = \bigcup_{i=1}^N C_i$, where $C_i = C_i' \cap (2Q \cap E)$, and C_i' are connected components of $3Q \cap E$. Let us fix $x_i \in C_i$, $i = 1, \dots, N$. Since E is connected, for every $i, j \in \{1, \dots, N\}$ there exists a continuous map $\gamma_{i,j}: [0, 1] \to E$, such that $\gamma_{i,j}(0) = x_i$, $\gamma_{i,j}(1) = x_j$. Moreover, for every i, j, there exists $k_{i,j} \in \mathbf{N}$ such that $\gamma_{i,j}([0, 1]) \subseteq k_{i,j} Q$. By taking $k = \max_{i,j} k_{i,j}$, we obtain that $2Q \cap E$ is contained in a single connected component of $kQ \cap E$. ■

LEMMA 2.4. Let A, ω be open subsets of $\mathbf{R}^n$. Assume that ω is bounded and that A is connected and has Lipschitz boundary at each point of $\partial A \cap \bar{\omega}$. Then there exists a bounded and connected open set $B \subseteq A$, with the cone property, having Lipschitz boundary at each point of $\partial B \cap \bar{\omega}$, such that $\omega \cap B = \omega \cap A$.

Proof. Since the set $U = \{x \in \partial A : A$ has Lipschitz boundary at $x\}$ is open in the relative topology of ∂A, and $\partial A \cap \bar{\omega}$ is a compact subset of U, there exist two bounded open subsets ω_1, ω_2 of $\mathbf{R}^n$, such that $\omega \subset\subset \omega_1 \subset\subset \omega_2$ and A has Lipschitz boundary at the points of $\partial A \cap \overline{\omega_2}$. By lemma 2.2, let $C_1, \ldots, C_N$, be the connected components of $A \cap \omega_2$ that intersect $A \cap \omega_1$, and let $x_i \in C_i$, for every $i = 1, \ldots, N$. Since A is connected, for every $i, j \in \{1, \ldots, N\}$ there exists a continuous map $\gamma_{i,j}: [0, 1] \to A$, such that $\gamma_{i,j}(0) = x_i$, $\gamma_{i,j}(1) = x_j$. We denote by S the bounded connected set

$$S = \left(\bigcup_{i=1}^{N} C_i \right) \cup \left(\bigcup_{i,j=1}^{N} \gamma_{ij}([0, 1]) \right),$$

and by $K = \bar{S}$ its closure, which is a compact connected subset of $\bar{A}$. Since $\partial A \cap K \subseteq \partial A \cap \overline{\omega_2}$, for every $x \in \partial A$ there exists an open neighbourhood U_x of x, such that $U_x \cap A$ satisfies the cone property. Moreover, for every $x \in A \cap K$, let U_x be an open ball centred at x and contained in A. Since K is compact, there exist $x_1, \ldots, x_M$, such that $K \subseteq \bigcup_{i=1}^{M} U_{x_i}$.

Let us define $B = \bigcup_{i=1}^{M} (U_{x_i} \cap A)$. By our construction, B is a bounded open subset of A. Moreover, the cone property holds for B, since it holds for each set $U_{x_i} \cap A$. It is easy to check that B is connected, since for every i the set $U_{x_i} \cap A$ is connected and has nonempty intersection with the connected set S. Let us prove now that $\omega_1 \cap B = \omega_1 \cap A$. The inclusion $\subseteq$ is trivial, since $B \subseteq A$. On the other side, if $x \in \omega_1 \cap A$, then $x \in C_{i0}$ for some $i_0 = 1, \ldots, N$. Hence $x \in K \cap A \subseteq \bigcup_{i=1}^{M} (U_{x_i} \cap A) = B$, i.e. $x \in \omega_1 \cap B$. Since $\omega \subset\subset \omega_1$, the conclusions of the lemma follow immediately. ∎

LEMMA 2.5. Let B, ω be open subsets of $\mathbf{R}^n$. Assume that ω is bounded and that B has Lipschitz boundary at each point of $\partial B \cap \bar{\omega}$. Then, there exists a linear and continuous operator $S: W^{1,p}(B) \to W^{1,p}(\omega)$ such that for every $u \in W^{1,p}(B)$

$$Su = u \qquad \text{a.e. in } B \cap \omega, \tag{2.4}$$

$$\|Su\|_{L^p(\omega)} \leq c\|u\|_{L^p(B)}, \tag{2.5}$$

$$\|Su\|_{W^{1,p}(\omega)} \leq c\|u\|_{W^{1,p}(B)}, \tag{2.6}$$

where $c = c(n, p, B, \omega)$.

Since the problem can be localized by means of a partition of unity, the proof of lemma 2.5 can be obtained by applying the standard reflection technique (see for instance [2, theorems 4.26, 4.28 and Section 4.29]) to the neighbourhoods of the points of $\partial B \cap \bar{\omega}$, and is therefore omitted.

In the next lemma, we shall use the notation

$$(f)_A = \fint_A f\,dx = \frac{1}{|A|} \int_A f\,dx$$

to denote the average of a function $f \in L^1(A)$, where $A \subseteq \mathbf{R}^n$ is bounded and $|A|$ is its Lebesgue measure.

The following lemma, which is crucial for the proof of theorem 2.1, states the existence of an extension operator that has separate estimates in terms of the gradients.

LEMMA 2.6. Let A, ω be open subsets of $\mathbf{R}^n$. Assume that ω is bounded and that A is connected and has Lipschitz boundary at each point of $\partial A \cap \bar{\omega}$. Then, there exists a linear and continuous operator $\tau: W^{1,p}(A) \to W^{1,p}(\omega)$ such that, for every $u \in W^{1,p}(A)$

$$\tau u = u \qquad \text{a.e. in } A \cap \omega, \tag{2.7}$$

$$\int_\omega |\tau u|^p \, \mathrm{d}x \le c_1 \int_A |u|^p \, \mathrm{d}x, \tag{2.8}$$

$$\int_\omega |D(\tau u)|^p \, \mathrm{d}x \le c_2 \int_A |Du|^p \, \mathrm{d}x, \tag{2.9}$$

where c_1, c_2 depend only on n, p, A, ω.

Proof. By lemma 2.4, there exists a bounded and connected open set $B \subseteq A$, with the cone property, such that $\omega \cap B = \omega \cap A$. Moreover, B has Lipschitz boundary at each point of $\partial B \cap \bar{\omega}$. Then, by applying lemma 2.5 to the sets B, ω, there exists a linear and continuous operator $S: W^{1,p}(B) \to W^{1,p}(\omega)$ satisfying (2.4), ..., (2.6). For every $u \in W^{1,p}(A)$ we set

$$\tau u = S(u_{|B} - (u)_B) + (u)_B,$$

where $u_{|B}$ denotes the restriction of u to the set B. From the properties of S we have that $\tau u \in W^{1,p}(\omega)$ and $\tau u = u$ a.e. in $B \cap \omega = A \cap \omega$, that is (2.7). Moreover, by (2.5) and Hölder's inequality, we have

$$\begin{aligned}\int_\omega |\tau u|^p \, \mathrm{d}x &= \int_\omega |S(u - (u)_B) + (u)_B|^p \, \mathrm{d}x \\ &\le c \int_\omega |S(u - (u)_B)|^p \, \mathrm{d}x + c \int_\omega |(u)_B|^p \, \mathrm{d}x \\ &\le c \int_B |u|^p \, \mathrm{d}x \le c \int_A |u|^p \, \mathrm{d}x,\end{aligned}$$

where, for simplicity, the letter c denotes a positive constant that depends only on n, p, B, ω, A, and can change from line to line. Condition (2.8) is then completely proven.

To show (2.9), let us remark that, since B is a bounded open set satisfying the cone property, the imbedding of $W^{1,p}(B)$ into $L^p(B)$ is compact (Rellich theorem, see, for instance, [2, theorem 6.2]). Therefore, the following Poincaré inequality holds (see, for instance, [17, theorem 4.2.1]): for every $u \in W^{1,p}(B)$

$$\int_B |u - (u)_B|^p \, \mathrm{d}x \le c \int_B |Du|^p \, \mathrm{d}x,$$

where $c = c(n, p, B)$. By taking (2.6) and Poincaré inequality into account, we have finally

$$\begin{aligned}\int_\omega |D(\tau u)|^p \, \mathrm{d}x &= \int_\omega |D(S(u - (u)_B))|^p \, \mathrm{d}x \\ &\le c \int_B |u - (u)_B|^p \, \mathrm{d}x + c \int_B |Du|^p \, \mathrm{d}x \\ &\le c \int_B |Du|^p \, \mathrm{d}x \le c \int_A |Du|^p \, \mathrm{d}x.\end{aligned}$$

The proof of lemma 2.6 is then complete. ∎

From now on we shall make use of the following notation. For every set $A \subseteq \mathbf{R}^n$, for every $\alpha \in \mathbf{Z}^n$, and for every real number $h > 0$ we denote by

$$A_h^\alpha = \alpha + hA \tag{2.10}$$

the translated image of the set $hA = \{hx: x \in A\}$ by the integer vector α. Moreover, we indicate by $\pi_h^\alpha: \mathbf{R}^n \to \mathbf{R}^n$ the invertible affine map defined by

$$\pi_h^\alpha(x) = \alpha + hx, \tag{2.11}$$

for every $x \in \mathbf{R}^n$. When $h = 1$, we shall simply write $A_1^\alpha = A^\alpha$ and $\pi_1^\alpha = \pi^\alpha$, while for $\alpha = 0 \in \mathbf{Z}^n$ we shall put $A_h^0 = A_h$ and $\pi_h^0 = \pi_h$.

LEMMA 2.7. Let E be a periodic, connected open subset of $\mathbf{R}^n$, with Lipschitz boundary. Let Ω, Ω' be open subsets of $\mathbf{R}^n$ such that $\Omega' \subset\subset \Omega$ and $\operatorname{dist}(\Omega', \partial\Omega) > 2\sqrt{n}k$, where $k \geq 3$ is the integer given by lemma 2.3. Then there exist two positive constants k_1, k_2, and a linear continuous operator $L: W^{1,p}(\Omega \cap E) \to W^{1,p}(\Omega')$, such that

$$Lu = u \qquad \text{a.e. in } \Omega' \cap E, \tag{2.12}$$

$$\int_{\Omega'} |Lu|^p \, dx \leq k_1 \int_{\Omega \cap E} |u|^p \, dx, \tag{2.13}$$

$$\int_{\Omega'} |D(Lu)|^p \, dx \leq k_2 \int_{\Omega \cap E} |Du|^p \, dx, \tag{2.14}$$

for every $u \in W^{1,p}(\Omega \cap E)$. The constants k_1, k_2 depend only on E, n, and p, but are independent of Ω and Ω'.

Proof. Let C be the connected component of $kQ \cap E$ containing $2Q \cap E$, given by lemma 2.3. Since E has Lipschitz boundary, the set C has Lipschitz boundary at each point of $\partial C \cap 2\bar{Q}$ and we can apply lemma 2.6 to the sets $A = C$ and $\omega = 2Q$. Therefore, there exists a linear and continuous operator $\tau: W^{1,p}(C) \to W^{1,p}(2Q)$ such that, for every $u \in W^{1,p}(C)$

$$\tau u = u \qquad \text{a.e. in } 2Q \cap C = 2Q \cap E, \tag{2.15}$$

$$\int_{2Q} |\tau u|^p \, dx \leq c_1 \int_C |u|^p \, dx \leq c_1 \int_{kQ \cap E} |u|^p \, dx, \tag{2.16}$$

$$\int_{2Q} |D(\tau u)|^p \, dx \leq c_2 \int_C |Du|^p \, dx \leq c_2 \int_{kQ \cap E} |Du|^p \, dx, \tag{2.17}$$

where c_1, c_2 depend only on n, p, E.

Now, let us consider the open cover of $\mathbf{R}^n$ given by the cubes $(Q_2^\alpha)_{\alpha \in \mathbf{Z}^n}$ (see notation (2.10)), and for every set $A \subseteq \mathbf{R}^n$ let us define $I(A) = \{\alpha \in \mathbf{Z}^n: Q_2^\alpha \cap A \neq \emptyset\}$. Under our assumptions, for every $\alpha \in I(\Omega')$ we have $Q_{2k}^\alpha \subseteq \Omega$.

For every $\alpha \in I(\Omega')$ we define by $\tau^\alpha: W^{1,p}(C^\alpha) \to W^{1,p}(Q_2^\alpha)$ the extension operator obtained by translating the operator τ, i.e. for every $u \in W^{1,p}(C^\alpha)$

$$\tau^\alpha u = (\tau(u \circ \pi^\alpha)) \circ \pi^{-\alpha} \tag{2.18}$$

(see notation (2.11)). For simplicity, for $u \in W^{1,p}(\Omega \cap E)$ we shall denote by u^α the function

$$u^\alpha = \tau^\alpha(u_{|C^\alpha}) \in W^{1,p}(Q_2^\alpha). \tag{2.19}$$

In order to define a global extension operator $L: W^{1,p}(\Omega \cap E) \to W^{1,p}(\Omega')$, we consider a partition of unity $(\varphi^\alpha)_\alpha$ associated to the open cover $(Q_2^\alpha)_\alpha$, such that $\varphi^\beta = \varphi^\alpha \circ \pi^{\alpha-\beta}$, for every $\alpha, \beta \in \mathbf{Z}^n$. A partition of unity of this kind can be constructed, for instance, by choosing a nonnegative function $\varphi \in \mathcal{C}_0^\infty(\mathbf{R}^n)$, with $\operatorname{spt} \varphi \subset\subset (2Q)$ and $\varphi > 0$ on $\frac{3}{2}Q$, and by setting

$$\varphi^\alpha(x) = \frac{\varphi(x-\alpha)}{\sum_{\beta \in \mathbf{Z}^n} \varphi(x-\beta)} \tag{2.20}$$

for every $\alpha \in \mathbf{Z}^n$ and for every $x \in \mathbf{R}^n$. In this way $\sum_{\alpha \in I(\Omega')} \varphi^\alpha(x) = 1$ for every $x \in \Omega'$, and there exists a positive constant M such that

$$(|\varphi^\alpha(x)| + |D\varphi^\alpha(x)|) \leq M, \tag{2.21}$$

for every $\alpha \in \mathbf{Z}^n$ and for every $x \in \mathbf{R}^n$.

Now, for every $u \in W^{1,p}(\Omega \cap E)$ we set

$$Lu = \sum_{\alpha \in I(\Omega')} u^\alpha \varphi^\alpha, \tag{2.22}$$

with u^α, φ^α given by (2.19), (2.20), respectively. It is clear that L is a linear operator from $W^{1,p}(\Omega \cap E)$ to $W^{1,p}(\Omega')$ and that condition (2.12) is satisfied, since by (2.19), (2.18), and (2.15), we have

$$\sum_{\alpha \in I(\Omega')} u^\alpha(x)\varphi^\alpha(x) = \sum_{\alpha \in I(\Omega')} u(x)\varphi^\alpha(x) = u(x)$$

for a.e. $x \in \Omega' \cap E$. To prove (2.13), let us notice that, by (2.21), for every $u \in W^{1,p}(\Omega \cap E)$ and for every $\beta \in I(\Omega')$,

$$\int_{Q_2^\beta} |Lu|^p \,\mathrm{d}x \leq N^{p-1}M^p \sum_{\alpha \in I(Q_2^\beta)} \int_{Q_2^\alpha} |u^\alpha|^p \,\mathrm{d}x, \tag{2.23}$$

where, from now on, N denotes the cardinality of the set $I(Q_2^\beta)$. Since by (2.19) and (2.16) we have for every $\alpha \in I(Q_2^\beta)$

$$\int_{Q_2^\alpha} |u^\alpha|^p \,\mathrm{d}x \leq c_1 \int_{Q_k^\alpha \cap E} |u|^p \,\mathrm{d}x,$$

and since $\alpha \in I(Q_2^\beta)$ implies $Q_k^\alpha \subseteq Q_{2k}^\beta$, then from (2.23) we obtain

$$\int_{Q_2^\beta} |Lu|^p \,\mathrm{d}x \leq N^p M^p c_1 \int_{Q_{2k}^\beta \cap E} |u|^p \,\mathrm{d}x.$$

By taking the sum over $\beta \in I(\Omega')$ in the preceding inequality, we have

$$\begin{aligned} \int_{\Omega'} |Lu|^p \,\mathrm{d}x &\leq \sum_{\beta \in I(\Omega')} \int_{Q_2^\beta} |Lu|^p \,\mathrm{d}x \\ &\leq N^p M^p c_1 \sum_{\beta \in I(\Omega')} \int_{Q_{2k}^\beta \cap E} |u|^p \,\mathrm{d}x \\ &\leq N^p M^p c_1 c(k) \int_{\Omega \cap E} |u|^p \,\mathrm{d}x, \end{aligned} \tag{2.24}$$

where $c(k)$ is a constant depending only on k and n, such that each point $x \in \mathbf{R}^n$ is contained in at most $c(k)$ cubes of the form $(Q_{2k}^{\beta})_{\beta \in \mathbf{Z}^n}$. Hence (2.13) is proven with $k_1 = N^p M^p c_1 c(k)$.

In order to prove (2.14), for every $u \in W^{1,p}(\Omega \cap E)$ and for every $\beta \in I(\Omega')$ we write

$$\int_{Q_2^\beta} |D(Lu)|^p \, dx \le 2^{p-1} \int_{Q_2^\beta} \left| \sum_{\alpha \in I(Q_2^\beta)} Du^\alpha \varphi^\alpha \right|^p dx + 2^{p-1} \int_{Q_2^\beta} \left| \sum_{\alpha \in I(Q_2^\beta)} u^\alpha D\varphi^\alpha \right|^p dx. \tag{2.25}$$

By (2.17), the first term can be estimated as follows

$$\begin{aligned} \int_{Q_2^\beta} \left| \sum_{\alpha \in I(Q_2^\beta)} Du^\alpha \varphi^\alpha \right|^p dx &\le N^{p-1} \sum_{\alpha \in I(Q_2^\beta)} \int_{Q_2^\beta \cap Q_2^\alpha} |Du^\alpha \varphi^\alpha|^p \, dx \\ &\le N^{p-1} M^p c_2 \sum_{\alpha \in I(Q_2^\beta)} \int_{Q_k^\alpha \cap E} |Du|^p \, dx \\ &\le N^{p-1} M^p c_2 \int_{Q_{2k}^\beta \cap E} |Du|^p \, dx. \end{aligned} \tag{2.26}$$

Since $\sum_{\alpha \in I(Q_2^\beta)} D\varphi^\alpha(x) = 0$ in Q_2^β, the second integrand on the right-hand side of (2.25) can be written as

$$\begin{aligned} \sum_{\alpha \in I(Q_2^\beta)} u^\alpha D\varphi^\alpha &= \sum_{\alpha \in I(Q_2^\beta)} [(u^\alpha - u^\beta) D\varphi^\alpha + u^\beta D\varphi^\alpha] \\ &= \sum_{\alpha \in I(Q_2^\beta)} (u^\alpha - u^\beta) D\varphi^\alpha \end{aligned}$$

for a.e. $x \in Q_2^\beta$. Hence, by (2.23) we have

$$\begin{aligned} \int_{Q_2^\beta} \left| \sum_{\alpha \in I(Q_2^\beta)} u^\alpha D\varphi^\alpha \right|^p dx &= \int_{Q_2^\beta} \left| \sum_{\alpha \in I(Q_2^\beta)} (u^\alpha - u^\beta) D\varphi^\alpha \right|^p dx \\ &\le N^{p-1} M^p \sum_{\alpha \in I(Q_2^\beta)} \int_{Q_2^\alpha \cap Q_2^\beta} |u^\alpha - u^\beta|^p \, dx. \end{aligned}$$

Since $u^\alpha - u^\beta = 0$ a.e. in $Q_2^\alpha \cap Q_2^\beta \cap E$, the Poincaré inequality on $Q_2^\alpha \cap Q_2^\beta$ (see [17, theorem 4.4.2]) yields

$$\int_{Q_2^\alpha \cap Q_2^\beta} |u^\alpha - u^\beta|^p \, dx \le c \int_{Q_2^\alpha \cap Q_2^\beta} |Du^\alpha - Du^\beta|^p \, dx$$

for a suitable constant c depending only on n, p, E, which together with (2.17) implies that

$$\int_{Q_2^\beta} \left| \sum_{\alpha \in I(Q_2^\beta)} u^\alpha D\varphi^\alpha \right|^p dx \le c(n, p, E, M, N) c_2 \int_{Q_{2k}^\beta \cap E} |Du|^p \, dx, \tag{2.27}$$

for every $u \in W^{1,p}(\Omega \cap E)$ and for every $\beta \in I(\Omega')$. From (2.25), (2.26), and (2.27) we get finally

$$\int_{Q_2^\beta} |D(Lu)|^p \, dx \le c(n, p, E) \int_{Q_{2k}^\beta \cap E} |Du|^p \, dx.$$

Now, to conclude the proof of (2.14) it is enough to sum up over $\beta \in I(\Omega')$ in the preceding inequality, and the conclusion follows as in (2.24). ■

Proof of theorem 2.1. Let $\varepsilon > 0$, $k_0 = 4\sqrt{n}k$, where k is the integer given by lemma 2.3. By means of lemma 2.7 we can construct a linear operator $L_\varepsilon : W^{1,p}(\Omega \cap E_\varepsilon) \to W^{1,p}(\Omega(\varepsilon k_0/2))$ such that

$$L_\varepsilon u = u \qquad \text{a.e. in } \Omega(\varepsilon k_0/2) \cap E_\varepsilon, \tag{2.28}$$

$$\int_{\Omega(\varepsilon k_0/2)} |L_\varepsilon u|^p \,\mathrm{d}x \le k_1 \int_{\Omega \cap E_\varepsilon} |u|^p \,\mathrm{d}x, \tag{2.29}$$

$$\int_{\Omega(\varepsilon k_0/2)} |D(L_\varepsilon u)|^p \,\mathrm{d}x \le k_2 \int_{\Omega \cap E_\varepsilon} |Du|^p \,\mathrm{d}x, \tag{2.30}$$

for every $u \in W^{1,p}(\Omega \cap E_\varepsilon)$. In fact, since $u \circ \pi_\varepsilon \in W^{1,p}((1/\varepsilon)\Omega \cap E)$ for every $u \in W^{1,p}(\Omega \cap E_\varepsilon)$, and $\operatorname{dist}((1/\varepsilon)\Omega(\varepsilon k_0/2), \partial((1/\varepsilon)\Omega)) > k_0/2 = 2\sqrt{n}k$, lemma 2.7 ensures the existence of a linear operator $L: W^{1,p}((1/\varepsilon)\Omega \cap E) \to W^{1,p}((1/\varepsilon)\Omega(\varepsilon k_0/2))$, such that $Lv = v$ a.e. in $(1/\varepsilon)\Omega(\varepsilon k_0/2) \cap E$, for every $v \in W^{1,p}((1/\varepsilon)\Omega \cap E)$. Moreover, the following estimates hold

$$\int_{(1/\varepsilon)\Omega(\varepsilon k_0/2)} |Lv|^p \,\mathrm{d}x \le k_1 \int_{(1/\varepsilon)\Omega \cap E} |v|^p \,\mathrm{d}x,$$

$$\int_{(1/\varepsilon)\Omega(\varepsilon k_0/2)} |D(Lv)|^p \,\mathrm{d}x \le k_2 \int_{(1/\varepsilon)\Omega \cap E} |Dv|^p \,\mathrm{d}x,$$

for every $v \in W^{1,p}((1/\varepsilon)\Omega \cap E)$, where k_1, k_2 are the constants given in lemma 2.7 and, in particular, are independent of ε.

Now, let us set $L_\varepsilon u = (L(u \circ \pi_\varepsilon)) \circ \pi_{1/\varepsilon}$. It is clear that $L_\varepsilon u \in W^{1,p}(\Omega(\varepsilon k_0/2))$ and that it satisfies (2.28), ..., (2.30).

To complete the proof of theorem 2.1, we have to construct an extension operator $T_\varepsilon : W^{1,p}(\Omega \cap E_\varepsilon) \to W^{1,p}_{\mathrm{loc}}(\Omega)$. To this aim we choose a locally finite open cover $(A_i)_{i \in \mathbf{N}}$ of Ω such that $A_0 = \Omega(\varepsilon k_0/2)$, $A_i \subset\subset \Omega$, $A_i \cap \overline{\Omega(\varepsilon k_0)} = \varnothing$ for every $i \ne 0$. Let $(\varphi_i)_{i \in \mathbf{N}}$ be a partition of unity associated to the sequence (A_i), i.e. a sequence of functions $\varphi_i \in \mathcal{C}_0^\infty(\mathbf{R}^n)$, with $\operatorname{spt} \varphi_i \subset\subset A_i$, and $\sum_{i=0}^\infty \varphi_i(x) = 1$ for every $x \in \Omega$. In particular we have $\varphi_0(x) \equiv 1$ in $\Omega(\varepsilon k_0)$.

Now, for every $\varepsilon > 0$, $i \in \mathbf{N}\setminus\{0\}$, by lemma 2.5 applied to $B = \Omega \cap E_\varepsilon$ and $\omega = A_i$, there exists a linear and continuous operator $L_{\varepsilon i}: W^{1,p}(\Omega \cap E_\varepsilon) \to W^{1,p}(A_i)$ such that $L_{\varepsilon i} u = u$ a.e. in $A_i \cap E_\varepsilon$, and for which estimates of the type (2.5), (2.6) hold. For $i = 0$ we choose $L_{\varepsilon 0} = L_\varepsilon$, where L_ε is the operator satisfying (2.28), ..., (2.30). Finally, for every $u \in W^{1,p}(\Omega \cap E_\varepsilon)$ we set

$$T_\varepsilon u = \sum_{i=0}^{\infty} (L_{\varepsilon i} u)\varphi_i$$

where the function $(L_{\varepsilon i} u)\varphi_i \in W_0^{1,p}(A_i)$ is extended to the whole set Ω by the constant 0. It is easy to check that $T_\varepsilon u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, that T_ε is linear and continuous from $W^{1,p}(\Omega \cap E_\varepsilon)$ into $W^{1,p}_{\mathrm{loc}}(\Omega)$, and that (2.1) is satisfied. Moreover, since $\varphi_0 \equiv 1$ in $\Omega(\varepsilon k_0)$ and $\Omega(\varepsilon k_0) \cap \operatorname{spt} \varphi_i = \varnothing$ for every $i \ge 1$, we have $T_\varepsilon u = L_\varepsilon u$ a.e. in $\Omega(\varepsilon k_0)$; hence (2.2), (2.3) follow immediately from (2.29), (2.30). ■

In this section we consider a sequence of minimum problems on perforated domains and we study the asymptotic behaviour of the corresponding minima and minimizers by means of Γ-convergence techniques (see definition 3.4 and theorem 3.5), and by using the extension operators T_ε constructed in theorem 2.1.

Given a periodic set $A \subseteq \mathbf{R}^n$, we say that a function $\varphi: A \to \mathbf{R}$ is periodic, if $\varphi(x + e_i) = \varphi(x)$ for every $x \in A$ and for every $i = 1, 2, \ldots, n$, where (e_i) is the canonical basis of $\mathbf{R}^n$.

Let us fix now a periodic, connected, open set $E \subseteq \mathbf{R}^n$, with Lipschitz boundary, a real number p, with $1 < p < +\infty$, and a function $f: E \times \mathbf{R}^n \to [0, +\infty[$ satisfying the following conditions:

$$f(\cdot, \xi) \text{ is measurable and periodic, for every } \xi \in \mathbf{R}^n; \tag{3.1}$$

$$f(x, \cdot) \text{ is strictly convex, for a.e. } x \in E; \tag{3.2}$$

$$\lambda_1 |\xi|^p \leq f(x, \xi) \leq \lambda_2 (1 + |\xi|^p) \text{ for a.e. } x \in E, \text{ for every } \xi \in \mathbf{R}^n; \tag{3.3}$$

where $0 < \lambda_1 \leq \lambda_2$ are fixed constants. Moreover let $\Omega \subseteq \mathbf{R}^n$ be a bounded open set and $(g_\varepsilon)_{\varepsilon > 0}$ be a sequence of functions in $L^p(\Omega)$ such that

$$g_\varepsilon \to g_0, \qquad \text{strongly in } L^p(\Omega), \tag{3.4}$$

as $\varepsilon \to 0^+$. For every $\varepsilon > 0$ we consider the minimum problem

$$m_\varepsilon = \min_{u \in W^{1,p}(\Omega \cap E_\varepsilon)} \left[\int_{\Omega \cap E_\varepsilon} f\left(\frac{x}{\varepsilon}, Du\right) \mathrm{d}x + \int_{\Omega \cap E_\varepsilon} |g_\varepsilon - u|^p \, \mathrm{d}x \right], \tag{3.5}$$

that, under our assumptions, has always one and only one solution $u_\varepsilon \in W^{1,p}(\Omega \cap E_\varepsilon)$. The main result of this section is the following theorem.

THEOREM 3.1. Assume that conditions (3.1), ..., (3.4) hold and let $u_\varepsilon \in W^{1,p}(\Omega \cap E_\varepsilon)$ be the unique solution to problem (3.5). Moreover let $T_\varepsilon: W^{1,p}(\Omega \cap E_\varepsilon) \to W^{1,p}_{\mathrm{loc}}(\Omega)$ be the extension operator introduced in theorem 2.1. Then the sequence $(T_\varepsilon u_\varepsilon)_\varepsilon$ converges strongly in $L^p_{\mathrm{loc}}(\Omega)$ (and weakly in $W^{1,p}_{\mathrm{loc}}(\Omega)$) to the unique solution $u \in W^{1,p}(\Omega)$ to the problem

$$m = \min_{u \in W^{1,p}(\Omega)} \left[\int_\Omega f_0(Du) \, \mathrm{d}x + |Q \cap E| \int_\Omega |g_0 - u|^p \, \mathrm{d}x \right], \tag{3.6}$$

where $f_0: \mathbf{R}^n \to \mathbf{R}$ is given by

$$f_0(\xi) = \inf \left\{ \int_{Q \cap E} f(x, Du) \, \mathrm{d}x : u - \xi \cdot x \in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^n),\ u - \xi \cdot x \text{ periodic} \right\}, \tag{3.7}$$

for every $\xi \in \mathbf{R}^n$. Moreover (m_ε) converges to m, as $\varepsilon \to 0^+$.

Remark 3.2. If $f(x, \cdot)$ is differentiable for a.e. $x \in \mathbf{R}^n$, the minimum point $u_\varepsilon \in W^{1,p}(\Omega \cap E_\varepsilon)$ of problem (3.5) is characterized as the unique solution of the associated Euler equation

$$\begin{cases} -\mathrm{div}\left(D_\xi f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right)\right) + p|u_\varepsilon - g_\varepsilon|^{p-2}(u_\varepsilon - g_\varepsilon) = 0 & \text{in } \Omega \cap E_\varepsilon, \\ D_\xi f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) \cdot n_\varepsilon = 0 & \text{on } \partial(\Omega \cap E_\varepsilon), \end{cases} \tag{3.8}$$

where n_ε is the outward unit normal to $\partial(\Omega \cap E_\varepsilon)$. In particular, if $p = 2$, $f(x, \xi) = |\xi|^2 \mathbf{1}_E(x)$, where $\mathbf{1}_E$ is 1 on E and 0 outside, and if $g_\varepsilon = g \in L^2(\Omega)$, then (3.8) coincides with the Laplace equation (1.1) with Neumann boundary conditions.

Remark 3.3. If, in the hypotheses of theorem 3.1, the strict convexity assumption (3.2) is weakened into simple convexity, we can still obtain the following results.

For every $\varepsilon > 0$ let $u_\varepsilon \in W^{1,p}(\Omega \cap E_\varepsilon)$ be a minimum point of problem (3.5) (which, in general, does not have a unique solution). Then, for every open set $\Omega' \subset\subset \Omega$ the sequence $\|T_\varepsilon u_\varepsilon\|_{W^{1,p}(\Omega')}$ is bounded independently of ε, for $\varepsilon < (1/k_0)\,\mathrm{dist}(\Omega', \partial\Omega)$, and hence there exists a subsequence $(T_{\varepsilon_h} u_{\varepsilon_h})_h$ that converges strongly in $L^p_{\mathrm{loc}}(\Omega)$ (and weakly in $W^{1,p}_{\mathrm{loc}}(\Omega)$). Moreover, every subsequence with these properties actually converges to a solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of (3.6). Finally, the whole sequence $(m_\varepsilon)_\varepsilon$ of the minimum values still converges to m, as $\varepsilon \to 0^+$. These results can be obtained by slightly modifying the proof of theorem 3.1.

For the reader's convenience, we include hereafter the definition and main properties of Γ-convergence, on which relies the proof of theorem 3.1.

Definition 3.4 (see [12]). Let (X, τ) be a metric space and F_h, F_0 functionals from X into $\bar{\mathbf{R}}$. We say that $(F_\varepsilon)_\varepsilon$ $\Gamma(\tau)$-converges to F_0, i.e.

$$F_0(x) = \Gamma(\tau) - \lim_{\varepsilon \to 0^+} F_\varepsilon(x)$$

for every $x \in X$, if for every $x \in X$ the following conditions are fulfilled

$$\forall\, \varepsilon_h \to 0^+, \qquad \forall\, x_h \to x \qquad F(x) \le \liminf_{h \to \infty} F_{\varepsilon_h}(x_h), \tag{3.9}$$

$$\forall\, \varepsilon_h \to 0^+ \qquad \exists\, x_h \to x \qquad \text{such that } F(x) = \lim_h F_{\varepsilon_h}(x_h). \tag{3.10}$$

THEOREM 3.5 (see [12, corollary 2.4]). Let (X, τ) be a metric space and F_ε, F_0 functionals from X into $\bar{\mathbf{R}}$, and let $x_\varepsilon \in X$ be a minimizer for F_ε, i.e.

$$F_\varepsilon(u_\varepsilon) = \min\{F_\varepsilon(x) : x \in X\}.$$

If (F_ε) $\Gamma(\tau)$-converges to F_0 and (x_ε) converges to $x_0 \in X$, then

$$F_0(x_0) = \lim_{\varepsilon \to 0^+} F_\varepsilon(x_\varepsilon) = \min\{F_0(x) : x \in X\}.$$

In order to prove theorem 3.1 by means of theorem 3.5, for every $\varepsilon > 0$ we introduce the functionals F_ε, $G_\varepsilon : L^p_{\mathrm{loc}}(\Omega) \times \mathfrak{A} \to [0, +\infty]$ defined as

$$F_\varepsilon(u, A) = \begin{cases} \displaystyle\int_{A \cap E_\varepsilon} f\left(\frac{x}{\varepsilon}, Du\right) \mathrm{d}x & \text{if } u_{|A \cap E_\varepsilon} \in W^{1,p}(A \cap E_\varepsilon), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.11}$$

$$G_\varepsilon(u, A) = F_\varepsilon(u, A) + \int_{A \cap E_\varepsilon} |g_\varepsilon - u|^p \, \mathrm{d}x \qquad \text{if } u \in L^p_{\mathrm{loc}}(\Omega), \tag{3.12}$$

for every $A \in \mathcal{A}$, where $\mathcal{A}$ denotes the family of all open subsets of Ω. With this notation we have

$$m_\varepsilon = \min_{u \in L^p_{\mathrm{loc}}(\Omega)} G_\varepsilon(u, \Omega). \tag{3.13}$$

We remark that, in general, problem (3.13) has no uniqueness of solutions. Moreover, if $u_\varepsilon \in W^{1,p}(\Omega \cap E_\varepsilon)$ is a minimum point of (3.5), then every L^p_{loc}-extension of u_ε to the whole set Ω (in particular $T_\varepsilon u_\varepsilon$, with T_ε given by theorem 2.1) is a solution of (3.13).

In the following, if $(H_\varepsilon)_{\varepsilon>0}$ is a family of functionals defined on $L^p_{\mathrm{loc}}(\Omega)$, by $\Gamma(L^p_{\mathrm{loc}}) - \lim_{\varepsilon\to 0^+} H_\varepsilon$ we denote the Γ-limit of H_ε in the strong topology of $L^p_{\mathrm{loc}}(\Omega)$, according to definition 3.4. By $\Gamma(L^p) - \lim_{\varepsilon\to 0^+} H_\varepsilon$ we denote the Γ-limit with respect to the topology on $L^p_{\mathrm{loc}}(\Omega)$ induced by the extended distance

$$d_{L^p}(u, v) = \left(\int_\Omega |u - v|^p \,\mathrm{d}x\right)^{1/p}.$$

The behaviour of F_ε, G_ε with respect to Γ-convergence is given by the following two propositions.

PROPOSITION 3.6. Assume that (3.1), ..., (3.3) hold and let $f_0 \colon \mathbf{R}^n \to [0, +\infty[$ be the function defined by (3.7). Then

$$\Gamma(L^p) - \lim_{\varepsilon\to 0^+} F_\varepsilon(u, A) = F_0(u, A), \tag{3.14}$$

for every $u \in L^p_{\mathrm{loc}}(\Omega)$ and for every $A \in \mathcal{A}$, where $F_0 \colon L^p_{\mathrm{loc}}(\Omega) \times \mathcal{A} \to [0, +\infty]$ is given by

$$F_0(u, A) = \begin{cases} \displaystyle\int_A f_0(Du)\,\mathrm{d}x & \text{if } u \in W^{1,p}(A), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.15}$$

Moreover, f_0 is strictly convex and satisfies the following growth conditions

$$\lambda_0|\xi|^p \le f_0(\xi) \le \lambda_2(1 + |\xi|)^p, \tag{3.16}$$

for every $\xi \in \mathbf{R}^n$, with $\lambda_0 = \lambda_1/k_2$, and k_2 given by theorem 2.1.

Proof. Let us begin by proving that

$$\Gamma(L^p) - \lim_{\varepsilon\to 0^+} F_\varepsilon(u, A) = +\infty \tag{3.17}$$

whenever $u \notin W^{1,p}(A)$. Let us fix $A \in \mathcal{A}$. According to definition 3.4, to prove (3.17) it is enough to show that if $\varepsilon_h \to 0^+$ and u_h is a sequence in $L^p_{\mathrm{loc}}(\Omega)$ converging to $u \in L^p_{\mathrm{loc}}(\Omega)$ in the strong topology of $L^p(\Omega)$, such that

$$\liminf_{h\to\infty} F_{\varepsilon_h}(u_h, A) < +\infty, \tag{3.18}$$

then $u \in W^{1,p}(A)$. By (3.3) and (3.18) it follows that

$$\liminf_{h\to\infty} \int_{A\cap E_{\varepsilon_h}} |Du_h|^p \,\mathrm{d}x \le \frac{1}{\lambda_1} \liminf_{h\to\infty} F_{\varepsilon_h}(u_h, A),$$

and hence that, up to a subsequence

$$\int_{A \cap E_{\varepsilon_h}} |Du_h|^p \, dx \leq c,$$

with c independent of h. For every $h \in \mathbf{N}$ let us consider the function $T_{\varepsilon_h} u_h \in W^{1,p}_{\text{loc}}(A)$, where $T_\varepsilon \colon W^{1,p}(A \cap E_\varepsilon) \to W^{1,p}_{\text{loc}}(A)$ is the operator given by theorem 2.1 (with Ω replaced by A). By (2.1), ..., (2.3) we have that $T_{\varepsilon_h} u_h = u_h$ a.e. in $A \cap E_{\varepsilon_h}$, and

$$\| T_{\varepsilon_h} u_h \|_{W^{1,p}(A')} \leq c,$$

for every open set $A' \subseteq A$, with $\operatorname{dist}(A', \partial A) > \varepsilon_h k_0$, where the constant c is independent of A' and h.

If A' has Lipschitz boundary, by Rellich theorem, $(T_{\varepsilon_h} u_h)$ converges, up to a subsequence, to a function $v \in W^{1,p}(A')$ strongly in $L^p(A')$ and weakly in $W^{1,p}(A')$.

If we now consider an invading sequence of smooth open subsets of Ω, by a diagonal process we can extract a subsequence of $(T_{\varepsilon_h} u_h)$ (still denoted by $T_{\varepsilon_h} u_h$) that converges to a function $v \in W^{1,p}_{\text{loc}}(A)$, strongly in $L^p_{\text{loc}}(A)$ and weakly in $W^{1,p}_{\text{loc}}(A)$. It is easy to show that $v = u$ a.e. in A. In fact, for every open set $A' \subset\subset A$ we have

$$\int_{A' \cap E_{\varepsilon_h}} |u - v|^p \, dx \leq c \int_{A' \cap E_{\varepsilon_h}} |u - u_h|^p \, dx + c \int_{A' \cap E_{\varepsilon_h}} |T_{\varepsilon_h} u_h - v|^p \, dx,$$

from which, by taking the limit as h tends to ∞, it follows

$$|Q \cap E| \int_{A'} |u - v|^p \, dx \leq 0.$$

Since this holds for every $A' \subset\subset A$, we have proven that $u = v$ a.e. in A.

Now, since

$$\|u\|_{W^{1,p}(A')} = \|v\|_{W^{1,p}(A')} \leq \liminf_{h \to \infty} \| T_{\varepsilon_h} u_h \|_{W^{1,p}(A')} \leq c$$

for every $A' \subset\subset A$, it follows that $u \in W^{1,p}(A)$.

Now (3.14) and (3.15) can be obtained as a direct consequence of [5, theorem 4], while the strict convexity of f_0 follows easily from its definition (3.7).

To conclude the proof of the theorem it remains to prove (3.16). To this aim, let us fix $A \in \mathcal{Q}$, $\xi \in \mathbf{R}^n$ and $u(x) = \xi \cdot x$ for a.e. $x \in \Omega$. By (3.9) with $u_{\varepsilon_h} = u$ and (3.3), we have

$$\int_A f_0(\xi) \, dx \leq \lambda_2 \int_A (1 + |\xi|^p) \, dx,$$

that is

$$f_0(\xi) \leq \lambda_2 (1 + |\xi|^p). \tag{3.19}$$

Moreover, by (3.10) for every $\varepsilon_h \to 0^+$ there exists a sequence (u_h) in $L^p_{\text{loc}}(\Omega)$ that converges to u in $L^p(\Omega)$, such that

$$\int_A f_0(\xi) \, dx = \lim_{h \to \infty} F_{\varepsilon_h}(u_h, A).$$

By arguing as in the proof of (3.14), we obtain in particular that, since $(T_{\varepsilon_h} u_h)$ converges to u also weakly in $W^{1,p}_{loc}(\Omega)$, by (2.3) and (3.3)

$$\int_{A'} |\xi|^p \, dx = \int_{A'} |Du|^p \, dx \leq \liminf_{h \to \infty} \int_{A'} |D(T_{\varepsilon_h} u_h)|^p \, dx$$
$$\leq k_2 \liminf_{h \to \infty} \int_{A \cap E_{\varepsilon_h}} |Du_h|^p \, dx \leq \frac{k_2}{\lambda_2} \int_A f_0(\xi) \, dx,$$

for every open set $A' \subset\subset A$. This implies immediately

$$\frac{\lambda_1}{k_2} |\xi|^p \leq f_0(\xi),$$

that together with (3.19) gives (3.16) and concludes the proof of the theorem. ■

Let $G_0 : L^p_{loc}(\Omega) \times \mathcal{A} \to [0, +\infty]$ be the functional defined by

$$G_0(u, A) = F_0(u, A) + |Q \cap E| \int_A |g_0 - u|^p \, dx, \tag{3.20}$$

where g_0 is the limit of (g_ε) given by (3.4).

Proposition 3.7. Assume that (3.1), ..., (3.4) hold. Then

$$\Gamma(L^p_{loc}) - \lim_{\varepsilon \to 0^+} G_\varepsilon(u, \Omega) = G_0(u, \Omega) \tag{3.21}$$

for every $u \in L^p_{loc}(\Omega)$.

Proof. Since

$$\int_{A \cap E_{\varepsilon_h}} |g_{\varepsilon_h} - u_h|^p \, dx \to |Q \cap E| \int_A |g_0 - u|^p \, dx$$

for every $\varepsilon_h \to 0^+$ and for every (u_h) in $L^p_{loc}(\Omega)$ converging to $u \in L^p_{loc}(\Omega)$, strongly in $L^p(\Omega)$, from proposition 3.6 we obtain that

$$\Gamma(L^p) - \lim_{\varepsilon \to 0^+} G_\varepsilon(u, A) = G_0(u, A) \tag{3.22}$$

for every $A \in \mathcal{A}$ and for every $u \in L^p_{loc}(\Omega)$. To prove (3.21), we observe that, by (3.22), condition (3.10) of definition 3.4 holds trivially for G_ε, with respect to the topology of $L^p_{loc}(\Omega)$. To prove (3.9), let us fix (ε_h) tending to 0^+, u, $u_h \in L^p_{loc}(\Omega)$ with (u_h) tending to u in $L^p_{loc}(\Omega)$. Since (u_h) converges to u in $L^p(\Omega')$ for every open set $\Omega' \subset\subset \Omega$, by (3.22) we have

$$G_0(u, \Omega') \leq \liminf_{h \to \infty} G_{\varepsilon_h}(u_h, \Omega') \leq \liminf_{h \to \infty} G_{\varepsilon_h}(u_h, \Omega).$$

By taking the supremum over $\Omega' \subset\subset \Omega$, the conclusion follows directly. ■

Proof of theorem 3.1. Let $u_\varepsilon \in W^{1,p}(\Omega \cap E_\varepsilon)$ be the unique solution to problem (3.5) and let $T_\varepsilon : W^{1,p}(\Omega \cap E_\varepsilon) \to W^{1,p}_{loc}(\Omega)$ be the extension operator introduced in theorem 2.1. Our first aim is to prove that $(T_\varepsilon u_\varepsilon)$ converges strongly in L^p_{loc} to a function $u \in L^p_{loc}(\Omega)$. By (3.3), ..., (3.5) we have that $(\|u_\varepsilon\|_{W^{1,p}(\Omega \cap E_\varepsilon)})$ is bounded independently of ε, and hence, by (2.2), (2.3)

$$\|T_\varepsilon u_\varepsilon\|_{W^{1,p}(A)} \leq c$$

for every open set $A \subset\subset \Omega$ such that $\operatorname{dist}(A, \partial\Omega) > \varepsilon k_0$, with a constant c independent of ε and A. If A has Lipschitz boundary, by Rellich's theorem there exists a subsequence $(T_{\varepsilon_h} u_{\varepsilon_h})$ that converges to a function $u \in W^{1,p}(A)$ strongly in $L^p(A)$. If we now consider an invading sequence of open subsets of Ω, using a diagonal process we extract a subsequence of $(T_{\varepsilon_h} u_{\varepsilon_h})$ (still denoted by $T_{\varepsilon_h} u_{\varepsilon_h}$) that converges to a function $u \in W^{1,p}_{\text{loc}}(\Omega)$, strongly in $L^p_{\text{loc}}(\Omega)$. Now, by theorem 3.5 and proposition 3.7 all the assertions of theorem 3.1 follow for the subsequence ε_h. More precisely, u turns out to be a solution to problem (3.6), and $m_{\varepsilon_h} \to m$ as $h \to \infty$. Since problem (3.6) has a unique solution by the strict convexity of f_0 (see proposition 3.6), the whole sequence $(T_\varepsilon u_\varepsilon)$ tends to u strongly in $L^p_{\text{loc}}(\Omega)$, and also $m_\varepsilon \to m$ as $\varepsilon \to 0^+$. The proof of theorem 3.1 is then complete. ∎

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