NEW LOWER SEMICONTINUITY RESULTS FOR POLYCONVEX INTEGRALS

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Abstract

We study integral functionals of the form $F(u,\Omega) = \int_\Omega f(\nabla u)\,dx$, defined for $u \in C^1(\Omega;\mathbf{R}^k)$, $\Omega \subseteq \mathbf{R}^n$. The function f is assumed to be polyconvex and to satisfy the inequality $f(A) \geq c_0|\mathcal{M}(A)|$ for a suitable constant $c_0 > 0$, where $\mathcal{M}(A)$ is the n-vector whose components are the determinants of all minors of the $k \times n$ matrix A. We prove that F is lower semicontinuous on $C^1(\Omega;\mathbf{R}^k)$ with respect to the strong topology of $L^1(\Omega;\mathbf{R}^k)$. Then we consider the relaxed functional $\mathcal{F}$, defined as the greatest lower semicontinuous functional on $L^1(\Omega;\mathbf{R}^k)$ which is less than or equal to F on $C^1(\Omega;\mathbf{R}^k)$. For every $u \in BV(\Omega;\mathbf{R}^k)$ we prove that $\mathcal{F}(u,\Omega) \geq \int_\Omega f(\nabla u)\,dx + c_0|D^s u|(\Omega)$, where $Du = \nabla u\,dx + D^s u$ is the Lebesgue decomposition of the Radon measure Du. Moreover, under suitable growth conditions on f, we show that $\mathcal{F}(u,\Omega) = \int_\Omega f(\nabla u)\,dx$ for every $u \in W^{1,p}(\Omega;\mathbf{R}^k)$, with $p \geq \min\{n,k\}$. We prove also that the functional $\mathcal{F}(u,\Omega)$ can not be represented by an integral for an arbitrary function $u \in BV_{\rm loc}(\mathbf{R}^n;\mathbf{R}^k)$. In fact, two examples show that, in general, the set function $\Omega \mapsto \mathcal{F}(u,\Omega)$ is not subadditive when $u \in BV_{\rm loc}(\mathbf{R}^n;\mathbf{R}^k)$, even if $u \in W^{1,p}_{\rm loc}(\mathbf{R}^n;\mathbf{R}^k)$ for every $p < \min\{n,k\}$. Finally, we examine in detail the properties of the functions $u \in BV(\Omega;\mathbf{R}^k)$ such that $\mathcal{F}(u,\Omega) = \int_\Omega f(\nabla u)\,dx$, particularly in the model case $f(A) = |\mathcal{M}(A)|$.

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Introduction

The aim of this paper is to study some lower semicontinuity properties of the functional

$$F(u) = \int_\Omega f(\nabla u(x))\, dx\,, \tag{0.1}$$

where Ω is a bounded open subset of $\mathbf{R}^n$, $u\colon\Omega \to \mathbf{R}^k$ is a vector valued function, ∇u denotes the Jacobian matrix of u, and f is a non-negative function defined in the space $\mathbf{M}^{k\times n}$ of all $k\times n$ matrices. We shall always assume that f is polyconvex in the sense of Ball (see [1] and [5]), i.e., there exists a convex function g, defined in the space Ξ of all n-vectors of $\mathbf{R}^n\times\mathbf{R}^k$, such that $f(A) = g(\mathcal{M}(A))$ for every $A \in \mathbf{M}^{k\times n}$, where $\mathcal{M}(A)$ denotes the n-vector whose components are the determinants of all minors of the matrix A (with the appropriate sign, see (1.1) and (1.2)), including the minor of order 0, whose determinant, by convention, is set to be equal to 1.

Our model example, in this investigation, is the functional

$$A(u) = \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\, dx\,, \tag{0.2}$$

which represents, when $u \in C^1(\Omega;\mathbf{R}^k)$, the n-dimensional area of the graph of u. More generally, we shall consider polyconvex integrals of the form (0.1) such that

$$f(A) \ge c_0|\mathcal{M}(A)| \qquad \forall A \in \mathbf{M}^{k\times n}\,, \tag{0.3}$$

for a suitable constant $c_0 > 0$.

The classical lower semicontinuity theorems with respect to the weak topology of $W^{1,p}(\Omega;\mathbf{R}^k)$, due to Morrey (see [21], [22], [5]), can not be applied to the study of minimum problems involving the functional F. Indeed, the lower bound (0.3) guarantees that $\big(\mathcal{M}(\nabla u_h)\big)$ is bounded in $L^1(\Omega;\Xi)$ along every minimizing sequence (u_h), and this implies only that a subsequence of (u_h) converges in $L^1(\Omega;\mathbf{R}^k)$. Therefore, in this paper we study the lower semicontinuity properties of F with respect to the strong convergence in $L^1(\Omega;\mathbf{R}^k)$.

A slight modification of Counterexample 7.4 of [2] shows that the area functional A is not lower semicontinuous on $W^{1,1}(\Omega;\mathbf{R}^k)$ with respect to the strong topology of $L^1(\Omega;\mathbf{R}^k)$. More precisely, it is possible to construct a sequence (u_h), converging to a smooth function u in the weak topology of $W^{1,p}(\Omega;\mathbf{R}^k)$ for every $p < \min\{n,k\}$, such that $A(u) > \lim_{h\to\infty} A(u_h)$. The main feature of this counterexample is the use of functions $u_h \in W^{1,p}(\Omega;\mathbf{R}^k)$ for which $A(u_h)$ loses its geometrical meaning.

To overcome this difficulty, we propose a different approach. We keep the definition (0.2) of $A(u)$ only when $u \in C^1(\Omega;\mathbf{R}^k)$, so that $A(u)$ is the n-dimensional area of the graph of u, and we extend, formally, the definition of A to every $u \in L^1(\Omega;\mathbf{R}^k)$ by setting $A(u) = +\infty$ for $u \notin C^1(\Omega;\mathbf{R}^k)$. Then we consider the relaxed functional $\mathcal{A}\colon L^1(\Omega;\mathbf{R}^k) \to [0,+\infty]$, defined as the greatest lower semicontinuous functional on $L^1(\Omega;\mathbf{R}^k)$ which is less than or equal to A (for more detailed information on the relaxation method in the calculus of variations we refer to [8], [4], [5]). Similarly, we keep the definition (0.1) of $F(u)$ only when $u \in C^1(\Omega;\mathbf{R}^k)$, and we extend the definition of F to every $u \in L^1(\Omega;\mathbf{R}^k)$ by setting $F(u) = +\infty$ for $u \notin C^1(\Omega;\mathbf{R}^k)$. The corresponding relaxed functional will be denoted by $\mathcal{F}$.

We prove (Theorems 2.4 and 2.5) that the functionals A and F are lower semicontinuous on $C^1(\Omega;\mathbf{R}^k) \cap L^1(\Omega;\mathbf{R}^k)$ with respect to the strong topology of $L^1(\Omega;\mathbf{R}^k)$. This implies that $\mathcal{A}(u) = A(u)$ and $\mathcal{F}(u) = F(u)$ for every $u \in C^1(\Omega;\mathbf{R}^k) \cap L^1(\Omega;\mathbf{R}^k)$.

The main difficulty of our problem lies in the fact that we are considering, in particular, functionals with linear growth with respect to the n-vector of the minors $\mathcal{M}(A)$. The situation is completely different, if we suppose

$$f(A) \geq c_0|\mathcal{M}(A)|^p \qquad \forall A \in \mathbf{M}^{k\times n}$$

for suitable constants $c_0 > 0$ and $p > 1$. Indeed, if (u_h) is a sequence in $C^1(\Omega;\mathbf{R}^k)$, converging in $L^1(\Omega;\mathbf{R}^k)$ to a function $u \in C^1(\Omega;\mathbf{R}^k)$, and $\big(\mathcal{M}(\nabla u_h)\big)$ is bounded in $L^p(\Omega;\Xi)$, then $u \in W^{1,p}(\Omega;\mathbf{R}^k)$, $\mathcal{M}(\nabla u) \in L^p(\Omega;\Xi)$, and $\big(\mathcal{M}(\nabla u_h)\big)$ converges to $\mathcal{M}(\nabla u)$ weakly in $L^p(\Omega;\Xi)$ (see [10] and [18]), so that the lower semicontinuity of F in $L^1(\Omega;\mathbf{R}^k)$ follows from the lower semicontinuity of the functional $w \mapsto \int_\Omega g(w)\,dx$ in the weak topology of $L^p(\Omega;\Xi)$. This idea can not be applied when $p = 1$, because easy counterexamples show that the boundedness of $\big(\mathcal{M}(\nabla u_h)\big)$ in $L^1(\Omega;\Xi)$ does not imply the convergence of $\big(\mathcal{M}(\nabla u_h)\big)$ to $\mathcal{M}(\nabla u)$ in the weak sense of distributions.

Our lower semicontinuity results on $C^1(\Omega;\mathbf{R}^k)$ show that $\mathcal{A}$ and $\mathcal{F}$ are not only the greatest lower semicontinuous functionals less than or equal to A and F, but they are also extensions of the functionals A and F outside $C^1(\Omega;\mathbf{R}^k)$. Following a tradition that goes back to Lebesgue [16] for the area functional, and that was adopted by Serrin [24] in the study of general variational integrals, we consider $\mathcal{A}(u)$ and $\mathcal{F}(u)$ as the only reasonable variational definition of the functionals A and F when $u \notin C^1(\Omega;\mathbf{R}^k)$.

The problem is now to describe the behaviour of the functionals $\mathcal{A}(u)$ and $\mathcal{F}(u)$ when $u \notin C^1(\Omega;\mathbf{R}^k)$, and, possibly, to give an explicit integral representation of these functionals, at least for a wide class of functions u. It is easy to see that, if $\mathcal{A}(u) < +\infty$ or $\mathcal{F}(u) < +\infty$, then $u \in BV(\Omega;\mathbf{R}^k)$ (Remark 2.6), thus we can restrict our investigation to the space $BV(\Omega;\mathbf{R}^k)$. We prove (Theorems 2.7 and 2.8) the following estimates from below for every $u \in BV(\Omega;\mathbf{R}^k)$:

$$\mathcal{A}(u) \geq \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\,dx + |D^s u|(\Omega)\,,$$

$$\mathcal{F}(u) \geq \int_\Omega f(\nabla u(x))\,dx + c_0|D^s u|(\Omega)\,,$$

where c_0 is the constant in (0.3), and $Du = \nabla u\,dx + D^s u$ is the Lebesgue decomposition of the $\mathbf{M}^{k\times n}$-valued Radon measure Du, with $\nabla u \in L^1(\Omega;\mathbf{M}^{k\times n})$ and $D^s u$ singular with respect to the Lebesgue measure. This leads to the equality

$$\mathcal{A}(u) = \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\,dx \tag{0.4}$$

when $u \in W^{1,p}(\Omega;\mathbf{R}^k)$ for some $p \geq \min\{n,k\}$ (Theorem 2.9), and shows that the functional $u \mapsto \int_\Omega |\mathcal{M}(\nabla u)|\,dx$ is lower semicontinuous on $W^{1,p}(\Omega;\mathbf{R}^k)$ with respect to the strong topology of $L^1(\Omega;\mathbf{R}^k)$, when $p \geq \min\{n,k\}$ (Corollary 2.10). Similar properties hold for $\mathcal{F}$, if $f(A) \leq c_1\big(|A|^p + 1\big)$ (Theorem 2.12 and Corollary 2.13).

A difficult open problem (see [7]) concerns the dependence on Ω of the functionals $\mathcal{A}(u)$ and $\mathcal{F}(u)$, which will be now denoted by $\mathcal{A}(u,\Omega)$ and $\mathcal{F}(u,\Omega)$. We prove, with an explicit example in the case $n = k = 2$, that the set function $\Omega \mapsto \mathcal{A}(u,\Omega)$ is not subadditive when u is an arbitrary function of $BV_{\rm loc}(\mathbf{R}^n;\mathbf{R}^k)$ (Theorem 3.1). Therefore, while in the scalar case $k = 1$ the functional $\mathcal{A}(u,\Omega)$ can be represented by an integral over Ω, involving a function and a measure, both depending on u (see [15]), this is no longer true in the vector case $k \geq 2$.

When $n > 2$ and $k > 2$ it is possible to give a counterexample to the subadditivity of $\Omega \mapsto \mathcal{A}(u,\Omega)$ even when $u \in W^{1,p}_{\rm loc}(\mathbf{R}^n;\mathbf{R}^k)$ for all $p < \min\{n,k\}$ (Theorem 4.1). The same example shows also that (0.4) does not hold, in general, when $p < \min\{n,k\}$ (Lemma 4.2).

Finally, we prove (Proposition 5.8) that, if the polyconvex function f satisfies the inequalities $c_0|\mathcal{M}(A)| \leq f(A) \leq c_1\big(|\mathcal{M}(A)| + 1\big)$, with $0 < c_0 \leq c_1$, then for every function $u \in BV(\Omega;\mathbf{R}^k)$ the condition

$$\mathcal{A}(u,\Omega) = \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\,dx < +\infty \tag{0.5}$$

implies

$$\mathcal{F}(u,\Omega) = \int_\Omega f(\nabla u(x))\,dx\,.$$

Moreover, we prove that (0.5) is satisfied if and only if $u \in W^{1,1}(\Omega;\mathbf{R}^k)$, $\mathcal{M}(\nabla u) \in L^1(\Omega;\Xi)$, and there exists a sequence (u_h) in $C^1(\Omega;\mathbf{R}^k)$, converging to u in $L^1(\Omega;\mathbf{R}^k)$, such that $\big(\mathcal{M}(\nabla u_h)\big)$ converges to $\mathcal{M}(\nabla u)$ in $L^1(\Omega;\Xi)$ (Theorem 5.4). This implies that, if (0.5) holds for a set Ω, then the same equality holds for every open subset of Ω (Remark 5.7).

We conclude the paper by proving that, in the case $n=2$, if $u \in BV(\Omega_1\cup\Omega_2;\mathbf{R}^k)\cap L^\infty(\Omega_1\cup\Omega_2;\mathbf{R}^k)$ satisfies (0.5) for $\Omega=\Omega_1$ and $\Omega=\Omega_2$, then (0.5) holds for every open set Ω with $\Omega \subset\subset \Omega_1\cup\Omega_2$.

Our results about the functional $\mathcal{F}$, stated in the introduction for $F(u) = \int_\Omega f(\nabla u)\,dx$, are all valid in the case $F(u)=\int_\Omega f(x,u,\nabla u)\,dx$, under suitable assumptions on f that are provided in the following sections.

The lower semicontinuity results proved in this paper are obtained by using some fundamentals ideas and techniques of geometric measure theory introduced in the study of polyconvex integrals by Giaquinta, Modica, and Souček in [10], [11], [12]. In particular, we use the compactness of the space $\mathrm{cart}(\Omega;\mathbf{R}^k)$ of Cartesian currents introduced in [10], and the relationships between Cartesian currents and graphs discussed in the same paper.

The counterexamples to the subadditivity of $\Omega \mapsto \mathcal{A}(u,\Omega)$ are based on some ideas of De Giorgi and on some technical arguments from geometric measure theory. In the counterexample concerning Sobolev functions we use also a construction which is reminiscent of the "dipole construction" considered by Brezis, Coron, and Lieb in [3]. Finally, the properties of the pairs (u,Ω) which satisfy (0.5) are obtained by using some results about convex functions of measures proved by Reshetnyak in [23].

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1. Definitions and preliminary results

Measures. Let U be a locally compact Hausdorff space with a countable base and let X be a finite dimensional vector space, with dual X'. A Radon measure on U with values in X is a continuous linear functional on the space $C^0_c(U;X')$ of all continuous X'-valued functions with compact support in U. The space of all Radon measures on U with values in X is denoted by $\mathcal{M}(U;X)$. Every Radon measure $\mu \in \mathcal{M}(U;X)$ will be identified with the corresponding countably additive X-valued set function, still denoted by μ, defined on the family of all relatively compact Borel subsets of U, so that $\mu(f) = \int_U f\,d\mu$ for every $f \in C^0_c(U;X')$.

Given a norm $|\cdot|$ on X, the variation of a Radon measure $\mu \in \mathcal{M}(U;X)$ will be denoted by $|\mu|$. It is well known that $|\mu|$ is a positive measure defined in the family of all Borel subsets of U. If $|\mu|(U) < +\infty$, we say that the Radon measure μ is bounded. In this case the integral $\mu(f) = \int_U f\,d\mu$ is well defined for every bounded Borel function $f\colon U \to X'$.

As $\mathcal{M}(U;X)$ is the dual of $C^0_c(U;X')$, we can consider the weak* topology on $\mathcal{M}(U;X)$. By definition, a sequence (μ_h) in $\mathcal{M}(U;X)$ converges to $\mu \in \mathcal{M}(U;X)$ in the weak* topology on $\mathcal{M}(U;X)$ if $\int_U f\,d\mu_h$ converges to $\int_U f\,d\mu$ for every $f \in C^0_c(U;X')$.

We shall frequently use the following lower semicontinuity theorem due to Reshetnyak.

Theorem 1.1. *Let $g\colon U\times X \to [0,+\infty]$ be a function such that:*

(i) *g is lower semicontinuous on $U\times X$;*

(ii) *for every $x \in U$ the function $\xi \mapsto g(x,\xi)$ is convex and positively homogeneous of degree one on X.*

Let $G\colon \mathcal{M}(U;X) \to [0,+\infty]$ be the functional defined by

$$G(\mu) \;=\; \int_U g\big(x, \tfrac{d\mu}{d|\mu|}(x)\big)\, d|\mu|(x)\,,$$

where $\frac{d\mu}{d|\mu|}$ denotes the Radon-Nikodym derivative of the vector measure μ with respect to the scalar measure $|\mu|$. Then G is sequentially lower semicontinuous on $\mathcal{M}(U;X)$ with respect to the weak* convergence.

Proof. In [23] the theorem is proved under the additional assumption that g is continuous on $U\times X$. The same arguments can be adapted to the general case. For a different proof see also [4], Corollary 3.4.2. $\square$

We shall also use the following version of a continuity theorem due to Reshetnyak.

Theorem 1.2. *Let $g: U\times X \to [0,+\infty[$ be a continuous function and let G be the functional defined in Theorem 1.1. If the norm $|\cdot|$ on X comes from a scalar product, then $G(\mu) = \lim_{h\to\infty} G(\mu_h)$ for every sequence (μ_h) in $\mathcal{M}(U;X)$, converging to a bounded Radon measure μ in the weak* topology of $\mathcal{M}(U;X)$, and such that $\big(|\mu_h|(U)\big)$ converges to $|\mu|(U)$.*

Proof. See [23], Theorem 3, and [17], Appendix. □

Let V be another locally compact Hausdorff space with a countable base, let $f: U \to V$ be a continuous function, and let μ be a bounded Radon measure on U with values in X. The image measure $f_*\mu$ is the X-valued Radon measure on V defined by

$$(f_*\mu)(\varphi) = \mu(f^*\varphi) \qquad \forall \varphi \in C^0_0(V;X')\,,$$

where $f^*: C^0_c(V;X') \to C^0(U;X')$ is defined by $f^*\varphi = \varphi\circ f$.

The Lebesgue measure in $\mathbf{R}^n$ will be denoted by $\mathcal{L}^n$, while the n-dimensional Hausdorff measure (in an arbitrary metric space) will be denoted by $\mathcal{H}^n$. If μ is any measure and A is any μ-measurable set, the measure $\mu\llcorner A$ is defined by $(\mu\llcorner A)(B) = \mu(A\cap B)$ for every μ-measurable set B. We say that two μ-measurable sets A_1 and A_2 are μ-equivalent, and use the notation $A_1 \simeq A_2$ (in the sense of μ), if $|\mu|(A_1 \triangle A_2) = 0$, where $\triangle$ denotes the symmetric difference of sets. It is clear that, if $A_1 \simeq A_2$ in the sense of μ, then $\mu\llcorner A_1 = \mu\llcorner A_2$.

Functions with bounded variation. Let U be a bounded open subset of $\mathbf{R}^n$. The space $BV(U;\mathbf{R}^k)$ of $\mathbf{R}^k$-valued functions with bounded variation is defined as the set of all functions $u \in L^1(\Omega;\mathbf{R}^k)$ whose distributional gradient Du is a bounded Radon measure on U with values in the space $\mathbf{M}^{k\times n}$ of all $k\times n$ matrices. Given $u\in BV(U;\mathbf{R}^k)$, the measure Du can be written in a unique way as

$$Du(B) \;=\; \int_B \nabla u\, dx \;+\; D^s u(B)$$

for every Borel set $B \subseteq U$, where $\nabla u \in L^1(U;\mathbf{M}^{k\times n})$ and $D^s u$ is an $\mathbf{M}^{k\times n}$-valued Radon measure, which is singular with respect to the Lebesgue measure (Lebesgue-Nikodym decomposition).

On $\mathbf{M}^{k\times n}$ we shall consider the Hilbert-Schmidt norm defined by $|A|^2 = \mathrm{tr}(AA^*)$. With this choice of the norm we have

$$|Du|(B) \;=\; \int_B |\nabla u|\, dx \;+\; |D^s u|(B)$$

for every Borel set $B \subseteq U$.

Currents. Given an open subset U of $\mathbf{R}^m$, the space of all n-forms with C^∞ coefficients and with compact support in U will be denoted by $\mathcal{D}^n(U)$. The space $\mathcal{D}_n(U)$ of all n-currents on U is defined as the dual of $\mathcal{D}^n(U)$. The mass $\mathbf{M}_U(T)$ of a current $T \in \mathcal{D}^n(U)$ is given by

$$\mathbf{M}_U(T) \;=\; \sup\{T(\omega) : \omega \in \mathcal{D}^n(U),\ |\omega(z)| \leq 1\ \ \forall z \in U\}\,,$$

where $|\cdot|$ is the Hilbert norm in the space of all n-covectors of $\mathbf{R}^m$.

It is well known that, if a current $T \in \mathcal{D}_n(U)$ has finite mass, then T is a bounded Radon measure on U with values in the space of all n-covectors of $\mathbf{R}^m$. Consequently, for every n-form $\omega \in \mathcal{D}^n(U)$ we have

$$T(\omega) = \int \langle \tfrac{dT}{d|T|}\,, \omega\rangle \, d|T|\,,$$

where $\frac{dT}{d|T|}$ denotes the Radon-Nikodym derivative of the vector measure T with respect to the scalar measure $|T|$, and $\langle\cdot,\cdot\rangle$ is the duality pairing between n-vectors and n-covectors. If T has finite mass, the previous formula defines unambiguously $T(\omega)$ for every n-form ω with bounded continuous coefficients on U.

Following the terminology of [25], Chapter 3, we say that a subset M of $\mathbf{R}^m$ is countably n-rectifiable if $M \subseteq \bigcup_{j=0}^\infty N_j$, where $\mathcal{H}^n(N_0) = 0$ and where each N_j, $j \geq 1$, is an n-dimensional embedded C^1-manifold of $\mathbf{R}^m$. If M is countably n-rectifiable, then the approximate tangent space $\mathbf{T}_M(x)$ exists for $\mathcal{H}^n$-a.e. $x \in M$ (see [25], Theorem 11.6). An orientation ξ of a countably n-rectifiable set M is an $\mathcal{H}^n$-measurable n-vector field on M such that for $\mathcal{H}^n$-a.e. $x \in M$ we have $\xi(x) = \tau_1 \wedge \cdots \wedge \tau_n$, where $\{\tau_1, \ldots, \tau_n\}$ forms an orthonormal basis for the approximate tangent space $\mathbf{T}_M(x)$. A countably n-rectifiable set together with an orientation is called an oriented countably n-rectifiable set.

We refer to [9] and [25] for a complete treatment of the theory of currents and of rectifiable sets.

If U is an open subset of $\mathbf{R}^m$ and M an oriented countably n-rectifiable subset of U, with orientation ξ, the integration on M is the current $[\![M]\!]$ defined by

$$[\![M]\!](\omega) = \int_M \langle \xi, \omega \rangle \, d\mathcal{H}^n \qquad \forall \omega \in \mathcal{D}^n(U)\,.$$

In the particular case $m = n$, for any measurable subset A of $\mathbf{R}^n$ we consider the canonical orientation given by the n-vector $e_1 \wedge \cdots \wedge e_n$, where $e_1, \ldots, e_n$ is the canonical basis of $\mathbf{R}^n$. Consequently, the current $[\![A]\!]$ is given by

$$[\![A]\!](\varphi \, dx^1 \wedge \cdots \wedge dx^n) = \int_A \varphi(x) \, dx$$

for every $\varphi \in \mathcal{D}(U)$.

More generally, given an oriented countably n-rectifiable set $M \subseteq U$, with orientation ξ, and a multiplicity function θ (i.e., an $\mathcal{H}^n$-measurable locally integrable function θ defined on M with positive integer values), we define the current $T = \tau(M, \theta, \xi)$ by

$$T(\omega) = \int_M \langle \xi, \omega \rangle \, \theta \, d\mathcal{H}^n \qquad \forall \omega \in \mathcal{D}^n(U)\,.$$

These currents will be called (integer multiplicity) rectifiable n-currents. Note that, if $T = \tau(M, \theta, \xi)$, then $|T| = \theta \, \mathcal{H}^n \llcorner M$ and $\frac{dT}{d|T|} = \xi$ $\mathcal{H}^n$-a.e. on M.

If V is an open subset of some Euclidean space $\mathbf{R}^k$ and $f\colon U \to V$ is a proper function of class C^∞, the "push-forward" $f_\#$ of a current $T \in \mathcal{D}_n(U)$ is the current $f_\# T \in \mathcal{D}_n(V)$ defined by $(f_\# T)(\omega) = T(f^\# \omega)$ for every $\omega \in \mathcal{D}^n(V)$, where $f^\# \omega \in \mathcal{D}^n(U)$ denotes the "pull-back" of the form $\omega \in \mathcal{D}^n(V)$. If T has a finite mass, then $f_\# T$ is well defined by the previous formula for every f of class C^1 with bounded derivatives, even if f is not proper. It is clear that, in this case, the current $f_\# T$ has a finite mass too. As T is a bounded Radon measure on U with values in the space of all n-vectors of $\mathbf{R}^m$, we can consider as well the image measure $f_* T$, which is a bounded Radon measure on V with values in the space of all n-vectors of $\mathbf{R}^m$. Note that, in general, $f_\# T \neq f_* T$, since, clearly, $f^\# \omega \neq f^* \omega$ for an arbitrary n-form.

Currents in product spaces. We intoduce now some notation particularly suited for dealing with n-forms and n-currents defined on the product space $\mathbf{R}^n \times \mathbf{R}^k$, when we want to stress the different role of the variables $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^k$.

Let $e_1, \ldots, e_n$ and $\varepsilon_1, \ldots, \varepsilon_k$ be the canonical bases of $\mathbf{R}^n$ and $\mathbf{R}^k$. If $1 \leq p \leq n$, we define

$$I_{p,n} = \{\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathbf{N}^p : 1 \leq \alpha_1 < \cdots < \alpha_p \leq n\}\,,$$

and we set $I_{0,n} = \{0\}$; if $\alpha \in I_{p,n}$, we set $|\alpha| = p$. If $|\alpha| = 1$, then $\alpha = (i)$ for some i between 1 and n: in this case we shall simply write i instead of α.

For any $\alpha \in I_{p,n}$, with $1 \leq p \leq n$, we define

$$e_\alpha = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_p}\,, \qquad dx^\alpha = dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p}\,,$$

and we set $e_0 = 1$ and $dx^0 = 1$. The definitions of ε_β and dy^β for $\beta \in I_{q,k}$ are analogous.

A basis for the space of n-vectors of $\mathbf{R}^n \times \mathbf{R}^k$ is given by $\{e_\alpha \wedge \varepsilon_\beta : |\alpha| + |\beta| = n\}$, so that any n-vector ξ may be written in a unique way as

$$\xi = \sum_{|\alpha|+|\beta|=n} \xi^{\alpha\beta}\, e_\alpha \wedge \varepsilon_\beta\,;$$

the dual basis for n-covectors is $\{dx^\alpha \wedge dy^\beta : |\alpha| + |\beta| = n\}$.

With the previous notation, given an open subset U of $\mathbf{R}^n \times \mathbf{R}^k$, every $\omega \in \mathcal{D}^n(U)$ may be written in a unique way as

$$\omega = \sum_{|\alpha|+|\beta|=n} \omega_{\alpha\beta}\, dx^\alpha \wedge dy^\beta\,,$$

where $\omega_{\alpha\beta} \in C_c^\infty(U)$. If $T \in \mathcal{D}_n(U)$, for every α, β with $|\alpha| + |\beta| = n$ we denote by $T^{\alpha\beta}$ the scalar distribution defined for every $\varphi \in C_c^\infty(U)$ by

$$T^{\alpha\beta}(\varphi) = T(\varphi\, dx^\alpha \wedge dy^\beta)\,,$$

so that for every $\omega \in \mathcal{D}^n(U)$

$$T(\omega) = \sum_{|\alpha|+|\beta|=n} T^{\alpha\beta}(\omega_{\alpha\beta})\,;$$

the distributions $T^{\alpha\beta}$ are called the components of T.

Graphs. We denote henceforth by Ω a bounded open subset of $\mathbf{R}^n$ and by U the cylinder $\Omega \times \mathbf{R}^k$. For any subset A of Ω and for any function $u\colon A \to \mathbf{R}^k$ we denote its graph by G_u, i.e.,

$$G_u = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^k : x \in A\,,\ y = u(x)\}\,.$$

If $u:\Omega\to\mathbf{R}^k$ is continuously differentiable, for any $(x,y)\in G_u$ a basis for the tangent space $\mathbf{T}_{G_u}(x,y)$ is given by the vectors

$$v_i(x)=e_i+\sum_{j=1}^{k}\frac{\partial u^j}{\partial x^i}(x)\varepsilon_j\,,\qquad i=1,\dots,n\,,$$

and, setting $\mu(x)=v_1(x)\wedge\cdots\wedge v_n(x)$, the natural orientation of $\mathbf{T}_{G_u}(x,y)$ is given by

$$\xi(x,y)=\frac{\mu(x)}{|\mu(x)|}\,.$$

With this choice of ξ, the set G_u becomes an oriented n-rectifiable set, with which we associate the current $[\![G_u]\!]$. Note that the orientation chosen for G_u implies that $p_\#\,[\![G_u]\!]=[\![\Omega]\!]$, where p is the canonical projection from $\mathbf{R}^n{\times}\mathbf{R}^k$ onto $\mathbf{R}^n$, and that $[\![G_u]\!]=\mathcal{U}_\#\,[\![\Omega]\!]$, where $\mathcal{U}(x)=(x,u(x))$.

If $\alpha\in I_{n-p,n}$, we denote by $\hat\alpha\in I_{p,n}$ the increasing complement of α in $\{1,\dots,n\}$; we note in particular that $\hat 0=(1,\dots,n)$. Also, we denote by $\epsilon(\alpha)$ the sign of the permutation of $(1,\dots,n)$ into $(\alpha,\hat\alpha)$, with the convention $\epsilon(0)=1$.

The n-vector $\mu(x)$ introduced above can be written explicitly as

$$\mu(x)=\sum_{|\alpha|+|\beta|=n}\mu^{\alpha\beta}(x)\,e_\alpha\wedge\varepsilon_\beta\,,\tag{1.1}$$

where

$$\mu^{\alpha\beta}(x)=\epsilon(\alpha)\det\Big(\frac{\partial u^{\beta_i}}{\partial x^{\hat\alpha_j}}(x)\Big)\,,\tag{1.2}$$

with the convention $\mu^{\hat 0 0}(x)=1$. Note in particular that $D_iu^j(x)=(-1)^{n-i}\mu^{\hat\imath j}(x)$.

Since the n-vector $\mu(x)$ depends on x only through the matrix $\nabla u(x)$, we are led to defining for every $k{\times}n$ matrix A the n-vector

$$\mathcal{M}(A)=\sum_{|\alpha|+|\beta|=n}\mathcal{M}^{\alpha\beta}(A)\,e_\alpha\wedge\varepsilon_\beta\,,$$

where

$$\mathcal{M}^{\alpha\beta}(A)=\epsilon(\alpha)\det\left(A_{\beta_i\hat\alpha_j}\right),\tag{1.3}$$

with the convention $\mathcal{M}^{\hat 0 0}(A)=1$. With this notation we have $\mu(x)=\mathcal{M}(\nabla u(x))$ for every $u\in C^1(\Omega;\mathbf{R}^k)$.

Setting $T = [\![G_u]\!]$, we remark that $|T| = \mathcal{H}^n \llcorner G_u$ and $p_*|T| = p_*(\mathcal{H}^n \llcorner G_u) = |\mu|\,\mathcal{L}^n \llcorner \Omega$, i.e.,

$$\mathcal{H}^n(G_u \cap (A{\times}\mathbf{R}^k)) = \int_A |\mu(x)|\,dx \tag{1.4}$$

for every $\mathcal{L}^n$-measurable subset A of Ω. Moreover, the components $T^{\alpha\beta}$ are Radon measures satisfying

$$T^{\alpha\beta}(\varphi) = \int_\Omega \varphi(x,u(x))\mu^{\alpha\beta}(x)\,dx$$

for every $\varphi \in C_0^\infty(U)$. This is equivalent to saying that $\mu^{\alpha\beta} = p_*\,T^{\alpha\beta}$ and $T^{\alpha\beta} = \mathcal{U}_*\,\mu^{\alpha\beta}$, where, as usual, we identify the locally integrable function $\mu^{\alpha\beta}$ with the Radon measure $\int_B \mu^{\alpha\beta} dx$. Therefore $\mu = p_*\,T$ and $T = \mathcal{U}_*\,\mu$.

By Stokes Theorem we have $\partial\,[\![G_u]\!] = 0$ in $\Omega{\times}\mathbf{R}^k$. We also remark that the canonical orientation ξ satisfies $\xi^{\hat{0}0}(x,y) = |\mu(x)|^{-1} > 0$ for every $(x,y) \in G_u$. This is related to the fact that p maps $\mathbf{T}_{G_u}(x,y)$ onto $\mathbf{R}^n$. Indeed, if V is an n-dimensional linear subspace of $\mathbf{R}^n{\times}\mathbf{R}^k$, with orthonormal basis $\{\tau_1,\dots,\tau_n\}$, and $\eta = \tau_1 \wedge \ldots \wedge \tau_n$, then the conditions $\eta^{\hat{0}0} \neq 0$ and $p(V) = \mathbf{R}^n$ are equivalent.

We consider now the case of a graph of a not necessarily smooth function: let A be a measurable subset of Ω and u be any function from A to $\mathbf{R}^k$. If G_u is a countably n-rectifiable set such that p maps $\mathbf{T}_{G_u}(x,y)$ onto $\mathbf{R}^n$ for $\mathcal{H}^n$-a.e. $(x,y) \in G_u$, then there exists a unique orientation ξ of G_u such that $\xi^{\hat{0}0} > 0$ $\mathcal{H}^n$-a.e. on G_u. This choice of ξ will be considered as the natural orientation of G_u, and again the current $[\![G_u]\!]$ satisfies $p_\#\,[\![G_u]\!] = [\![A]\!]$.

Cartesian currents. To generalize the notion of Cartesian graph, Giaquinta, Modica, and Souček introduced in [10] and [12] the space $\mathrm{cart}(\Omega;\mathbf{R}^k)$ of all rectifiable n-currents T on $U = \Omega{\times}\mathbf{R}^k$ such that $p_\# T = [\![\Omega]\!]$, $T^{\hat{0}0} \geq 0$, $\partial T = 0$, $\mathbf{M}_U(T) < +\infty$, and

$$\sup\{T(|y|\varphi(x,y)\,dx^1 \wedge \cdots \wedge dx^n) : \varphi \in C_0^\infty(U)\,,\ \sup|\varphi| \leq 1\} < +\infty\,.$$

In the case $T = [\![G_u]\!]$ the last expression is just the L^1 norm of u; if, in addition, $u \in C^1(\Omega;\mathbf{R}^k)$, then $\mathbf{M}_U([\![G_u]\!])$ is the n-dimensional area of the graph G_u. This implies that $[\![G_u]\!] \in \mathrm{cart}(\Omega;\mathbf{R}^k)$ for every $u \in C^1(\Omega;\mathbf{R}^k)$ with $\int_\Omega |u|\,dx < +\infty$ and $\mathcal{H}^n(G_u) < +\infty$. Note that, if $u \notin C^1(\Omega;\mathbf{R}^k)$, even if these two conditions are satisfied and G_u is n-rectifiable and can be oriented in the natural way, we cannot deduce that $[\![G_u]\!] \in \mathrm{cart}(\Omega;\mathbf{R}^k)$, because the condition $\partial\,[\![G_u]\!] = 0$ may fail when u is not smooth (consider, for instance, the case of a piecewise constant function; an example of a function u such that $\partial[\![G_u]\!] \neq 0$, although u belongs to a Sobolev space, may be found in Section 3).

Given $T = \tau(M,\theta,\xi) \in \operatorname{cart}(\Omega;\mathbf{R}^k)$, a point $(x,y) \in M$ is said to be regular if p maps $\mathbf{T}_M(x,y)$ onto $\mathbf{R}^n$. The set of regular points will be denoted by M_r, and its complement in M by M_s. Moreover, we define $\Omega_s = p(M_s)$, $\Omega_r = \Omega \setminus \Omega_s$, $T_r = \tau(M_r,\theta,\xi)$, and $T_s = \tau(M_s,\theta,\xi)$. Note that $\mathbf{M}_U(T) = \mathbf{M}_U(T_r) + \mathbf{M}_U(T_s)$.

We also introduce the Radon measure μ_T on Ω, with values in the space of n-vectors of $\mathbf{R}^n{\times}\mathbf{R}^k$, defined by $\mu_T = p_* T$; its components are the scalar Radon measures $\mu_T^{\alpha\beta}$ on Ω defined by $\mu_T^{\alpha\beta} = p_* T^{\alpha\beta}$. Note that, if $T = [\![G_u]\!]$ for some $u \in C^1(\Omega;\mathbf{R}^k)$, then μ_T coincides with the function μ given by (1.1).

The following theorems are a summary of the results of [10], [11], [12] that are relevant to our paper.

Theorem 1.3. *Let $T = \tau(M,\theta,\xi)$ be a current in* $\operatorname{cart}(\Omega;\mathbf{R}^k)$. *Then $\mathcal{L}^n(\Omega_s) = 0$ and there exists a function $\tilde{u}_T\colon \Omega_r \to \mathbf{R}^k$ such that $G_{\tilde{u}_T} \simeq M_r$ in the sense of $\mathcal{H}^n$. Moreover $\theta = 1$ $\mathcal{H}^n$-a.e. on M_r, so that $T_r = [\![G_{\tilde{u}_T}]\!]$.*

Proof. See [10], Section 3, Theorem 2, and [12], Theorem 1.

Remark 1.4. If $u\colon \Omega \to \mathbf{R}^k$ is another function such that $G_u \simeq M_r$, then $G_u \simeq G_{\tilde{u}_T}$ in the sense of $\mathcal{H}^n$, hence $u = \tilde{u}_T$ $\mathcal{L}^n$-a.e. on Ω. The converse is not always true: there are simple examples, even with $n = k = 1$, where $u = \tilde{u}_T$ $\mathcal{L}^n$-a.e. on Ω, but $\mathcal{H}^n(G_u \Delta G_{\tilde{u}_T}) > 0$ and $[\![G_u]\!] \neq T_r$. However, if G_u is countably n-rectifiable and p maps $\mathbf{T}_{G_u}(x,y)$ onto $\mathbf{R}^n$ for $\mathcal{H}^n$-a.e. $(x,y) \in G_u$, then the equality $u = \tilde{u}_T$ $\mathcal{L}^n$-a.e. on Ω implies $\mathcal{H}^n(G_u \Delta G_{\tilde{u}_T}) = 0$ by the area formula, hence $T_r = [\![G_u]\!]$. In particular, this condition is satisfied when u is locally Lipschitz on Ω, and $u = \tilde{u}_T$ $\mathcal{L}^n$-a.e. on Ω.

Theorem 1.5. *Let $T = \tau(M,\theta,\xi)$ be a current in* $\operatorname{cart}(\Omega;\mathbf{R}^k)$ *and let $u_T\colon \Omega \to \mathbf{R}^k$ be any $\mathcal{L}^n$-measurable function such that $u_T = \tilde{u}_T$ $\mathcal{L}^n$-a.e. on Ω_r. Then*

(a) *$u_T \in BV(\Omega;\mathbf{R}^k)$ and $D_i u_T^j = (-1)^{n-i}\mu_T^{\hat{\imath}j}$ on Ω for every $1 \le i \le n$, $1 \le j \le k$;*

(b) *$\mu_T \llcorner \Omega_r$ is absolutely continuous and $\mu_T \llcorner \Omega_s$ is singular with respect to $\mathcal{L}^n$;*

(c) *$|Du_T| \le |\mu_T| \le p_*|T| = p_*(\mathcal{H}^n \llcorner M)$ on Ω, and $|D^s u_T| = |Du_T| \llcorner \Omega_s$;*

(d) *$\mu_T(B \cap \Omega_r) = \int_B \mathcal{M}(\nabla u_T(x))\,dx$ and $|\mu_T|(B \cap \Omega_r) = \int_B |\mathcal{M}(\nabla u_T(x))|\,dx$ for every Borel subset B of Ω;*

(e) $|\mu_T| = p_*|T| = p_*(\mathcal{H}^n \llcorner M_r)$ *on* Ω_r *;*

(f) $\xi(x, u_T(x)) = \frac{d\mu_T}{d|\mu_T|}(x) = \frac{\mathcal{M}(\nabla u_T(x))}{|\mathcal{M}(\nabla u_T(x))|}$ *for* $\mathcal{L}^n$*-a.e.* $x \in \Omega_r$ *;*

(g) $T_r(\omega) = \int_\Omega \langle \mathcal{M}(\nabla u_T(x)), \omega(x, u_T(x)) \rangle \, dx$ *for every* $\omega \in \mathcal{D}^n(U)$ *.*

Proof. (a) See [10], Section 3, Theorem 3.
(b) Since $\mathcal{L}^n(\Omega_s) = 0$, the measure $\mu_T \llcorner \Omega_s$ is singular with respect to $\mathcal{L}^n$. Let $J^M p$ be the Jacobian of the map p on M (see [25], Section 12). For every Borel subset B of Ω_r the area formula gives

$$\int_{M \cap p^{-1}(B)} J^M p(z) \, d\mathcal{H}^n(z) \; = \; \int_B \mathcal{H}^0(M \cap p^{-1}(x)) \, d\mathcal{L}^n(x) \, .$$

As $J^M p(z) > 0$ for every $z \in M_r$, the condition $\mathcal{L}^n(B) = 0$ implies $\mathcal{H}^n(M_r \cap p^{-1}(B)) = 0$. Since $B \cap p(M_s) = \emptyset$, and hence $M \cap p^{-1}(B) = M_r \cap p^{-1}(B)$, we obtain

$$\mu_T(B) \; = \; \int_{M \cap p^{-1}(B)} \xi(z) \, d\mathcal{H}^n(z) \; = \; \int_{M_r \cap p^{-1}(B)} \xi(z) \, d\mathcal{H}^n(z) \; = \; 0$$

for every Borel subset B of Ω_r with $\mathcal{L}^n(B) = 0$. This proves that $\mu_T \llcorner \Omega_r$ is absolutely continuous with respect to $\mathcal{L}^n$.
(c) The first inequality follows from (a). For the second one it is enough to observe that $|p_* \mu| \leq p_*|\mu|$ for every vector measure μ on $\Omega \times \mathbf{R}^k$ with finite total variation. The equality concerning $D^s u_T$ follows from (b).
(d) The formula for μ_T is proved in [11], Theorem 5. The formula for $|\mu_T|$ follows easily from the previous one.
(e) For every Borel subset B of Ω_r we have $M \cap p^{-1}(B) = M_r \cap p^{-1}(B) \simeq G_{\tilde{u}_T} \cap p^{-1}(B)$ in the sense of $\mathcal{H}^n$, hence

$$\mu_T(B) \; = \; \int_{G_{\tilde{u}_T} \cap p^{-1}(B)} \xi(z) \, d\mathcal{H}^n(z) \, .$$

Since p is one-to-one from $G_{\tilde{u}_T}$ to Ω_r and $|\xi(z)| = 1$ for $\mathcal{H}^n$-a.e. $z \in G_{\tilde{u}_T}$, we conclude that $|\mu_T|(B) = \mathcal{H}^n(G_{\tilde{u}_T}(z) \cap p^{-1}(B)) = \mathcal{H}^n(M_r \cap p^{-1}(B))$.
(f) Since $\big(p(z), \tilde{u}_T(p(z))\big) = z$ for $\mathcal{H}^n$-a.e. $z \in M_r$, and $|\mu_T| = p_*(\mathcal{H}^n \llcorner M_r)$, we have

$$\mu_T(B) \; = \; \int_{M_r \cap p^{-1}(B)} \xi(z) \, d\mathcal{H}^n(z) \; = \; \int_B \xi(x, \tilde{u}_T(x)) \, d|\mu_T|(x)$$

for every Borel subset B of Ω_r. This implies that $\xi(x, \tilde{u}_T(x)) = \frac{d\mu_T}{d|\mu_T|}(x)$ for $|\mu_T|$-a.e. $x \in \Omega_r$, hence $\xi(x, u_T(x)) = \frac{d\mu_T}{d|\mu_T|}(x)$ for $\mathcal{L}^n$-a.e. $x \in \Omega_r$ (recall that $|\mu_T| \geq \mathcal{L}^n$ on Ω thanks to (d)). The other equality follows easily from (d).
(g) See [11], Theorem 5. □

Theorem 1.6. *Let* (u_h) *be a sequence in* $C^1(\Omega;\mathbf{R}^k)$ *converging strongly in* $L^1(\Omega;\mathbf{R}^k)$ *to some function* u*, and such that the graphs* G_{u_h} *have equibounded* $\mathcal{H}^n$ *measure. Then there exist a subsequence, still denoted* (u_h)*, and a current* $T\in \operatorname{cart}(\Omega;\mathbf{R}^k)$*, such that:*

(a) $[\![G_{u_h}]\!] \rightharpoonup T$ *weakly in* $\mathcal{D}_n(U)$*;*

(b) $u=u_T$ $\mathcal{L}^n$*-a.e. on* Ω*, hence* $u\in BV(\Omega;\mathbf{R}^k)$*;*

(c) $Du_h \rightharpoonup Du$ *in the weak** *topology of* $\mathcal{M}(\Omega;\mathbf{M}^{k\times n})$*.*

If, in addition, u *is locally Lipschitz on* Ω*, then* $T_r = [\![G_u]\!]$*.*

Proof. The compactness property is proved in [10], Section 3, Theorem 1. For (b) and (c) see [10], Section 3, Theorem 3. The last assertion follows from Remark 1.4. □

2. Lower semicontinuity and relaxation

Let Ω be a bounded open subset of $\mathbf{R}^n$. For every $u\in C^1(\Omega;\mathbf{R}^k)$ the n-dimensional area of the graph of u is given by

$$A(u)=\mathbf{M}_U(\,[\![G_u]\!]\,)=\mathcal{H}^n(G_u)=\int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\,dx\,.$$

We define the functional A on $L^1(\Omega;\mathbf{R}^k)$ by setting

$$A(u)=\begin{cases}\displaystyle\int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\,dx\,, & \text{if } u\in C^1(\Omega;\mathbf{R}^k),\\ +\infty\,, & \text{otherwise,}\end{cases}$$

and we consider the corresponding relaxed functional $\mathcal{A}\colon L^1(\Omega;\mathbf{R}^k)\to[0,+\infty]$, defined as the greatest lower semicontinuous functional on $L^1(\Omega;\mathbf{R}^k)$ which is less than or equal to A.

Besides the area functionals A and $\mathcal{A}$, with the same methods we can study a wide class of polyconvex functionals. We recall that a function $f\colon\mathbf{M}^{k\times n}\to[0,+\infty]$ is said to be *polyconvex* if there exists a convex function $g\colon\Xi\to[0,+\infty]$, defined in the space Ξ of all n-vectors of $\mathbf{R}^n{\times}\mathbf{R}^k$, such that $f(A)=g(\mathcal{M}(A))$ for every $A\in\mathbf{M}^{k\times n}$. As $\mathcal{M}^{\hat{0}0}(A)=1$ for every $A\in\mathbf{M}^{k\times n}$, the definition of g is relevant only on the hyperplane $\Xi^{\hat{0}0}=\{\xi\in\Xi:\xi^{\hat{0}0}=1\}$. By changing, if needed, g out of this hyperplane, we may always assume that g is positively homogeneous of degree one on Ξ.

Let $f\colon \Omega\times\mathbf{R}^k\times\mathbf{M}^{k\times n} \to [0,+\infty[$ be a function such that:

(i) for every $(x,y)\in\Omega\times\mathbf{R}^k$ the function $A\mapsto f(x,y,A)$ is polyconvex on $\mathbf{M}^{k\times n}$;

(ii) there exists a constant $c_0>0$ such that $f(x,y,A)\ge c_0|\mathcal{M}(A)|$ for every $x\in\Omega$, $y\in\mathbf{R}^k$, $A\in\mathbf{M}^{k\times n}$;

(iii) for every $x_0\in\Omega$, $y_0\in\mathbf{R}^k$ and for every $\varepsilon>0$ there exists $\delta>0$ such that $f(x,y,A)\ge(1-\varepsilon)f(x_0,y_0,A)$ for every $x\in\Omega$, $y\in\mathbf{R}^k$, $A\in\mathbf{M}^{k\times n}$, with $|x-x_0|<\delta$ and $|y-y_0|<\delta$;

(iv) the function $(x,y)\mapsto f(x,y,0)$ is locally bounded on $\Omega\times\mathbf{R}^k$.

We shall consider the functional $F\colon L^1(\Omega;\mathbf{R}^k)\to[0,+\infty]$ defined by

$$F(u)=\begin{cases}\displaystyle\int_\Omega f\big(x,u(x),\nabla u(x)\big)\,dx\,, & \text{if } u\in C^1(\Omega;\mathbf{R}^k),\\ +\infty\,, & \text{otherwise,}\end{cases}$$

and the corresponding relaxed functional $\mathcal{F}\colon L^1(\Omega;\mathbf{R}^k)\to[0,+\infty]$, defined as the greatest lower semicontinuous functional on $L^1(\Omega;\mathbf{R}^k)$ which is less than or equal to F. We shall use the notation $A(u,\Omega)$, $\mathcal{A}(u,\Omega)$, $F(u,\Omega)$, $\mathcal{F}(u,\Omega)$ when we want to stress the dependence of these functionals on the set Ω.

The results contained in the following proposition are needed in all estimates concerning the relaxed functional $\mathcal{F}$. As in Section 1, the cylinder $\Omega\times\mathbf{R}^k$ will be denoted by U.

Proposition 2.1. *Assume that conditions* (i), (ii), (iii), (iv) *are satisfied. Then there exists a function* $g\colon U\times\Xi\to[0,+\infty]$ *such that:*

(a) $f(x,y,A)=g(z,\mathcal{M}(A))$ *for every* $z=(x,y)\in U$ *and for every* $A\in\mathbf{M}^{k\times n}$*;*

(b) g *is lower semicontinuous on* $U\times\Xi$*;*

(c) *for every* $z\in U$ *the function* $\xi\mapsto g(z,\xi)$ *is convex and positively homogeneous of degree one on* Ξ*;*

(d) *for every* $z\in U$ *and for every* $\xi\in\Xi$ *we have* $g(z,\xi)\ge c_0|\xi|$, *where* $c_0>0$ *is the constant in condition* (iii).

Proof. Let m be the dimension of the vector space Ξ and let $\Xi^{\hat{0}0}=\{\xi\in\Xi:\xi^{\hat{0}0}=1\}$. For every $z=(x,y)\in U$ and for every $\xi\in\Xi^{\hat{0}0}$ we define

$$g_0(z,\xi)\;=\;\inf\sum_{i=1}^m\lambda^i f(x,y,A^i)\,,$$

where the infimum is taken over all families $(A^i)_{1\le i\le m}$ in $\mathbf{M}^{k\times n}$ and over all families $(\lambda^i)_{1\le i\le m}$ of non-negative real numbers such that

$$\sum_{i=1}^{m}\lambda^i = 1 \qquad \text{and} \qquad \sum_{i=1}^{m}\lambda^i\mathcal{M}(A^i) = \xi\,.$$

By Theorem 1.3 of Chapter 4 of [5], for every $z\in U$ the function $\xi\mapsto g_0(z,\xi)$ is convex and finite, hence continuous, on $\Xi^{\hat{0}0}$ and $f(x,y,A) = g_0(z,\mathcal{M}(A))$ for every $z=(x,y)\in U$ and for every $A\in\mathbf{M}^{k\times n}$. By (ii) we have also $g_0(z,\xi)\ge c_0|\xi|$ for every $z\in U$ and for every $\xi\in\Xi^{\hat{0}0}$.

Let us prove that g_0 is lower semicontinuous on $U\times\Xi^{\hat{0}0}$. Condition (iii) implies that for every $z_0\in U$ and for every $\varepsilon>0$ there exists $\delta>0$ such that $g_0(z,\xi)\ge(1-\varepsilon)g_0(z_0,\xi)$ for every $\xi\in\Xi^{\hat{0}0}$ and for every $z\in U$ with $|z-z_0|<\delta$. As the function $\xi\mapsto g_0(z_0,\xi)$ is continuous, for every $\xi_0\in\Xi^{\hat{0}0}$ there exists $\eta>0$ such that $g_0(z_0,\xi)\ge g_0(z_0,\xi_0)-\varepsilon$ for every $\xi\in\Xi^{\hat{0}0}$ with $|\xi-\xi_0|<\eta$. Therefore $g_0(z,\xi)\ge(1-\varepsilon)g_0(z_0,\xi_0)-\varepsilon$ whenever $|z-z_0|<\delta$ and $|\xi-\xi_0|<\eta$, which proves the lower semicontinuity of g_0.

Let us define now $g\colon U\times\Xi\to[0,+\infty]$ by

$$g(z,\xi) = \begin{cases} \xi^{\hat{0}0}g_0\big(z,\dfrac{\xi}{\xi^{\hat{0}0}}\big)\,, & \text{if } \xi^{\hat{0}0}>0,\\[2ex] \displaystyle\lim_{\varrho\to0^+}\varrho g_0\big(z,\frac{\xi+\varrho e^{\hat{0}}}{\varrho}\big)\,, & \text{if } \xi^{\hat{0}0}=0,\\[2ex] +\infty\,, & \text{if } \xi^{\hat{0}0}<0,\end{cases}$$

where $e^{\hat{0}} = e^1\wedge\cdots\wedge e^n$ is the n-vector of $\mathbf{R}^n$ associated with the canonical basis $e^1,\dots,e^n$. It is easy to see that for every $z\in U$ the function $\xi\mapsto g(z,\xi)$ is convex and positively homogeneous of degree one. Moreover, as $g(z,\xi)=g_0(z,\xi)$ for every $\xi\in\Xi^{\hat{0}0}$, and $\mathcal{M}(A)\in\Xi^{\hat{0}0}$ for every $A\in\mathbf{M}^{k\times n}$, we have $f(x,y,A)=g_0(z,\mathcal{M}(A))=g(z,\mathcal{M}(A))$ for every $z=(x,y)\in U$ and for every $A\in\mathbf{M}^{k\times n}$.

Let us prove that g is lower semicontinuous at each point of $U\times\Xi$. Let $z_h\to z$ and $\xi_h\to\xi$ be two sequences in U and Ξ respectively. We want to prove that

$$g(z,\xi)\le\liminf_{h\to\infty}g(z_h,\xi_h)\,. \tag{2.1}$$

This inequality is obvious if $\xi^{\hat{0}0}\ne0$, so we consider only the case $\xi^{\hat{0}0}=0$. By the definition of g, it is clearly enough to consider sequences (ξ_h) with $\xi_h^{\hat{0}0}\ge0$ for every h.

Let us fix $\varrho > 0$. As g is lower semicontinuous at the point $\xi + \varrho e^{\hat{0}}$, whose $\hat{0}0$ coordinate is different from 0, we have

$$g\big(z, \xi + \varrho e^{\hat{0}}\big) \le \liminf_{h\to\infty} g\big(z_h, \xi_h + (\varrho - \xi_h^{\hat{0}0})e^{\hat{0}}\big)\,. \tag{2.2}$$

Since $\xi_h^{\hat{0}0} \le \varrho$ for h large enough, and g is convex and positively homogeneous of degree one, taking the equality $\mathcal{M}(0) = e^{\hat{0}}$ into account we obtain

$$\begin{aligned} g(z_h, \xi_h + (\varrho - \xi_h^{\hat{0}0})e^{\hat{0}}) &\le g(z_h, \xi_h) + (\varrho - \xi_h^{\hat{0}0}) g(z_h, e^{\hat{0}}) = \\ &= g(z_h, \xi_h) + (\varrho - \xi_h^{\hat{0}0}) f(x_h, y_h, 0)\,, \end{aligned} \tag{2.3}$$

where $(x_h, y_h) = z_h$. By (iv) we have also

$$\limsup_{h\to\infty} (\varrho - \xi_h^{\hat{0}0}) f(x_h, y_h, 0) \le c\varrho$$

for some constant c, so that from (2.2) and (2.3) we obtain

$$g(z, \xi + \varrho e^{\hat{0}}) \le \liminf_{h\to\infty} g(z_h, \xi_h) + c\varrho\,.$$

As ϱ tends to 0 we get (2.1), so that g is lower semicontinuous, and the proposition is proved. □

The following remark provides the well known sequential characterization of the relaxed functionals $\mathcal{A}$ and $\mathcal{F}$.

Remark 2.2. It is easy to prove that for every $u \in L^1(\Omega; \mathbf{R}^k)$ the value of $\mathcal{A}(u)$ is uniquely determined by the following conditions (see, e.g., [4], Proposition 1.3.3):

(a) for every sequence (u_h) in $C^1(\Omega; \mathbf{R}^k)$ converging to u in $L^1(\Omega; \mathbf{R}^k)$ we have

$$\mathcal{A}(u) \le \liminf_{h\to\infty} A(u_h)\,;$$

(b) there exists a sequence (u_h) in $C^1(\Omega; \mathbf{R}^k)$, converging to u in $L^1(\Omega; \mathbf{R}^k)$, such that

$$\mathcal{A}(u) = \lim_{h\to\infty} A(u_h)\,.$$

A similar characterization holds for $\mathcal{F}$.

The following lemma deals with the case of bounded functions.

Lemma 2.3. *Let* $u \in L^\infty(\Omega;\mathbf{R}^k)$. *Then there exists a sequence* (u_h) *in* $C^1(\Omega;\mathbf{R}^k)$, *bounded in* $L^\infty(\Omega;\mathbf{R}^k)$ *and converging to* u *in* $L^1(\Omega;\mathbf{R}^k)$, *such that* $\mathcal{A}(u) = \lim_{h\to\infty} A(u_h)$.

Proof. By Remark 2.2(b) there exists a sequence (v_h) in $C^1(\Omega;\mathbf{R}^k)$, converging to u in $L^1(\Omega;\mathbf{R}^k)$, such that

$$\mathcal{A}(u) = \lim_{h\to\infty} A(v_h) = \lim_{h\to\infty} \mathcal{H}^n(G_{v_h})\,. \tag{2.4}$$

Let $\varphi\colon \mathbf{R}^k \to \mathbf{R}^k$ be a globally Lipschitz bounded function of class C^1, with Lipschitz constant 1, such that $\varphi(y) = y$ when $|y| \le \|u\|_{L^\infty(\Omega;\mathbf{R}^k)}$. Let us define $u_h(x) = \varphi(v_h(x))$. Then the sequence (u_h) is bounded in $L^\infty(\Omega;\mathbf{R}^k)$ and converges to u in $L^1(\Omega;\mathbf{R}^k)$. Moreover, each function u_h belongs to $C^1(\Omega;\mathbf{R}^k)$, and $\mathcal{H}^n(G_{u_h}) \le \mathcal{H}^n(G_{v_h})$, since the map $\Phi(x,y) = (x,\varphi(y))$ is a contraction in $\mathbf{R}^n\times\mathbf{R}^k$ and $G_{u_h} = \Phi(G_{v_h})$. Therefore, (2.4) implies $\mathcal{A}(u) \ge \limsup_{h\to\infty} \mathcal{H}^n(G_{u_h}) = \limsup_{h\to\infty} A(u_h)$. The conclusion follows now from Remark 2.2(a). □

We prove now the lower semicontinuity of the functionals $\mathcal{A}$ and $\mathcal{F}$ on $C^1(\Omega;\mathbf{R}^k)$ with respect to the strong topology of $L^1(\Omega;\mathbf{R}^k)$. This implies that $\mathcal{A}(u) = A(u)$ and $\mathcal{F}(u) = F(u)$ for every $u \in C^1(\Omega;\mathbf{R}^k) \cap L^1(\Omega;\mathbf{R}^k)$.

Theorem 2.4. *The functional* A *is lower semicontinuous on* $C^1(\Omega;\mathbf{R}^k) \cap L^1(\Omega;\mathbf{R}^k)$ *with respect to the strong topology of* $L^1(\Omega;\mathbf{R}^k)$.

Proof. Let $u_h \to u$ in $L^1(\Omega;\mathbf{R}^k)$ with $u_h,\ u \in C^1(\Omega;\mathbf{R}^k)$. We may assume that $\lim_{h\to\infty} A(u_h)$ exists and is finite. Thus, by Theorem 1.6, we may also assume that $[\![G_{u_h}]\!] \rightharpoonup T$ weakly in $\mathcal{D}_n(U)$, for some $T \in \operatorname{cart}(\Omega;\mathbf{R}^k)$ satisfying $T_r = [\![G_u]\!]$. By the lower semicontinuity of the mass we have

$$A(u) = \mathbf{M}_U(T_r) \le \mathbf{M}_U(T) \le \lim_{h\to\infty} \mathbf{M}_U(\,[\![G_{u_h}]\!]\,) = \lim_{h\to\infty} A(u_h)\,,$$

which concludes the proof. □

Theorem 2.5. *The functional* F *is lower semicontinuous on* $C^1(\Omega;\mathbf{R}^k) \cap L^1(\Omega;\mathbf{R}^k)$ *with respect to the strong topology of* $L^1(\Omega;\mathbf{R}^k)$.

Proof. Let $u_h \to u$ in $L^1(\Omega;\mathbf{R}^k)$ with $u_h,\ u \in C^1(\Omega;\mathbf{R}^k)$. We may assume that $\lim_{h\to\infty} F(u_h)$ exists and is finite, and, therefore, $A(u_h)$ is bounded uniformly with respect to h. Thus, by Theorem 1.6, we may also assume that $[\![G_{u_h}]\!] \rightharpoonup T$ weakly in $\mathcal{D}_n(U)$, for some $T = \tau(M,\theta,\xi) \in \mathrm{cart}(\Omega;\mathbf{R}^k)$ satisfying $T_r = [\![G_u]\!]$. Writing, for simplicity, T_h instead of $[\![G_{u_h}]\!]$, we have

$$F(u_h) \;=\; \int_{\mathbf{R}^n\times\mathbf{R}^k} g(z, \tfrac{dT_h}{d|T_h|}(z))\, d|T_h|(z)\,,$$

where g is the function given by Proposition 2.1. Moreover

$$F(u) \;=\; \int_\Omega g\big((x,u(x)),\mathcal{M}(\nabla u(x))\big)\,dx \;=\; \int_{M_r} g(z,\xi(z))\,d\mathcal{H}^n(z) \;\leq$$

$$\leq\; \int_M g(z,\xi(z))\,d\mathcal{H}^n(z) \;=\; \int_{\mathbf{R}^n\times\mathbf{R}^k} g(z, \tfrac{dT}{d|T|}(z))\,d|T|(z)\,.$$

By Reshetnyak's semicontinuity theorem (Theorem 1.1) we have

$$\int_{\mathbf{R}^n\times\mathbf{R}^k} g(z, \tfrac{dT}{d|T|}(z))\,d|T|(z) \;\leq\; \liminf_{h\to\infty} \int_{\mathbf{R}^n\times\mathbf{R}^k} g(z, \tfrac{dT_h}{d|T_h|}(z))\,d|T_h|(z)\,,$$

which proves the assertion. □

Remark 2.6. As $|\mathcal{M}(\nabla u)| \geq |\nabla u|$ we have $A(u) \geq \int_\Omega |\nabla u|\,dx$ for every $u \in C^1(\Omega;\mathbf{R}^k)$. By Remark 2.2(b) this implies that, if $u \in L^1(\Omega;\mathbf{R}^k)$ and $\mathcal{A}(u) < +\infty$, then $u \in BV(\Omega;\mathbf{R}^k)$ and $\mathcal{A}(u) \geq |Du|(\Omega)$ (see [14], Theorem 1.9). Similarly, as the constant c_0 in (ii) is positive, we have that, if $u \in L^1(\Omega;\mathbf{R}^k)$ and $\mathcal{F}(u) < +\infty$, then $u \in BV(\Omega;\mathbf{R}^k)$ and $\mathcal{F}(u) \geq c_0|Du|(\Omega)$. The following theorems provide better estimates from below for the relaxed functionals $\mathcal{A}$ and $\mathcal{F}$.

Theorem 2.7. *For every $u \in BV(\Omega;\mathbf{R}^k)$ we have*

$$\mathcal{A}(u) \geq \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\,dx \;+\; |D^s u|(\Omega)\,.$$

Proof. We may assume that $\mathcal{A}(u) < +\infty$; then there exists a sequence (u_h) in $C^1(\Omega;\mathbf{R}^k)$ converging strongly in $L^1(\Omega;\mathbf{R}^k)$ to u, and such that $\mathcal{A}(u) = \lim_{h\to\infty} \mathcal{H}^n(G_{u_h})$. By Theorem 1.6, we may assume that $[\![G_{u_h}]\!] \rightharpoonup T$ weakly in $\mathcal{D}_n(U)$, for some $T \in \mathrm{cart}(\Omega;\mathbf{R}^k)$ satisfying $u_T = u$ $\mathcal{L}^n$-a.e. on Ω_r. By the lower semicontinuity of the mass we have

$$\mathbf{M}_U(T) \leq \lim_{h\to\infty} \mathbf{M}_U(\,[\![G_{u_h}]\!]\,) = \lim_{h\to\infty} \mathcal{H}^n(G_{u_h}) = \mathcal{A}(u)\,.$$

Since $\mu_T = p_* T$, we obtain $\mathbf{M}_U(T) = |T|(U) \geq |\mu_T|(\Omega) = |\mu_T|(\Omega_r) + |\mu_T|(\Omega_s)$. By Theorem 1.5 we deduce that

$$|\mu_T|(\Omega_r) = \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\,dx\,, \qquad |\mu_T|(\Omega_s) \geq |Du|(\Omega_s) = |D^s u|(\Omega)\,,$$

and the result follows immediately. □

In the case of the functional $\mathcal{F}$ we obtain the following result.

Theorem 2.8. *For every* $u \in BV(\Omega;\mathbf{R}^k)$ *we have*

$$\mathcal{F}(u) \geq \int_\Omega f\big(x,u(x),\nabla u(x)\big)\,dx \;+\; c_0|D^s u|(\Omega)\,,$$

where c_0 *is the constant in condition* (ii).

Proof. We may assume that $\mathcal{F}(u)<+\infty$; then there exists a sequence (u_h) in $C^1(\Omega;\mathbf{R}^k)$ converging to u strongly in $L^1(\Omega;\mathbf{R}^k)$, and such that $\mathcal{F}(u)=\lim\limits_{h\to\infty}F(u_h)$. By (ii) the sequence $(A(u_h))$ is bounded uniformly with respect to h. Thus, by Theorem 1.6, we may also assume that $[\![G_{u_h}]\!] \rightharpoonup T$ weakly in $\mathcal{D}_n(U)$, for some $T=\tau(M,\theta,\xi)\in \mathrm{cart}(\Omega;\mathbf{R}^k)$ satisfying $u_T=u$ $\mathcal{L}^n$-a.e. on Ω_r. Writing, for simplicity, T_h instead of $[\![G_{u_h}]\!]$, we have

$$F(u_h) \;=\; \int_{\mathbf{R}^n\times\mathbf{R}^k} g(z,\tfrac{dT_h}{d|T_h|}(z))\,d|T_h|(z)$$

where g is the function given by Proposition 2.1. By Reshetnyak's semicontinuity theorem (Theorem 1.1) we have

$$\int_{\mathbf{R}^n\times\mathbf{R}^k} g(z,\tfrac{dT}{d|T|}(z))\,d|T|(z) \;\leq\; \lim_{h\to\infty}\int_{\mathbf{R}^n\times\mathbf{R}^k} g(z,\tfrac{dT_h}{d|T_h|}(z))\,d|T_h|(z) \;=\; \mathcal{F}(u)\,.$$

Therefore

$$\mathcal{F}(u) \;\geq\; \int_{M_r} g(z,\xi(z))\,d\mathcal{H}^n(z) \;+\; \int_{M_s} g(z,\xi(z))\,d\mathcal{H}^n(z)\,. \tag{2.5}$$

Let $p\colon \mathbf{R}^n\times\mathbf{R}^k\to\mathbf{R}^n$ be the canonical projection. Since $\big(p(z),\tilde u_T(p(z))\big)=z$ for $\mathcal{H}^n$-a.e. $z\in M_r$, by Theorem 1.5 we have

$$\begin{aligned}\int_{M_r} g(z,\xi(z))\,d\mathcal{H}^n(z) \;&=\; \int_{\Omega_r} g\big((x,\tilde u_T(x)),\xi(x,\tilde u_T(x))\big)\,d|\mu_T|(x) \;=\\ &=\; \int_\Omega g\left((x,u(x)),\tfrac{\mathcal{M}(\nabla u(x))}{|\mathcal{M}(\nabla u(x))|}\right)|\mathcal{M}(\nabla u(x))|\,dx\,.\end{aligned}$$

Since $g(z,\xi)$ is positively homogeneous of degree one in ξ, we obtain

$$\int_{M_r} g(z,\xi(z))\,d\mathcal{H}^n(z) \;=\; \int_\Omega g\big((x,u(x)),\mathcal{M}(\nabla u(x))\big)\,dx \;=\; \int_\Omega f\big(x,u(x),\nabla u(x)\big)\,dx\,. \tag{2.6}$$

On the other hand, as $|\xi(z)| = 1$ $\mathcal{H}^n$-a.e. on M, the lower bound in Proposition 2.1(d) implies

$$\int_{M_s} g(z,\xi(z))\,d\mathcal{H}^n(z) \;\ge\; c_0\mathcal{H}^n(M_s)\,.$$

Since, by Theorem 1.3, $M_s \simeq M \cap p^{-1}(\Omega_s)$ in the sense of $\mathcal{H}^n$, from Theorem 1.5 we obtain $\mathcal{H}^n(M_s) = p_*(\mathcal{H}^n \llcorner M)(\Omega_s) \ge |\mu_T|(\Omega_s) \ge |Du|(\Omega_s) = |D^s u|(\Omega)$. Therefore

$$\int_{M_s} g(z,\xi(z))\,d\mathcal{H}^n(z) \;\ge\; c_0|D^s u|(\Omega)\,. \tag{2.7}$$

The conclusion follows now from (2.5), (2.6), (2.7). □

The following theorem, which is an easy consequence of Theorem 2.7, shows that the relaxed area functional $\mathcal{A}(u)$ can be represented by an integral when u belongs to the Sobolev space $W^{1,p}(\Omega;\mathbf{R}^k)$, with p large enough. We shall see in Lemma 4.2 that this is no longer true if p is small.

Theorem 2.9. *Let $u \in W^{1,p}(\Omega;\mathbf{R}^k)$, with $p \ge \min\{n,k\}$. Then*

$$\mathcal{A}(u) \;=\; \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\,dx\,.$$

Proof. By Theorem 2.7 we have

$$\mathcal{A}(u) \;\ge\; \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\,dx\,.$$

To prove the opposite inequality, let (u_h) be a sequence in $C^1(\Omega;\mathbf{R}^k)$ converging to u in the strong topology of $W^{1,p}(\Omega;\mathbf{R}^k)$ (see [19]). Since $p \ge \min\{n,k\}$, we have also $\mathcal{M}(\nabla u_h) \to \mathcal{M}(\nabla u)$ in $L^1(\Omega;\Xi)$, hence

$$\mathcal{A}(u) \;\le\; \liminf_{h\to\infty} A(u_h) \;=\; \lim_{h\to\infty} \int_\Omega \big|\mathcal{M}(\nabla u_h(x))\big|\,dx \;=\; \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\,dx\,,$$

which concludes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 2.9 and of the definition of $\mathcal{A}$.

Corollary 2.10. *Assume that* $p \geq \min\{n,k\}$. *Then the functional* $u \mapsto \int_\Omega |\mathcal{M}(\nabla u)|\,dx$ *is lower semicontinuous on* $W^{1,p}(\Omega;\mathbf{R}^k)$ *with respect to the strong topology of* $L^1(\Omega;\mathbf{R}^k)$.

Remark 2.11. Using the idea of Counterexample 7.4 of [2], it is possible to construct an example, which shows that, if $1 \leq p < \min\{n,k\}$, then the functional $u \mapsto \int_\Omega |\mathcal{M}(\nabla u)|\,dx$ is not lower semicontinuous on $W^{1,p}(\Omega;\mathbf{R}^k)$ with respect to the strong topology of $L^1(\Omega;\mathbf{R}^k)$.

In the case of the functional $\mathcal{F}$ we obtain the following result.

Theorem 2.12. *Assume that conditions* (i), (ii), (iii) *are satisfied. Assume, in addition, that the function* $(y,A) \mapsto f(x,y,A)$ *is continuous on* $\mathbf{R}^k \times \mathbf{M}^{k\times n}$ *for* $\mathcal{L}^n$*-a.e.* $x \in \Omega$, *and that there exist two constants* $c_1 > 0$ *and* $p \geq \min\{n,k\}$ *such that* $f(x,y,A) \leq c_1\big(|A|^p + 1\big)$ *for every* $x \in \Omega$, $y \in \mathbf{R}^k$, $A \in \mathbf{M}^{k\times n}$. *Then*

$$\mathcal{F}(u) \;=\; \int_\Omega f(x,u(x),\nabla u(x))\,dx$$

for every $u \in W^{1,p}(\Omega;\mathbf{R}^k)$.

Proof. Let us fix $u \in W^{1,p}(\Omega;\mathbf{R}^k)$. By Theorem 2.8 we have

$$\mathcal{F}(u) \;\geq\; \int_\Omega f(x,u(x),\nabla u(x))\,dx\,.$$

To prove the opposite inequality, we argue as in the proof of Theorem 2.9. Let (u_h) be a sequence in $C^1(\Omega;\mathbf{R}^k)$ converging to u in the strong topology of $W^{1,p}(\Omega;\mathbf{R}^k)$. By the Carathéodory continuity theorem (see, e.g., [26]) we have

$$\mathcal{F}(u) \;\leq\; \lim_{h\to\infty} \int_\Omega f(x,u_h(x),\nabla u_h(x))\,dx \;=\; \int_\Omega f(x,u(x),\nabla u(x))\,dx\,,$$

which concludes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 2.12 and of the definition of $\mathcal{F}$.

Corollary 2.13. *Assume that* $p \geq \min\{n, k\}$. *Under the hypotheses of Theorem* 2.12 *the functional* $u \mapsto \int_\Omega f(x, u, \nabla u)\, dx$ *is lower semicontinuous on* $W^{1,p}(\Omega; \mathbf{R}^k)$ *with respect to the strong topology of* $L^1(\Omega; \mathbf{R}^k)$.

The following theorem concerns piecewise constant functions such that the essential closures of the sets of the underlying partition have no triple intersections. In this case the inequality of Theorem 2.7 becomes an equality. This is no longer true if there are triple intersections, as we shall see in Theorem 3.1. We recall that a set with locally finite perimeter in $\mathbf{R}^n$, or a Caccioppoli set, is a set whose characteristic function belongs to $BV_{\rm loc}(\mathbf{R}^n)$. For the notion of the reduced boundary $\partial^* E$ of a set E with locally finite perimeter we refer to [14], Chapter 3.

Theorem 2.14. *Let* $(E_i)_{i\in I}$ *be a finite partition of* $\mathbf{R}^n$ *composed of sets with locally finite perimeter, let* $(a_i)_{i\in I}$ *be a finite family of points of* $\mathbf{R}^k$, *and let* $u \in BV_{\rm loc}(\mathbf{R}^n; \mathbf{R}^k)$ *be the function defined by* $u(x) = a_i$ *for* $x \in E_i$. *Suppose that for every* $x \in \overline{\Omega}$ *there exists* $r > 0$ *such that* $\mathcal{L}^n(B_r(x) \cap E_i) > 0$ *for at most two indices* i. *Then*

$$\mathcal{A}(u) \,=\, \mathcal{L}^n(\Omega) \,+\, \tfrac{1}{2} \sum_{i,j\in I} |a_i - a_j|\, \mathcal{H}^{n-1}(\partial^* E_i \cap \partial^* E_j \cap \Omega) \,=$$

$$=\, \mathcal{L}^n(\Omega) \,+\, |D^s u|(\Omega) \,=\, \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\, dx \,+\, |D^s u|(\Omega)\,,$$

provided that $\mathcal{L}^n(\partial\Omega) = 0$ *and* $\mathcal{H}^{n-1}(\partial^* E_i \cap \partial\Omega) = 0$ *for every* $i \in I$.

Proof. Let (φ_h) be a sequence of mollifiers, i.e., $\varphi_h \in \mathcal{D}(\mathbf{R}^n)$, $\varphi_h \geq 0$, $\mathrm{supp}(\varphi_h) \subseteq B_{1/h}(0)$, $\int \varphi_h\, dx = 1$, and let $u_h = u * \varphi_h$. Let us prove that $\mathrm{rank}(\nabla u_h(x)) \leq 1$ for every $x \in \overline{\Omega}$. For every $x_0 \in \overline{\Omega}$ there exists $r > 0$ such that $B_r(x_0)$ meets (on a set of positive measure) at most two sets of the partition, say E_i and E_j. By compactness we may assume that r is independent of x_0. For every $x \in B_{r/2}(x_0)$ and for every $h \geq 2/r$ we have

$$u_h(x) = a_i + (a_j - a_i) \int_{E_j} \varphi_h(x - y)\, dy\,,$$

hence

$$\nabla u_h(x) = (a_j - a_i) \otimes \int_{E_j} \nabla\varphi_h(x - y)\, dy\,.$$

This proves that $\mathrm{rank}(\nabla u_h(x)) \leq 1$ for every $x \in \overline{\Omega}$ and for every $h \geq 2/r$. Therefore

$$A(u_h) = \int_\Omega \big|\mathcal{M}(\nabla u_h(x))\big|\, dx \,=\, \int_\Omega \big(1 + |\nabla u_h(x)|^2\big)^{1/2}\, dx\,.$$

By using the results of [15], or by adapting the argument of [20], Theorem 1.8, to the vector case, we obtain that

$$\mathcal{A}(u) \le \liminf_{h\to\infty} A(u_h) = \mathcal{L}^n(\Omega) + |D^s u|(\Omega) = \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\, dx + |D^s u|(\Omega)\,.$$

The opposite inequality follows from Theorem 2.7. □

3. A critical example in dimension two

In this section we prove that, if $n > 1$ and $k > 1$, then there exists a function $u \in BV_{\rm loc}(\mathbf{R}^n, \mathbf{R}^k)$ such that the set function $\mathcal{A}(u,\cdot)$ is not subadditive. Clearly it is enough to consider the case $n = k = 2$.

Theorem 3.1. *Assume that $n = k = 2$. Let us consider a partition of $\mathbf{R}^2$ composed of three non-overlapping non-degenerate angular regions A, B, C, with common vertex at the origin. Let α, β, γ be three non-collinear points in $\mathbf{R}^2$, and let $u\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the function defined by $u(x) = \alpha$, if $x \in A$, $u(x) = \beta$, if $x \in B$, $u(x) = \gamma$, if $x \in C$. Then there exist three bounded open sets, Ω_1, Ω_2, Ω_3, in $\mathbf{R}^2$, such that $\Omega_3 \subset\subset \Omega_1 \cup \Omega_2$ and $\mathcal{A}(u,\Omega_3) > \mathcal{A}(u,\Omega_1) + \mathcal{A}(u,\Omega_2)$.*

For every $r > 0$ we denote by B_r the open ball in $\mathbf{R}^2$ with center 0 and radius r, and, for every $R > r > 0$, we denote by B_r^R the annulus $B_R \setminus \overline{B}_r$.

Lemma 3.2. *Let u be the function defined in Theorem* 3.1. *Then*

$$\mathcal{A}(u, B_r) \le \pi r^2 + 2r(|\alpha-\gamma| + |\beta-\gamma|)$$

for every $r > 0$.

Proof. For every $\varepsilon > 0$ let us consider the sets

$$A_\varepsilon = \{x \in \mathbf{R}^2 : {\rm dist}(x, \mathbf{R}^2 \setminus A) > \varepsilon\}\,, \qquad B_\varepsilon = \{x \in \mathbf{R}^2 : {\rm dist}(x, \mathbf{R}^2 \setminus B) > \varepsilon\}\,.$$

Let $u_\varepsilon\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the function defined by $u_\varepsilon(x) = \alpha$, if $x \in A_\varepsilon$, $u_\varepsilon(x) = \beta$, if $x \in B_\varepsilon$, $u_\varepsilon(x) = \gamma$, if $x \in \mathbf{R}^2 \setminus (A_\varepsilon \cup B_\varepsilon)$. By Theorem 2.14 we obtain

$$\mathcal{A}(u_\varepsilon, B_r) \le \pi r^2 + (2r + \pi\varepsilon)(|\alpha-\gamma| + |\beta-\gamma|)\,.$$

Since (u_ε) converges to u in $L^1(B_r; \mathbf{R}^2)$, the conclusion follows from the lower semicontinuity of $\mathcal{A}(\cdot, B_r)$. □

Lemma 3.3. *Let u be the function defined in Theorem* 3.1. *Then*

$$\mathcal{A}(u, B_r^R) \;=\; \pi(R^2 - r^2) \;+\; (R-r)(|\alpha-\beta| + |\beta-\gamma| + |\gamma-\alpha|)$$

for every $R > r > 0$.

Proof. It is enough to apply Theorem 2.14. □

Lemma 3.4. *Let u be the function defined in Theorem* 3.1. *Then*

$$\mathcal{A}(u, B_R) \;>\; \pi R^2 + \; R(|\alpha-\beta| + |\beta-\gamma| + |\gamma-\alpha|)$$

for every $R > 0$.

Proof. Assume, by contradiction, that

$$\mathcal{A}(u, B_R) \;\leq\; \pi R^2 + \; R(|\alpha-\beta| + |\beta-\gamma| + |\gamma-\alpha|)$$

for some $R > 0$. Then by Theorem 2.7 we have $\mathcal{A}(u, B_R) \;=\; \mathcal{L}^2(B_R) \;+\; |D^s u|(B_R)$. By Lemma 2.3 there exists a sequence (u_h) in $C^1(B_R; \mathbf{R}^2)$, bounded in $L^\infty(B_R; \mathbf{R}^2)$ and converging to u in $L^1(B_R; \mathbf{R}^2)$, such that

$$\mathcal{L}^2(B_R) \;+\; |D^s u|(B_R) \;=\; \lim_{h\to\infty} A(u_h, B_R)\,. \tag{3.1}$$

Since by Theorem 2.7 we have $\liminf_{h\to\infty} A(u_h, \Omega) \geq \mathcal{A}(u, \Omega) \geq \mathcal{L}^2(\Omega) + |D^s u|(\Omega)$ for every open subset Ω of B_R, from the additivity of $A(u_h, \cdot)$ and from (3.1) we obtain that

$$\lim_{h\to\infty} A(u_h, \Omega) \;=\; \mathcal{L}^2(\Omega) \;+\; |D^s u|(\Omega) \tag{3.2}$$

for every open set Ω contained in B_R such that $\mathcal{L}^2(B_R \cap \partial\Omega) = 0$ and $|D^s u|(B_R \cap \partial\Omega) = 0$.

Let $W_R \;=\; B_R \times \mathbf{R}^2$. By Theorem 1.6 we may assume that $[\![G_{u_h}]\!] \rightharpoonup T$ weakly in $\mathcal{D}_2(W_R)$, for some $T \in \mathrm{cart}(W_R)$ satisfying $u_T = u$ $\mathcal{L}^2$-a.e. on B_R. By the lower semicontinuity of the mass we have

$$\mathbf{M}_{W_R}(T) \;\leq\; \lim_{h\to\infty} \mathbf{M}_{W_R}(\,[\![G_{u_h}]\!]\,) \;=\; \lim_{h\to\infty} A(u_h, B_R)\,,$$

hence by (3.1)

$$\mathbf{M}_{W_R}(T) \;\leq\; \pi R^2 + \; R(|\alpha-\beta| + |\beta-\gamma| + |\gamma-\alpha|)\,. \tag{3.3}$$

Let a, b, c be the points on ∂B_R lying on the half lines separating B and C, C and A, A and B, respectively. Suppose that the triple (a,b,c) determines the counterclockwise orientation on ∂B_R, so that $\partial\,[\![A]\!] = [\![0,b]\!] - [\![0,c]\!]$ in B_R, and similar properties hold for B and C. We want to prove that

$$\begin{aligned} T = {} & [\![A]\!]\times[\![\alpha]\!] + [\![B]\!]\times[\![\beta]\!] + [\![C]\!]\times[\![\gamma]\!] + [\![0,a]\!]\times[\![\beta,\gamma]\!] + \\ & + [\![0,b]\!]\times[\![\gamma,\alpha]\!] + [\![0,c]\!]\times[\![\alpha,\beta]\!] + [\![0]\!]\times S \end{aligned} \tag{3.4}$$

on W_R, where S is the integration over the triangle with vertices α, β, γ, oriented in such a way that $\partial S = -\,[\![\alpha,\beta]\!] - [\![\beta,\gamma]\!] - [\![\gamma,\alpha]\!]$. This implies that

$$\mathbf{M}_{W_R}(T) = \pi R^2 + R(|\alpha-\beta| + |\beta-\gamma| + |\gamma-\alpha|) + \mathbf{M}_{\mathbf{R}^2}(S)$$

which contradicts (3.3).

In order to prove (3.4) we consider a covering of $B_R\backslash\{0\}$ composed of three overlapping open angular sectors Ω_a, Ω_b, Ω_c, with vertex at the origin. We shall assume that Ω_a contains the open segment $(0,a)$ and does not intersect the open segments $(0,b)$, $(0,c)$, and that similar properties hold for Ω_b and Ω_c.

Let us prove that

$$T = [\![B]\!]\times[\![\beta]\!] + [\![C]\!]\times[\![\gamma]\!] + [\![0,a]\!]\times[\![\beta,\gamma]\!] \quad \text{on } U_a = \Omega_a\times\mathbf{R}^2\,. \tag{3.5}$$

As $T\in \operatorname{cart}(W_R)$, we can write $T=\tau(M,\theta,\xi)$, where M is a countably 2-rectifiable subset of $W_R = B_R\times\mathbf{R}^2$. Moreover, since $u = u_T$ $\mathcal{L}^2$-a.e. on B_R, we can write $M = M_r\cup M_s$ and $T = T_r + T_s$, with $M_r \simeq G_u$ in the sense of $\mathcal{H}^2$, $T_r = [\![G_u]\!]$, and $\mathcal{L}^2(p(M_s)) = 0$, where p is the canonical projection from $B_R\times\mathbf{R}^2$ onto B_R (see Theorem 1.3 and Remark 1.4). As $[\![G_u]\!] = [\![B]\!]\times[\![\beta]\!] + [\![C]\!]\times[\![\gamma]\!]$ on U_a, in order to prove (3.5) it is enough to show that

$$T_s = [\![0,a]\!]\times[\![\beta,\gamma]\!] \quad \text{on } U_a\,. \tag{3.6}$$

Since $\partial T = 0$ on U_a and $\partial\,[\![G_u]\!] = -\,[\![0,a]\!]\times[\![\beta]\!] + [\![0,a]\!]\times[\![\gamma]\!]$ on U_a, we have that

$$\partial T_s = [\![0,a]\!]\times[\![\beta]\!] - [\![0,a]\!]\times[\![\gamma]\!] \quad \text{on } U_a\,. \tag{3.7}$$

By the lower semicontinuity of the mass and by (3.2) we have

$$\begin{aligned} \mathcal{L}^2(\Omega_a) + \mathbf{M}_{U_a}(T_s) &= \mathbf{M}_{U_a}(T_r) + \mathbf{M}_{U_a}(T_s) = \mathbf{M}_{U_a}(T) \le \\ &\le \lim_{h\to\infty} \mathbf{M}_{U_a}(\,[\![G_{u_h}]\!]\,) = \lim_{h\to\infty} A(u_h,\Omega_a) = \mathcal{L}^2(\Omega_a) + |a|\cdot|\gamma-\beta|\,, \end{aligned}$$

hence

$$\mathbf{M}_{U_a}(T_s) \le |a|\cdot|\gamma-\beta|\,. \tag{3.8}$$

Moreover $\mathcal{L}^2(p(M_s))=0$ and, being (u_h) bounded in $L^\infty(B_R;\mathbf{R}^2)$, the support of T_s is bounded. We shall prove in Lemma 3.8 that these properties, together with (3.7) and (3.8), imply (3.6). From (3.5), and from the analogous statements for Ω_b and Ω_c, we obtain that

$$T = [\![A]\!]\times[\![\alpha]\!] + [\![B]\!]\times[\![\beta]\!] + [\![C]\!]\times[\![\gamma]\!] + [\![0,a]\!]\times[\![\beta,\gamma]\!] + \\ + [\![0,b]\!]\times[\![\gamma,\alpha]\!] + [\![0,c]\!]\times[\![\alpha,\beta]\!]$$

on $(B_R\setminus\{0\})\times\mathbf{R}^2$. Therefore, (3.4) holds for a suitable current $S\in\mathcal{R}_2(\mathbf{R}^2)$ with finite mass. As $\partial T=0$, we have $\partial S = -[\![\alpha,\beta]\!] - [\![\beta,\gamma]\!] - [\![\gamma,\alpha]\!]$. Then, by the constancy theorem, the current S coincides with the integration over the triangle with vertices α, β, γ, with a suitable orientation. This shows that (3.4) holds with the prescribed S and provides the desired contradiction. □

In order to conclude the proof of Lemma 3.4, we have to prove Lemma 3.8. This will require some auxiliary statements (Lemmas 3.5, 3.6, 3.7), which will be proved in the general case $U=\Omega\times\mathbf{R}^k$, Ω bounded open subset of $\mathbf{R}^n$, considered in the previous sections. According to the notation introduced in Section 1, for every $i=1,\dots,n$ we set $\hat\imath=(1,\dots,i-1,i+1,\dots,n)$.

Lemma 3.5. *Let $T=\tau(M,\theta,\xi)$ be an n-dimensional rectifiable current in U with finite mass and with $\partial T=0$. Assume that*

$$\xi(z)\in \operatorname{span}\{e_{\hat\imath}\wedge\varepsilon_1 : i=1,\dots,n\} \tag{3.9}$$

for $\mathcal{H}^n$-a.e. $z\in M$. Then $T=0$.

Proof. By (3.9) we have

$$T(\omega)=\sum_{i=1}^n T^{\hat\imath,1}(\omega_{\hat\imath,1}) \tag{3.10}$$

for every $\omega\in\mathcal{D}(U)$. Therefore

$$T^{\hat\imath,1}\Big(\frac{\partial\varphi}{\partial y^1}\Big) = (-1)^{n-1}T(d\varphi\wedge dx^{\hat\imath}) = (-1)^{n-1}\partial T(\varphi dx^{\hat\imath}) = 0$$

for every $i=1,\dots,n$ and for every $\varphi\in\mathcal{D}(U)$. This implies that the distribution $T^{\hat\imath,1}$ is invariant under translations along the y^1-axis. By (3.10) the same property holds for T. Since T has finite mass, we conclude that $T=0$. □

Lemma 3.6. *Assume that* $k = 1$. *Let* $T = \tau(M,\theta,\xi)$ *be an* n*-dimensional rectifiable current in* U *with finite mass and with* $\partial T = 0$. *If* $\mathcal{L}^n(p(M)) = 0$, *then* $T = 0$.

Proof. If $\mathcal{L}^n(p(M)) = 0$, from the area formula we obtain that for $\mathcal{H}^n$-a.e. $z \in M$ the tangent space $\mathbf{T}_M z$ is projected by p onto a proper subspace of $\mathbf{R}^n$. This means that $\xi^{\hat{0}0} = 0$ $\mathcal{H}^n$-a.e. on M, hence (3.9) is satisfied. The conclusion follows now from Lemma 3.5. □

Lemma 3.7. *Assume that* $k = 1$. *Let* α, $\beta \in \mathbf{R}$, *let* S *be an* $(n-1)$*-dimensional rectifiable current in* Ω, *with finite mass, such that* $\partial S = 0$ *in* Ω, *and let* $T = \tau(M,\theta,\xi)$ *be an* n*-dimensional rectifiable current in* U, *with finite mass, such that* $(-1)^n \partial T = S \times [\![\alpha]\!] - S \times [\![\beta]\!]$ *in* U. *If* $\mathcal{L}^n(p(M)) = 0$, *then* $T = S \times [\![\alpha, \beta]\!]$.

Proof. It is enough to apply Lemma 3.6 to the current $T - (-1)^n S \times [\![\alpha,\beta]\!]$. □

Lemma 3.8. *Assume that* $n = 2$ *and* $k = 2$. *Let* α *and* β *be two points in* $\mathbf{R}^2$, *and let* a *and* b *be two points of* $\partial\Omega \subseteq \mathbf{R}^2$ *such that* Ω *contains the (open) segment between* a *and* b. *Finally, let* $T = \tau(M,\theta,\xi)$ *be a* 2*-dimensional rectifiable current in* $U = \Omega \times \mathbf{R}^2$ *with bounded support. Assume that* $\mathcal{L}^2(p(M)) = 0$, $\partial T = [\![a,b]\!] \times [\![\alpha]\!] - [\![a,b]\!] \times [\![\beta]\!]$, *and* $\mathbf{M}_U(T) \le |b-a| \cdot |\beta - \alpha|$. *Then* $T = [\![a,b]\!] \times [\![\alpha,\beta]\!]$.

Proof. We may assume that $\alpha = 0$ and $\beta = \varepsilon_1$. Let $\pi: U \to \Omega \times \mathbf{R}$ be the projection defined by $\pi(x^1,x^2,y^1,y^2) = (x^1,x^2,y^1)$, and let $q: \Omega \times \mathbf{R} \to \Omega$ be the projection defined by $q(x^1,x^2,y^1) = (x^1,x^2)$. As T has bounded support, the map π is proper on $\mathrm{supp}(T)$, so that $\partial(\pi_\# T) = \pi_\#(\partial T) = [\![a,b]\!] \times [\![0]\!] - [\![a,b]\!] \times [\![1]\!]$. Since T is a 2-dimensional rectifiable current, $\pi_\# T$ is rectifiable too, and we have $\pi_\# T = \tau(N,\zeta,\eta)$, with $N \subseteq \pi(M)$. As $q(N) \subseteq q(\pi(M)) = p(M)$, it follows that $\mathcal{L}^2(q(N)) = 0$. By Lemma 3.7 we conclude that $\pi_\# T = [\![a,b]\!] \times [\![0,1]\!]$, hence $N \simeq [a,b] \times [0,1]$ in the sense of $\mathcal{H}^2$, and $\zeta = 1$, $\eta = \frac{b-a}{|b-a|} \wedge \varepsilon_1$ $\mathcal{H}^2$-a.e. on N. As $|b-a| = \mathcal{H}^2(N) \le \mathcal{H}^2(\pi(M)) \le \mathcal{H}^2(M) \le |b-a|$, we obtain $\pi(M) \simeq [a,b] \times [0,1]$ in the sense of $\mathcal{H}^2$ and, by the area formula, $\xi(z) = \pm \frac{b-a}{|b-a|} \wedge \varepsilon_1$ for $\mathcal{H}^2$-a.e. $z \in M$. If we apply Lemma 3.5 to the current $T - [\![a,b]\!] \times [\![0,\varepsilon_1]\!]$, we obtain immediately $T = [\![a,b]\!] \times [\![0,\varepsilon_1]\!]$. □

Proof of Theorem 3.1. Let us fix $\varrho > 0$. Since, by Lemma 3.4,

$$\mathcal{A}(u, B_\varrho) > \pi\varrho^2 + \varrho(|\alpha - \beta| + |\beta - \gamma| + |\gamma - \alpha|),$$

we can choose r close to 0 and R close to ϱ, with $0 < r < \varrho < R$, so that

$$\pi r^2 + 2r(|\alpha - \gamma| + |\beta - \gamma|) + \pi R^2 + R(|\alpha - \beta| + |\beta - \gamma| + |\gamma - \alpha|) < \mathcal{A}(u, B_\varrho)\,.$$

Then $B_\varrho \subset\subset B_r \cup B^R_{r/2}$ and $\mathcal{A}(u, B_\varrho) > \mathcal{A}(u, B_r) + \mathcal{A}(u, B^R_{r/2})$ by Lemmas 3.2 and 3.3. □

4. An example with Sobolev functions

If $n > 2$ and $k > 2$, it is possible to find an example of non subadditivity for $\mathcal{A}(u, \cdot)$ even among Sobolev functions. Throughout this section we always assume $n = k$. The function u of our example, already considered in [10], Section 3, Example 1, is $u(x) = x/|x|$, which belongs to $W^{1,p}_{\text{loc}}(\mathbf{R}^n; \mathbf{R}^n)$ for all $p < n$. Therefore, the example provided by the Theorem 4.1 and by Lemma 4.2 shows also that the condition $p \geq n$ in Theorem 2.9 can not be dropped.

Theorem 4.1. *Assume that $n = k \geq 3$ and let $u(x) = x/|x|$ for every $x \in \mathbf{R}^n$. Then there exist three bounded open sets, Ω_1, Ω_2, Ω_3, in $\mathbf{R}^n$, such that $\Omega_3 \subset\subset \Omega_1 \cup \Omega_2$ and $\mathcal{A}(u, \Omega_3) > \mathcal{A}(u, \Omega_1) + \mathcal{A}(u, \Omega_2)$.*

The proof of the theorem is based on Lemmas 4.2 and 4.3. Throughout this section we shall use the following notation: for every $r > 0$ we denote by B_r the open ball in $\mathbf{R}^n$ with center 0 and radius r, and by W_r the cylinder $B_r \times \mathbf{R}^n$.

Lemma 4.2. *Assume that $n = k \geq 2$ and let $u(x) = x/|x|$ for every $x \in \mathbf{R}^n$. There exists a constant $r_n > 0$, depending only on n, such that, if $r > r_n$, then*

$$\mathcal{A}(u, B_r) = \mathbf{M}_{W_r}(\,[\![G_u]\!]\,) + \omega_n = \int_{B_r} \big|\mathcal{M}(\nabla u(x))\big|\,dx + \omega_n\,,$$

where ω_n is the measure of the unit ball in $\mathbf{R}^n$.

Proof. As $u \in C^\infty(B_r \setminus \{0\}; \mathbf{R}^n)$, the second equality follows from (1.4). Let us prove that $\mathcal{A}(u, B_r) \leq \mathbf{M}_{W_r}(\,[\![G_u]\!]\,) + \omega_n$. For every $h > 2$ let $\varphi_h\colon \mathbf{R} \to [0, 1]$ be a function of class C^1, with $\varphi_h(t) = 0$ for $t \le 1/h^2$, $\varphi_h(t) = 1$ for $t \ge 1/h$, and $0 \le \varphi'(t) \le 2h$ for every t. Let us define $u_h(x) = \varphi_h(|x|)u(x)$. Then $u_h \in C^1(B_r; \mathbf{R}^n)$ and

$$u_h \to u \quad \text{in } W^{1,p}(B_r; \mathbf{R}^n) \qquad \text{for } p < n\,, \tag{4.1}$$

$$\int_{B_r} |\mathcal{M}^{0\hat{0}}(\nabla u_h)|\,dx \;=\; \int_{B_{1/h}} |\det \nabla u_h|\,dx \;=\; \omega_n \qquad \text{for } h \ge 1/r\,. \tag{4.2}$$

In particular, (u_h) converges to u in $L^1(B_r; \mathbf{R}^n)$; moreover, by (4.1) we have

$$\mathcal{M}^{\alpha\beta}(\nabla u_h) \to \mathcal{M}^{\alpha\beta}(\nabla u) \quad \text{in } L^1(B_r) \tag{4.3}$$

unless $\alpha = 0$ and $\beta = \hat{0} = (1, \ldots, n)$. For any $n \times n$ matrix A let $\mathcal{M}'(A)$ be the n-vector defined by $\mathcal{M}'(A) = \mathcal{M}(A) - \mathcal{M}^{0\hat{0}}(A)\,\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$, so that, with respect to the basis $(e_\alpha \wedge \varepsilon_\beta)$, $\mathcal{M}'(A)$ has the same components as $\mathcal{M}(A)$, except the component $0\hat{0}$, which is 0. Since

$$A(u_h, B_r) \;\le\; \int_{B_r} |\mathcal{M}'(\nabla u_h)|\,dx \;+\; \int_{B_r} |\mathcal{M}^{0\hat{0}}(\nabla u_h)|\,dx$$

and $\mathcal{M}^{0\hat{0}}(\nabla u) = \det(\nabla u) = 0$, by (4.2) and (4.3) we get

$$\mathcal{A}(u, B_r) \;\le\; \int_{B_r} |\mathcal{M}(\nabla u)|\,dx \;+\; \omega_n \;=\; \mathbf{M}_{W_r}(\,[\![G_u]\!]\,) + \omega_n\,. \tag{4.4}$$

To get the opposite inequality, let (v_h) be a sequence of regular functions converging to u in $L^1(B_r; \mathbf{R}^n)$ such that $\mathcal{A}(u, B_r) = \lim_{h\to\infty} A(v_h, B_r)$. By Lemma 2.3 we may assume that (v_h) is bounded in $L^\infty(B_r; \mathbf{R}^n)$. By Theorem 1.6 (applied to the annulus $B_r \setminus B_\varepsilon$ for every $\varepsilon > 0$), we may suppose also that $[\![G_{v_h}]\!] \rightharpoonup T$ weakly in $\mathcal{D}_n(W_r)$ for some $T \in \operatorname{cart}(B_r; \mathbf{R}^n)$ satisfying $T_r = [\![G_u]\!]$. Since (v_h) is bounded in $L^\infty(B_r; \mathbf{R}^n)$, the support of T is contained in the product $\overline{B}_r \times B_R$ for some $R > 0$. As in Theorem 2.4, by the lower semicontinuity of the mass we obtain

$$\mathbf{M}_{W_r}(\,[\![G_u]\!]\,) + \mathbf{M}_{W_r}(T_s) \;=\; \mathbf{M}_{W_r}(T_r) + \mathbf{M}_{W_r}(T_s) \;=\; \mathbf{M}_{W_r}(T) \;\le\; \mathcal{A}(u, B_r)\,, \tag{4.5}$$

so that (4.4) implies

$$\mathbf{M}_{W_r}(T_s) \le \omega_n\,. \tag{4.6}$$

Let us compute the boundary of $[\![G_u]\!]$. As

$$G_u \;=\; \{(x, x/|x|) : x \in \mathbf{R}^n \setminus \{0\}\} \;=\; \{(\varrho y, y) : \varrho > 0,\, y \in \partial B_1\}\,,$$

we can write $G_u = \Phi\big(]0,+\infty[\times\partial B_1\big)$, where $\Phi\colon \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}^n\times\mathbf{R}^n$ is the function defined by $\Phi(\varrho,y)=(\varrho y,y)$. Therefore we have $[\![G_u]\!] = \Phi_\#\big([\![0,+\infty]\!]\times\partial[\![B_1]\!]\big)$, and this implies

$$\partial[\![G_u]\!] \;=\; \Phi_\#\big(\partial([\![0,+\infty]\!]\times\partial[\![B_1]\!])\big) \;=\; \Phi_\#\big(-[\![0]\!]\times\partial[\![B_1]\!]\big) \;=\; -[\![0]\!]\times\partial[\![B_1]\!]\;.$$

Since $\partial T = 0$, we necessarily have

$$\partial T_s = -\partial[\![G_u]\!] \;=\; [\![0]\!]\times\partial[\![B_1]\!]\;. \tag{4.7}$$

So far we have proved that T_s has mass not larger than the volume of a unit n-ball, and its boundary (in W_r) is the boundary of a "vertical" unit n-ball above $x=0$. We will prove that, if the set $\{0\}\times\partial B_1$ is sufficiently far from the boundary of W_r, then

$$\mathbf{M}_{W_r}(T_s) \ge \mathbf{M}_{W_r}(\,[\![0]\!]\times[\![B_1]\!]\,) = \omega_n\,, \tag{4.8}$$

so that (4.5) implies

$$\mathcal{A}(u,B_r) \ge \mathbf{M}_{W_r}(\,[\![G_u]\!]\,) + \omega_n\,, \tag{4.9}$$

which, together with (4.4), concludes the proof of the lemma. Instead, if r is small, then (4.6) might be strict, as we shall see in Lemma 4.3, at least if $n\ge 3$.

If the the projection onto B_r of the support of T_s is compact in B_r, then T_s can be regarded as a current in $\mathbf{R}^n\times\mathbf{R}^n$ with $\mathbf{M}_{\mathbf{R}^n\times\mathbf{R}^n}(T_s)=\mathbf{M}_{W_r}(T_s)$, and its boundary in $\mathbf{R}^n\times\mathbf{R}^n$ is $[\![0]\!]\times\partial[\![B_1]\!]$, so that $T_s = [\![0]\!]\times[\![B_1]\!]$ by the minimality of the disk, and, consequently, (4.8) and (4.9) are proved. Therefore, it is enough to prove that this property of the support of T_s is always satisfied, if r is sufficiently large.

Assume, by contradiction, that the projection onto B_r of the support of T_s is not compact in B_r. Then we slice T_s along cylinders. Set $\psi(x,y)=|x|$; for $\mathcal{L}^1$-a.e. $t\in\,]0,r[$ (more precisely, for every $t\in\,]0,r[$ such that $\mathbf{M}_{W_r}(T_s\llcorner\{\psi=t\})=0$) the slice of T_s is the rectifiable current of dimension $n-1$ defined by

$$\langle T_s,\psi,t\rangle \;=\; -\partial(T_s\llcorner\{\psi>t\})\,.$$

Note that, by (4.7), we have $\mathbf{M}_{W_r}(\partial T_s\llcorner\{\psi=t\}) = 0$ and $(\partial T_s)\llcorner\{\psi>t\} = 0$, so that our definition of $\langle T_s,\psi,t\rangle$ coincides with the classical one (see [25], 28.7). Since $|\nabla\psi|\le 1$, by the properties of the slices (see [25], Lemma 28.5) we get $\partial\langle T_s,\psi,t\rangle = 0$ and

$$\mathbf{M}_{W_r}(T_s\llcorner\{\psi>t\}) \;\ge\; \int_t^r \mathbf{M}_{W_r}(\langle T_s,\psi,\tau\rangle)\,d\tau\,. \tag{4.10}$$

As the support of T_s is contained in $\overline{B}_r \times B_R$, the slices $\langle T_s, \psi, t\rangle$ have compact support in W_r (see [25], 28.8), therefore $\mathbf{M}_{W_r} = \mathbf{M}_{\mathbf{R}^n\times\mathbf{R}^n}$ for the slice, and by the Isoperimetric Theorem (see [25], Theorem 30.1) for $\mathcal{L}^1$-a.e. $t \in \,]0, r[$ there exists a current S_t with support in $\overline{B}_t \times B_R$, boundary $\partial S_t = -\langle T_s, \psi, t\rangle$, and satisfying

$$\mathbf{M}_{\mathbf{R}^n\times\mathbf{R}^n}(S_t) \le \gamma_n \Big[\mathbf{M}_{\mathbf{R}^n\times\mathbf{R}^n}(\langle T_s, \psi, t\rangle)\Big]^{n/(n-1)},$$

where γ_n, the isoperimetric constant, depends only on n. If $\mathbf{M}_{W_r}(T_s \llcorner \{\psi > t\}) > \mathbf{M}_{\mathbf{R}^n\times\mathbf{R}^n}(S_t)$, then the current $(T_s \llcorner \{\psi < t\}) + S_t$ has the same boundary in W_r as T_s, less mass, and compact support in W_r, thus its mass is at least ω_n by (4.7) and by the minimality of the disk. This implies $\mathbf{M}_{W_r}(T_s) > \omega_n$, a contradiction to (4.6). We may therefore assume that

$$\mathbf{M}_{W_r}(T_s \llcorner \{\psi > t\}) \le \mathbf{M}_{\mathbf{R}^n\times\mathbf{R}^n}(S_t) \le \gamma_n \Big[\mathbf{M}_{W_r}(\langle T_s, \psi, t\rangle)\Big]^{n/(n-1)} \tag{4.11}$$

for $\mathcal{L}^1$ almost all $t \in \,]0, r[$. If we set

$$\zeta(t) = \int_t^r \Big[\frac{1}{\gamma_n}\,\mathbf{M}_{W_r}(T_s \llcorner \{\psi > \tau\})\Big]^{1-1/n} d\tau\,,$$

by (4.10) and (4.11) we have

$$\big(-\zeta'(t)\big)^{n/(n-1)} \ge \frac{1}{\gamma_n}\,\zeta(t)\,, \tag{4.12}$$

and the assumption on the support of T_s implies that $\zeta(t) > 0$ for all $t < r$. Moreover, (4.6), (4.10), and (4.11) give $\zeta(0) \le \mathbf{M}_{W_r}(T_s) \le \omega_n$. An easy computation shows that (4.12) implies

$$0 \le \lim_{t\to r}\big(\zeta(t)\big)^{1/n} \le \big(\zeta(0)\big)^{1/n} - \frac{r}{n\gamma_n^{1-1/n}}\,,$$

and a contradiction arises if we take $r > r_n = n\omega_n^{1/n}\gamma_n^{1-1/n}$. This shows that, if $r > r_n$, then the projection onto B_r of the support of T_s is compact in B_r, and this concludes the proof of the lemma. □

We stick to the notation $u(x) = x/|x|$, $W_r = B_r \times \mathbf{R}^n$; the second step in the proof of Theorem 4.1 is provided by the following lemma.

Lemma 4.3. *Let $n = k \ge 3$. Then there exists a constant $c_n > 0$, depending only on n, such that*

$$\mathcal{A}(u, B_r) \le \mathbf{M}_{W_r}(\,[\![G_u]\!]\,) + c_n r \tag{4.13}$$

for every $r > 0$.

Proof. Without loss of generality we prove the proposition only in the case $n = k = 3$. Using spherical coordinates on $\mathbf{R}^3_x$ and rectangular coordinates on $\mathbf{R}^3_y$, the function $u(x) = x/|x|$ is rewritten as

$$\tilde{u}(\varrho, \vartheta, \varphi) = (\sin\vartheta\, \cos\varphi\,,\, \sin\vartheta\, \sin\varphi\,,\, \cos\vartheta)\,,$$

where $\varrho > 0$ is the distance from the origin, $0 \le \vartheta \le \pi$ is the anomaly from the north pole, and $0 \le \varphi < 2\pi$ is the longitude. Fix $0 < r_0 < r$ and $0 < \vartheta_0 < \pi/2$, set

$$g(\varrho) = \begin{cases} \varrho/r_0\,, & \text{if } 0 \le \varrho \le r_0, \\ 1\,, & \text{if } r_0 \le \varrho \le r, \end{cases}$$

$$f(\vartheta) = \begin{cases} \vartheta\,, & \text{if } 0 \le \vartheta \le \pi - \vartheta_0, \\ \frac{\pi - \vartheta_0}{\vartheta_0}(\pi - \vartheta)\,, & \text{if } \pi - \vartheta_0 \le \vartheta \le \pi, \end{cases}$$

and define

$$\tilde{v}(\varrho, \vartheta, \varphi) = \Big(\sin(g(\varrho)f(\vartheta))\cos\varphi\,,\, \sin(g(\varrho)f(\vartheta))\sin\varphi\,,\, \cos(g(\varrho)f(\vartheta))\Big)\,.$$

The function v, corresponding to $\tilde{v}$ in cartesian coordinates, coincides with u on a large portion of B_r, with two exceptions: in a cone of amplitude ϑ_0 about the south pole, which is reversed onto the complement of its outer surface in ∂B_1, and in the ball B_{r_0}, where some smoothing had to be done. We recall that for any function w, when passing to spherical coordinates, we have

$$\iiint |Dw|^2 dx^1 dx^2 dx^3 \;=\; \iiint \Big(|D_\varrho \tilde{w}|^2 + \frac{|D_\vartheta \tilde{w}|^2}{\varrho^2} + \frac{|D_\varphi \tilde{w}|^2}{\varrho^2\, \sin^2\vartheta}\Big)\varrho^2 \sin\vartheta\, d\varrho\, d\vartheta\, d\varphi\,. \tag{4.14}$$

Moreover, since $v(B_r) \subseteq \partial B_1$, we have $\det \nabla v(x) = 0$ everywhere in B_r. To estimate the integral of $|\nabla v - \nabla u|^2$, we remark that, defining $\Gamma_{\vartheta_0} = \{x \in B_r : \pi - \vartheta_0 \le \vartheta \le \pi\}$, the cone about the south pole, we have with an easy computation

$$|D_\varrho(\tilde{v} - \tilde{u})| \begin{cases} = 0 & \text{in } B_r \setminus B_{r_0}, \\ \le c/r_0 & \text{in } B_{r_0}, \end{cases}$$

$$|D_\vartheta(\tilde{v} - \tilde{u})| \begin{cases} = 0 & \text{in } B_r \setminus (B_{r_0} \cup \Gamma_{\vartheta_0}), \\ \le c & \text{in } B_{r_0} \setminus \Gamma_{\vartheta_0}, \\ \le c/\vartheta_0 & \text{in } \Gamma_{\vartheta_0}, \end{cases}$$

$$|D_\varphi(\tilde{v} - \tilde{u})| \begin{cases} = 0 & \text{in } B_r \setminus (B_{r_0} \cup \Gamma_{\vartheta_0}), \\ \le \vartheta & \text{in } B_{r_0} \setminus \Gamma_{\vartheta_0}, \\ \le c\frac{\pi - \vartheta}{\vartheta_0} & \text{in } \Gamma_{\vartheta_0}. \end{cases}$$

Then we have easily by (4.14)

$$\int_{B_r \setminus \Gamma_{\vartheta_0}} |\nabla v - \nabla u|^2 dx \le c\, r_0/\vartheta_0 \,, \qquad \int_{\Gamma_{\vartheta_0}} |\nabla v - \nabla u|^2 dx \le c\, r \,,$$

for some absolute constant c. Now take two positive sequences (r_h) and (ϑ_h), converging to 0 as $h \to \infty$, and such that $\lim_{h\to\infty} r_h/\vartheta_h = 0$, and let v_h be the corresponding functions obtained by taking $r_0 = r_h$ and $\vartheta_0 = \vartheta_h$ in the definition of the functions $g(\varrho)$ and $f(\vartheta)$. Our previous remarks, together with the fact that $\mathcal{L}^3(\Gamma_{\vartheta_h}) \to 0$, imply that (v_h) converges to u in $L^1(B_r; \mathbf{R}^3)$, $\det \nabla v_h = 0$ in B_r, and

$$\lim_{h\to\infty} \int_{B_r \setminus \Gamma_{\vartheta_h}} |\nabla v_h - \nabla u|^2 dx = 0 \,, \tag{4.15}$$

$$\int_{\Gamma_{\vartheta_h}} |\mathcal{M}(\nabla v_h)| \, dx \le c\, r \,. \tag{4.16}$$

As $n = 3$ and $\det \nabla v_h = 0$, (4.15) gives

$$\lim_{h\to\infty} \int_{B_r \setminus \Gamma_{\vartheta_h}} |\mathcal{M}(\nabla v_h) - \mathcal{M}(\nabla u)| \, dx = 0 \,,$$

hence

$$\lim_{h\to\infty} \int_{B_r \setminus \Gamma_{\vartheta_h}} |\mathcal{M}(\nabla v_h)| \, dx = \mathbf{M}_{W_r}(\llbracket G_u \rrbracket) \,. \tag{4.17}$$

As the functions v_h are Lipschitz continuous, by Theorem 2.9 we have

$$\mathcal{A}(v_h, B_r) = \int_{B_r} |\mathcal{M}(\nabla v_h)| \, dx \,,$$

so that, by the lower semicontinuity of $\mathcal{A}(\cdot, B_r)$, (4.13) follows from (4.16) and (4.17). □

Proof of Theorem 4.1. Let r_n and c_n be the constants appearing in Lemmas 4.2 and 4.3, and let $\varrho > r_n$. By Lemma 4.2 we have

$$\mathcal{A}(u, B_\varrho) = \int_{B_\varrho} |\mathcal{M}(\nabla u)| \, dx + \omega_n \,. \tag{4.18}$$

We can choose r close to 0 and R close to ϱ, with $0 < r < \varrho < R$, so that

$$\int_{B_\varrho} |\mathcal{M}(\nabla u)| \, dx + \omega_n > \int_{B_R} |\mathcal{M}(\nabla u)| \, dx + \int_{B_r} |\mathcal{M}(\nabla u)| \, dx + c_n r \,. \tag{4.19}$$

Let us define $\Omega_1 = B_r$, $\Omega_2 = B_R \backslash \overline{B}_{r/2}$, $\Omega_3 = B_\varrho$. Then $\Omega_3 \subset\subset \Omega_1 \cup \Omega_2$. Since u is regular on Ω_2, by Remark 2.2 we have

$$\mathcal{A}(u,\Omega_2) \le \int_{B_R} |\mathcal{M}(\nabla u)|\, dx\,, \tag{4.20}$$

while Lemma 4.3 gives

$$\mathcal{A}(u,\Omega_1) \le \int_{B_r} |\mathcal{M}(\nabla u)|\, dx \,+\, c_n r\,. \tag{4.21}$$

Therefore $\mathcal{A}(u,\Omega_3) > \mathcal{A}(u,\Omega_1) + \mathcal{A}(u,\Omega_2)$ by (4.18), (4.19), (4.20), (4.21). ☐

5. Absolute continuity of the relaxed functional

In this section we examine some conditions on $u \in BV(\Omega;\mathbf{R}^k)$, weaker than those considered in Theorem 2.12, under which the relaxed functional $\mathcal{F}(u,\Omega)$ can be written in the form

$$\mathcal{F}(u,\Omega) = \int_\Omega f\big(x, u(x), \nabla u(x)\big)\, dx\,. \tag{5.1}$$

In particular, we shall prove that, if

$$\mathcal{A}(u,\Omega) = \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\, dx \,<\, +\infty\,, \tag{5.2}$$

then (5.1) holds under very weak assumptions on f.

In addition to conditions (i), (ii), (iii), (iv) of Section 2, we shall now assume that

(v) for every $x_0 \in \Omega$, $y_0 \in \mathbf{R}^k$, $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|f(x_0,y,A) - f(x_0,y_0,A)\big| \le \varepsilon\big(|\mathcal{M}(A)| + 1\big)$$

for every $A \in \mathbf{M}^{k\times n}$ and for every $y \in \mathbf{R}^k$ with $|y - y_0| < \delta$;

(vi) there exists a constant $c_1 > 0$ such that $f(x,y,A) \le c_1\big(|\mathcal{M}(A)| + 1\big)$ for every $x \in \Omega$, $y \in \mathbf{R}^k$, $A \in \mathbf{M}^{k\times n}$.

We begin by improving the results of Proposition 2.1. We recall that Ξ is the space of all n-vectors of $\mathbf{R}^n{\times}\mathbf{R}^k$. We shall consider also the hyperplane $\Xi^{\hat{0}0} = \{\xi \in \Xi : \xi^{\hat{0}0} = 1\}$ and the half-space $\Xi^+ = \{\xi \in \Xi : \xi^{\hat{0}0} \ge 0\}$. The following lemma will be used to obtain upper bounds for the function g introduced in the proof of Proposition 2.1.

Lemma 5.1. *For every $\xi \in \Xi^{\hat{0}0}$ there exist a finite family $(A^i)_{i\in I}$ in $\mathbf{M}^{k\times n}$ and a finite family $(\lambda^i)_{i\in I}$ of positive real numbers such that*

$$\sum_{i\in I}\lambda^i = 1\,, \qquad \sum_{i\in I}\lambda^i\mathcal{M}(A^i) = \xi\,, \qquad \sum_{i\in I}\lambda^i|\mathcal{M}(A^i)| \le c|\xi|\,, \tag{5.3}$$

where $c = c_{n,k} \ge 1$ is a constant depending only on n and k.

Proof. Given an integer h between 0 and $m = \min\{n,k\}$, we shall consider the space Ξ_h of all n-vectors of $\mathbf{R}^n{\times}\mathbf{R}^k$ of the form

$$\xi = \sum_{\substack{|\alpha|+|\beta|=n\\ |\beta|\le h}} \xi^{\alpha\beta}\, e_\alpha\wedge\varepsilon_\beta\,,$$

and the affine subspace $\Xi^{\hat{0}0}_h = \Xi_h\cap\Xi^{\hat{0}0} = \{\xi\in\Xi_h : \xi^{\hat{0}0} = 1\}$. It is clear that $\Xi_m = \Xi$, so that $\Xi^{\hat{0}0}_m = \Xi^{\hat{0}0}$, while $\Xi_0 = \{te^{\hat{0}} : t\in\mathbf{R}\}$, where $e^{\hat{0}} = e^1\wedge\cdots\wedge e^n$, so that $\Xi^{\hat{0}0}_0 = \{e^{\hat{0}}\}$.

We want to prove, by induction on h, that (5.3) holds for every $\xi\in\Xi^{\hat{0}0}_h$, with a constant $c = c_{n,k,h}$. As $\Xi^{\hat{0}0}_0 = \{e^{\hat{0}}\}$ and $\mathcal{M}(0) = e^{\hat{0}}$, the proposition is true for $h=0$, with $c_{n,k,0} = 1$. Suppose that it is true for a given h, with $0\le h < m$. We want to prove that (5.3) holds, with a different constant c, for every $\xi\in\Xi^{\hat{0}0}_{h+1}$. Let us fix $\xi\in\Xi^{\hat{0}0}_{h+1}$. Then we can write

$$\xi = \sum_{\substack{|\alpha|=n-h-1\\ |\beta|=h+1}} \xi^{\alpha\beta}\, e_\alpha\wedge\varepsilon_\beta + \zeta\,,$$

with $\zeta\in\Xi^{\hat{0}0}_h$. Let $r = \binom{n}{h+1}\binom{k}{h+1}+1$. Given $|\alpha| = n-h-1$ and $|\beta| = h+1$, we can construct a matrix $A\in\mathbf{M}^{k\times n}$ with $\epsilon(\alpha)A_{\beta_1\hat{\alpha}_1} = r\xi^{\alpha\beta}$, with $A_{\beta_i\hat{\alpha}_i} = 1$ for $i = 2,\dots,h+1$, and with all other entries equal to zero. Let us denote this matrix by $A^{\alpha\beta}$. Then

$$\mathcal{M}(A^{\alpha\beta}) = r\xi^{\alpha\beta}\, e_\alpha\wedge\varepsilon_\beta + \eta_{\alpha\beta}\,,$$

with $\eta_{\alpha\beta}\in\Xi^{\hat{0}0}_h$, and $|\mathcal{M}(A^{\alpha\beta})| \le r\,2^h|\xi^{\alpha\beta}| + 2^h \le r\,2^{h+1}|\xi|$. This implies that

$$\xi = \sum_{\substack{|\alpha|=n-h-1\\ |\beta|=h+1}} \frac{\mathcal{M}(A^{\alpha\beta})}{r} + \frac{\eta}{r}\,,$$

where $\eta\in\Xi^{\hat{0}0}_h$ and $|\eta| \le r(r\,2^{h+1}+1)|\xi|$. By the inductive hypothesis there exist a finite family $(A^i)_{i\in I}$ in $\mathbf{M}^{k\times n}$ and a finite family $(\lambda^i)_{i\in I}$ of positive real numbers such that

$$\sum_{i\in I}\lambda^i = 1\,, \qquad \sum_{i\in I}\lambda^i\mathcal{M}(A^i) = \eta\,, \qquad \sum_{i\in I}\lambda^i|\mathcal{M}(A^i)| \le c_{n,k,h}|\eta|\,.$$

Let J be the disjoint union of the sets I and $H = \{(\alpha, \beta) : |\alpha| = n-h-1,\ |\beta| = h+1\}$, and let $(B^j)_{j\in J}$ be the family in $\mathbf{M}^{k\times n}$ defined by $B^j = A^j$, if $j \in I$, and by $B^j = A^{\alpha\beta}$, if $j = (\alpha, \beta) \in H$. Finally, let $(\mu^j)_{j\in J}$ be the family of positive real numbers defined by $\mu^j = \lambda^j/r$, if $j \in I$, and by $\mu^j = 1/r$, if $j \in H$. As H has exactly $r-1$ elements, we obtain

$$\sum_{j\in J}\mu^j = 1, \qquad \sum_{j\in J}\mu^j\mathcal{M}(A^j) = \frac{1}{r}\sum_{i\in I}\lambda^i\mathcal{M}(A^i) + \sum_{\substack{|\alpha|=n-h-1\\|\beta|=h+1}}\frac{\mathcal{M}(A^{\alpha\beta})}{r} = \xi,$$

$$\sum_{j\in J}\mu^j|\mathcal{M}(A^j)| \le \frac{1}{r}\sum_{i\in I}\lambda^i|\mathcal{M}(A^i)| + \frac{1}{r}\sum_{\substack{|\alpha|=n-h-1\\|\beta|=h+1}}|\mathcal{M}(A^{\alpha\beta})| \le$$

$$\le \frac{c_{n,k,h}}{r}|\eta| + r\,2^{h+1}|\xi| \le \big((r\,2^{h+1}+1)c_{n,k,h} + r\,2^{h+1}\big)|\xi|,$$

which concludes the proof of (5.3) in the case $h+1$. ☐

The following proposition improves the results of Proposition 2.1. As in the previous sections, the cylinder $\Omega\times\mathbf{R}^k$ will be denoted by U.

Proposition 5.2. *Assume that f satisfies conditions* (i)–(vi). *Then there exists a function $g\colon U\times\Xi \to [0, +\infty]$ such that:*

(a) *$f(x, y, A) = g(z, \mathcal{M}(A))$ for every $z = (x, y) \in U$ and for every $A \in \mathbf{M}^{k\times n}$;*

(b) *the function g is lower semicontinuous on $U\times\Xi$, and, for every $x \in \Omega$, the function $(y, \xi) \mapsto g((x, y), \xi)$ is continuous on $\mathbf{R}^k\times\Xi^+$;*

(c) *for every $z \in U$ the function $\xi \mapsto g(z, \xi)$ is convex and positively homogeneous of degree one on Ξ;*

(d) *there exists a constant $C_1 \ge c_0$ such that $c_0|\xi| \le g(z, \xi) \le C_1|\xi|$ for every $z \in U$ and for every $\xi \in \Xi^+$, where c_0 is the constant in condition* (ii).

Proof. Let g and g_0 be the functions defined in the proof of Proposition 2.1. Then (a) and (c) follow from Proposition 2.1, together with the lower semicontinuity of g and the lower bound in (d). The upper bound in (d) follows from Lemma 5.1.

By (ii) and (v) for every $x_0 \in \Omega$, $y_0 \in \mathbf{R}^k$, $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(x_0, y, A) \le (1+\varepsilon)f(x_0, y_0, A) + \varepsilon \qquad \text{and} \qquad f(x_0, y_0, A) \le (1+\varepsilon)f(x_0, y, A) + \varepsilon$$

for every $A \in \mathbf{M}^{k\times n}$ and for every $y \in \mathbf{R}^k$ with $|y - y_0| < \delta$. By Lemma 5.1 and by Proposition 2.1(d) for every $x_0 \in \Omega$, $y_0 \in \mathbf{R}^k$, $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|g_0((x_0, y), \xi) - g_0((x_0, y_0), \xi)\big| \le \varepsilon\big(|\xi| + 1\big)$$

for every $\xi \in \Xi^{\hat{0}0}$ and for every $y \in \mathbf{R}^k$ with $|y - y_0| < \delta$. By the definition of g this implies that for every $x_0 \in \Omega$, the function $(y, \xi) \mapsto g((x_0, y), \xi)$ is continuous on $\mathbf{R}^k \times \Xi^+$. □

The following lemma will be used in the proof of Theorem 5.4.

Lemma 5.3. *Let (v_h) be a sequence in $L^1(\Omega, \mathbf{R}^m)$ which converges in the weak* topology of $\mathcal{M}(\Omega; \mathbf{R}^m)$ to a function $v \in L^1(\Omega, \mathbf{R}^m)$. Assume that*

$$\int_\Omega \big(1 + |v(x)|^2\big)^{1/2}\, dx = \lim_{h\to\infty} \int_\Omega \big(1 + |v_h(x)|^2\big)^{1/2}\, dx\,.$$

Then (v_h) converges to v in the strong topology of $L^1(\Omega, \mathbf{R}^m)$.

Proof. Let u_h, $u\colon \Omega \to \mathbf{R}^{m+1} = \mathbf{R}^m \times \mathbf{R}$ be defined by $u_h(x) = (v_h(x), 1)$ and $u(x) = (v(x), 1)$. Then (u_h) converges to u in the weak* topology of $\mathcal{M}(\Omega; \mathbf{R}^{m+1})$ and

$$\int_\Omega |u(x)|\, dx = \lim_{h\to\infty} \int_\Omega |u_h(x)|\, dx\,. \tag{5.4}$$

Given $\varphi \in C^0_c(\Omega; \mathbf{R}^m)$, let us define the continuous function $\psi\colon \Omega \times \mathbf{R}^{m+1} \to \mathbf{R}$ by $\psi(x, \zeta) = |\hat{\zeta} - \zeta_{m+1}\varphi(x)|$, where $\hat{\zeta} = (\zeta_1, \ldots, \zeta_m)$. Since ψ is convex and positively homogeneous of degree one with respect to ζ, from (5.4) and from Reshetnyak's continuity theorem (Theorem 1.2) it follows that

$$\int_\Omega \psi(x, u(x))\, dx = \lim_{h\to\infty} \int_\Omega \psi(x, u_h(x))\, dx\,,$$

hence

$$\int_\Omega |v(x) - \varphi(x)|\, dx = \lim_{h\to\infty} \int_\Omega |v_h(x) - \varphi(x)|\, dx \tag{5.5}$$

for every $\varphi \in C^0_c(\Omega; \mathbf{R}^m)$. As $C^0_c(\Omega; \mathbf{R}^m)$ is dense in $L^1(\Omega, \mathbf{R}^m)$, it is easy to prove that (5.5) holds true for every $\varphi \in L^1(\Omega, \mathbf{R}^m)$. The conclusion follows now by taking $\varphi = v$. □

Theorem 5.4. *Let $u \in BV(\Omega;\mathbf{R}^k)$. The following conditions are equivalent:*

(a) $\mathcal{A}(u,\Omega) = \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\, dx < +\infty$;

(b) $u \in W^{1,1}(\Omega;\mathbf{R}^k)$, $\mathcal{M}(\nabla u) \in L^1(\Omega;\Xi)$, *and there exists a sequence* (u_h) *in* $C^1(\Omega;\mathbf{R}^k)$, *converging to* u *in* $L^1(\Omega;\mathbf{R}^k)$, *such that* $\big(\mathcal{M}(\nabla u_h)\big)$ *converges to* $\mathcal{M}(\nabla u)$ *in* $L^1(\Omega;\Xi)$.

Proof. It is clear that (b) implies (a) by Theorem 2.7. Let us prove the converse. Assume (a). Then $\mathcal{M}(\nabla u) \in L^1(\Omega;\Xi)$. Moreover $D^s u = 0$ by Theorem 2.7, hence $u \in W^{1,1}(\Omega;\mathbf{R}^k)$. By Remark 2.2(b) there exists a sequence (u_h) in $C^1(\Omega;\mathbf{R}^k)$, converging strongly to u in $L^1(\Omega;\mathbf{R}^k)$, such that

$$\int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\, dx = \lim_{h\to\infty} \int_\Omega \big|\mathcal{M}(\nabla u_h(x))\big|\, dx\,.$$

By Theorem 1.6 we may assume that $[\![G_{u_h}]\!] \rightharpoonup T$ weakly in $\mathcal{D}_n(U)$, for some $T \in \operatorname{cart}(\Omega;\mathbf{R}^k)$ satisfying $u_T = u$ $\mathcal{L}^n$-a.e. on Ω_r. By the lower semicontinuity of the mass and by Theorem 1.5 we have

$$\begin{aligned}\mathbf{M}_U(T_r) + \mathbf{M}_U(T_s) = \mathbf{M}_U(T) \le \lim_{h\to\infty} \mathbf{M}_U(\,[\![G_{u_h}]\!]\,) = \lim_{h\to\infty} A(u_h,\Omega) = \\ = \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\, dx = \mathbf{M}_U(T_r)\,,\end{aligned}$$

hence $T_s = 0$, $T = T_r$, and for every $\omega \in \mathcal{D}^n(U)$ we have

$$T(\omega) = \int_\Omega \big\langle \mathcal{M}(\nabla u(x)), \omega(x,u(x))\big\rangle\, dx$$

by Theorem 1.3(g). This implies that

$$\mathbf{M}_U(T) = \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\, dx\,,$$

thus (a) gives $\mathbf{M}_U(T) = \lim_{h\to\infty} \mathbf{M}_U(T_h)$. As (T_h) converges to T weakly in $\mathcal{D}_n(U)$, we conclude that $T(\omega) = \lim_{h\to\infty} T_h(\omega)$ for every bounded continuous function $\omega: U \to \Xi'$, where Ξ' is the space of all n-covectors of $\mathbf{R}^n \times \mathbf{R}^k$. In particular

$$\int_\Omega \big\langle \mathcal{M}(\nabla u(x)), \varphi(x)\big\rangle\, dx = \lim_{h\to\infty} \int_\Omega \big\langle \mathcal{M}(\nabla u_h(x)), \varphi(x)\big\rangle\, dx$$

for every bounded continuous function $\varphi: \Omega \to \Xi'$. Therefore $\big(\mathcal{M}(\nabla u_h)\big)$ converges to $\mathcal{M}(\nabla u)$ in the weak* topology of $\mathcal{M}(\Omega;\Xi)$. Since $\mathcal{M}^{\hat{0}0}(\nabla u_h) = \mathcal{M}^{\hat{0}0}(\nabla u) = 1$, Lemma 5.3 implies that $\big(\mathcal{M}(\nabla u_h)\big)$ converges to $\mathcal{M}(\nabla u)$ in $L^1(\Omega;\Xi)$. ☐

Remark 5.5. The proof of Theorem 5.4, together with Lemma 2.3, shows that, if $\mathcal{A}(u,\Omega) = \int_\Omega |\mathcal{M}(\nabla u(x))|\,dx$ and $u \in BV(\Omega;\mathbf{R}^k) \cap L^\infty(\Omega;\mathbf{R}^k)$, then there exists a sequence (u_h) in $C^1(\Omega;\mathbf{R}^k)$, bounded in $L^\infty(\Omega;\mathbf{R}^k)$ and converging to u in $L^1(\Omega;\mathbf{R}^k)$, such that $\big(\mathcal{M}(\nabla u_h)\big)$ converges to $\mathcal{M}(\nabla u)$ in $L^1(\Omega;\Xi)$.

Remark 5.6. The proof of Theorem 5.4 shows that, if $\mathcal{A}(u,\Omega) = \int_\Omega |\mathcal{M}(\nabla u(x))|\,dx$, then the current

$$T(\omega) = \int_\Omega \big\langle \mathcal{M}(\nabla u(x)), \omega(x,u(x))\big\rangle\,dx\,,$$

which is well defined by Theorem 5.4(b), belongs to $\mathrm{cart}(\Omega;\mathbf{R}^k)$. A conterexample in [13] shows that the converse is not true, even if $u \in W^{1,p}(\Omega;\mathbf{R}^k)$ for every $p < \min\{n,k\}$.

Remark 5.7. Theorem 5.4 implies that, if $\mathcal{A}(u,\Omega) = \int_\Omega |\mathcal{M}(\nabla u(x))|\,dx < +\infty$ for some open set Ω, then the same property holds for every open subset of Ω.

The following proposition shows that (5.2) implies (5.1) under very weak assumptions on f.

Proposition 5.8. *Assume that f satisfies conditions* (i)–(vi). *Then* (5.2) *implies* (5.1).

Proof. Assume that f satisfies (i)–(vi) and let g be the function given by Lemma 5.2. Then the functional

$$(v,w) \mapsto \int_\Omega g\big((x,v(x)),w(x)\big)\,dx \tag{5.6}$$

is continuous on $L^1(\Omega;\mathbf{R}^k)\times L^1(\Omega;\Xi^+)$ by the Carathéodory continuity theorem (see, e.g., [26]). If (5.2) holds, then by Theorem 5.4 there exists a sequence (u_h) in $C^1(\Omega;\mathbf{R}^k)$, converging to u strongly in $L^1(\Omega;\mathbf{R}^k)$, such that $\big(\mathcal{M}(\nabla u_h)\big)$ converges to $\mathcal{M}(\nabla u)$ in $L^1(\Omega;\Xi)$. By the continuity of (5.6) we have

$$\lim_{h\to\infty}\int_\Omega g\big((x,u_h(x)),\mathcal{M}(\nabla u_h(x))\big)\,dx = \int_\Omega g\big((x,u(x)),\mathcal{M}(\nabla u(x))\big)\,dx\,,$$

so that

$$\mathcal{F}(u,\Omega) \le \int_\Omega f\big(x,u(x),\nabla u(x)\big)\,dx\,.$$

As the opposite inequality is given by Theorem 2.8, we obtain (5.1). □

6. Further properties in dimension two

Throughout this section we shall assume $n = 2$. Given a bounded open set $\Omega_0 \subseteq \mathbf{R}^2$ and a function $u \in BV(\Omega_0; \mathbf{R}^k) \cap L^\infty(\Omega_0; \mathbf{R}^k)$, we shall prove a stability result for the class $\mathcal{E}(u, \Omega_0)$ of all open subsets Ω of Ω_0 such that

$$\mathcal{A}(u, \Omega) \;=\; \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\, dx\,. \tag{6.1}$$

The main result of this section is the following theorem, which can be regarded as an improvement, in the special case $n = 2$, of the stability result stated in Remark 5.7 .

Theorem 6.1. *Let Ω_1 and Ω_2 be two bounded open subsets of $\mathbf{R}^2$ let u be a function of $BV(\Omega_1 \cup \Omega_2; \mathbf{R}^k) \cap L^\infty(\Omega_1 \cup \Omega_2; \mathbf{R}^k)$ such that $\mathcal{A}(u, \Omega_1) = \int_{\Omega_1} |\mathcal{M}(\nabla u(x))|\, dx$ and $\mathcal{A}(u, \Omega_2) = \int_{\Omega_2} |\mathcal{M}(\nabla u(x))|\, dx$. Then*

$$\mathcal{A}(u, \Omega) \;=\; \int_\Omega \big|\mathcal{M}(\nabla u(x))\big|\, dx$$

for every open set Ω with $\Omega \subset\subset \Omega_1 \cup \Omega_2$.

We begin by proving the theorem in the case of two polyrectangles with a nice intersection, according to the following definition.

Definition 6.2. A *polyrectangle* of $\mathbf{R}^2$ is any finite union of open rectangles of $\mathbf{R}^2$ with sides parallel to the axes. Two polyrectangles A and B of $\mathbf{R}^2$ are said to have a *nice intersection* if, after renumbering the coordinate axes and changing, if necessary, their orientation, there exist a strip $S = \{x \in \mathbf{R}^2 : a < x^1 < b\}$, corresponding to an open interval $I =]a, b[$, and an open subset C of $\mathbf{R}$, such that $A \subseteq \{x \in \mathbf{R}^2 : x^1 > a\}$, $B \subseteq \{x \in \mathbf{R}^2 : x^1 < b\}$, and $A \cap S = B \cap S = I \times C$.

The following proposition deals with the special case of two polyrectangles with a nice intersection.

Proposition 6.3. *Let Ω_1 and Ω_2 be two polyrectangles of $\mathbf{R}^2$ with a nice intersection and let $u \in BV(\Omega_1 \cup \Omega_2; \mathbf{R}^k) \cap L^\infty(\Omega_1 \cup \Omega_2; \mathbf{R}^k)$ be a function such that $\mathcal{A}(u, \Omega_1) = \int_{\Omega_1} |\mathcal{M}(\nabla u(x))|\, dx$ and $\mathcal{A}(u, \Omega_2) = \int_{\Omega_2} |\mathcal{M}(\nabla u(x))|\, dx$. Then*

$$\mathcal{A}(u, \Omega_1 \cup \Omega_2) \;=\; \int_{\Omega_1 \cup \Omega_2} \big|\mathcal{M}(\nabla u(x))\big|\, dx\,. \tag{6.2}$$

To prove the proposition we need the following lemma.

Lemma 6.4. *Assume that* $\Omega = A\times B$, *where* A *and* B *are open subsets of* $\mathbf{R}$. *Then for every* $u \in L^1(\Omega;\mathbf{R}^k)$ *and for every* $\varepsilon > 0$ *there exists closed set* $A_\varepsilon \subseteq A$, *with* $\mathcal{L}^1(A \setminus A_\varepsilon) < \varepsilon$, *such that*

$$\lim_{h\to\infty} \int_B |u(x^1_h, x^2) - u(x^1_0, x^2)|\, dx^2 \;=\; 0 \tag{6.3}$$

for every sequence (x^1_h) *in* A_ε *converging to a point* $x^1_0 \in A_\varepsilon$.

Proof. Let $u \in L^1(\Omega;\mathbf{R}^k)$ and let $v\colon A \to L^1(B;\mathbf{R}^k)$ be the function defined by $v(x^1) = u(x^1,\cdot)$. Then v is Bochner integrable on A. By Lusin's theorem, there exists a closed set A_ε, with $\mathcal{L}^1(A\setminus A_\varepsilon) < \varepsilon$, such that the restriction of v to A_ε is continuous, and this implies (6.3). ☐

Proof of Proposition 6.3. By Theorem 5.4 we have $u \in W^{1,1}(\Omega_1 \cup \Omega_2;\mathbf{R}^k)$ and $\mathcal{M}(\nabla u) \in L^1(\Omega_1\cup\Omega_2;\Xi)$, where Ξ is now the set of all 2-vectors of $\mathbf{R}^2\times\mathbf{R}^k$. Moreover, by Theorem 5.4 and by Remark 5.5, there exists a sequence (v_h) in $C^1(\Omega_1;\mathbf{R}^k)$, bounded in $L^\infty(\Omega_1;\mathbf{R}^k)$ and converging to u strongly in $L^1(\Omega_1;\mathbf{R}^k)$, such that $\big(\mathcal{M}(\nabla v_h)\big)$ converges to $\mathcal{M}(\nabla u)$ strongly in $L^1(\Omega_1;\Xi)$. Similarly, there exists a sequence (w_h) in $C^1(\Omega_2;\mathbf{R}^k)$, bounded in $L^\infty(\Omega_2;\mathbf{R}^k)$ and converging to u strongly in $L^1(\Omega_2;\mathbf{R}^k)$, such that $\big(\mathcal{M}(\nabla w_h)\big)$ converges to $\mathcal{M}(\nabla u)$ strongly in $L^1(\Omega_2;\Xi)$. We want to construct a sequence (z_h), converging to u (or to a function sufficiently close to u), with $z_h = v_h$ on $\Omega_1 \setminus \Omega_2$ and $z_h = w_h$ on $\Omega_2 \setminus \Omega_1$, such that the integrals of the functions $|\mathcal{M}(\nabla z_h)|$ on the sets $\{z_h \neq v_h\}\cap\{z_h \neq w_h\}\cap\Omega$ are sufficiently small. We shall see later that this implies (6.2).

We may assume that there exist a strip $S = \{x \in \mathbf{R}^2 : a < x^1 < b\}$, corresponding to an an open interval $I =]a,b[$, and an open subset B of $\mathbf{R}$, such that $\Omega_1 \subseteq \{x \in \mathbf{R}^2 : x^1 < b\}$, $\Omega_2 \subseteq \{x \in \mathbf{R}^2 : x^1 > a\}$, and $\Omega_1 \cap \Omega_2 = \Omega_1 \cap S = \Omega_2 \cap S = I\times B$. As $D_1 u \in L^1(I\times B;\mathbf{R}^k)$, there exists a Borel set $A_0 \subseteq I$, with $\mathcal{L}^1(I \setminus A_0) = 0$ such that

$$\lim_{m\to\infty} \int_B \big|u(t_m, x^2) - u(t, x^2)\big|\, dx^2 \;=\; \lim_{m\to\infty} \int_B \int_t^{t_m} \big|D_1 u(x^1, x^2)\big|\, dx^1\, dx^2 \;=\; 0 \tag{6.4}$$

for every decreasing sequence (t_m) in A_0 converging to a point $t \in A_0$. By Lemma 6.4 there exists a Borel set $A \subseteq A_0$, with $\mathcal{L}^1(A) > 0$, such that

$$\int_B \big|\mathcal{M}(\nabla u(t, x^2))\big|\, dx^2 \;<\; +\infty\,, \tag{6.5}$$

$$\lim_{m\to\infty} \int_B \big|\mathcal{M}(\nabla u(t_m, x^2)) - \mathcal{M}(\nabla u(t, x^2))\big|\, dx^2 \;=\; 0 \tag{6.6}$$

for every sequence (t_m) in A converging to a point $t \in A$. By Fubini's theorem, passing, if necessary, to a subsequence, we may also assume that for every $x^1 \in A$

$$\int_B |u(x^1,x^2)|\,dx^2 \;<\; +\infty\,, \tag{6.7}$$

$$\lim_{h\to\infty} \int_B |v_h(x^1,x^2) - u(x^1,x^2)|\,dx^2 \;=\; 0\,, \tag{6.8}$$

$$\lim_{h\to\infty} \int_B |w_h(x^1,x^2) - u(x^1,x^2)|\,dx^2 \;=\; 0\,, \tag{6.9}$$

$$\lim_{h\to\infty} \int_B \big|\mathcal{M}(\nabla v_h(x^1,x^2)) - \mathcal{M}(\nabla u(x^1,x^2))\big|\,dx^2 \;=\; 0\,, \tag{6.10}$$

$$\lim_{h\to\infty} \int_B \big|\mathcal{M}(\nabla w_h(x^1,x^2)) - \mathcal{M}(\nabla u(x^1,x^2))\big|\,dx^2 \;=\; 0\,. \tag{6.11}$$

Let us fix a decreasing sequence (t_m) in A, converging to a point $t \in A$, and let us fix $m \in \mathbf{N}$. We define the sequence (z_h) in $W^{1,\infty}(\Omega_1 \cup \Omega_2; \mathbf{R}^k)$ by setting

$$z_h(x^1,x^2) = \begin{cases} v_h(x^1,x^2)\,, & \text{if } (x^1,x^2) \in \Omega_1 \text{ and } x^1 < t, \\ \frac{x^1-t_m}{t-t_m} v_h(t,x^2) \;+\; \frac{x^1-t}{t_m-t} w_h(t_m,x^2)\,, & \text{if } t \le x^1 \le t_m \text{ and } x^2 \in B, \\ w_h(x^1,x^2)\,, & \text{if } (x^1,x^2) \in \Omega_2 \text{ and } x^1 > t_m. \end{cases}$$

By (6.8) and (6.9) the sequence (z_h) converges in $L^1(\Omega_1 \cup \Omega_2; \mathbf{R}^k)$ to the function u_m defined by

$$u_m(x^1,x^2) = \begin{cases} u(x^1,x^2)\,, & \text{if } (x^1,x^2) \in \Omega_1 \text{ and } x^1 < t, \\ \frac{x^1-t_m}{t-t_m} u(t,x^2) \;+\; \frac{x^1-t}{t_m-t} u(t_m,x^2)\,, & \text{if } t \le x^1 \le t_m \text{ and } x^2 \in B, \\ u(x^1,x^2)\,, & \text{if } (x^1,x^2) \in \Omega_2 \text{ and } x^1 > t_m. \end{cases}$$

Moreover, setting $\Omega_1^t = \{x \in \Omega_1 : x^1 < t\}$ and $\Omega_2^{t_m} = \{x \in \Omega_2 : x^1 > t_m\}$, we have

$$\begin{aligned} \liminf_{h\to\infty} \int_{\Omega_1\cup\Omega_2} \big|\mathcal{M}(\nabla z_h(x))\big|\,dx \;&\le\; \lim_{h\to\infty} \int_{\Omega_1^t} \big|\mathcal{M}(\nabla v_h(x))\big|\,dx \,+ \\ +\; \lim_{h\to\infty} \int_{\Omega_2^{t_m}} \big|\mathcal{M}(\nabla w_h(x))\big|\,dx \;+\; \varepsilon_m \;&\le\; \int_{\Omega_1\cup\Omega_2} \big|\mathcal{M}(\nabla u(x))\big|\,dx \;+\; \varepsilon_m\,, \end{aligned} \tag{6.12}$$

where

$$\varepsilon_m \;=\; \limsup_{h\to\infty} \int_t^{t_m}\!\!\int_B \big|\mathcal{M}(\nabla z_h(x^1,x^2))\big|\,dx^2 dx^1\,.$$

We shall prove that

$$\lim_{m\to\infty} \varepsilon_m \;=\; 0\,. \tag{6.13}$$

Let us show that (6.13) implies (6.2). From (6.12) we obtain

$$\mathcal{A}(u_m, \Omega_1\cup\Omega_2) \;\le\; \int_{\Omega_1\cup\Omega_2} \big|\mathcal{M}(\nabla u(x))\big|\, dx \;+\; \varepsilon_m\,.$$

Since $\mathcal{A}(\cdot, \Omega_1\cup\Omega_2)$ is lower semicontinuous in $L^1(\Omega_1\cup\Omega_2;\mathbf{R}^k)$ and (u_m) tends to u in $L^1(\Omega_1\cup\Omega_2;\mathbf{R}^k)$, the previous inequality together with (6.13) yields

$$\mathcal{A}(u, \Omega_1\cup\Omega_2) \;\le\; \int_{\Omega_1\cup\Omega_2} \big|\mathcal{M}(\nabla u(x))\big|\, dx\,.$$

As the opposite inequality follows from Theorem 2.7, we obtain (6.2), and the proposition is proved.

Let us prove now (6.13). For every $h\in\mathbf{N}$, for every $x^1\in\,]t,t_m[$, and for every $x^2\in B$ we have

$$\begin{aligned} D_1 z_h(x^1,x^2) \;&=\; \delta_m^{-1}\big(w_h(t_m,x^2) - v_h(t,x^2)\big) \;=\; \delta_m^{-1}\big(w_h(t_m,x^2) - u(t_m,x^2)\big) + \\ &+ \delta_m^{-1}\big(u(t_m,x^2) - u(t,x^2)\big) \;+\; \delta_m^{-1}\big(u(t,x^2) - v_h(t,x^2)\big)\,, \end{aligned} \tag{6.14}$$

where $\delta_m = t_m - t$, while

$$D_2 z_h(x^1,x^2) \;=\; \tfrac{x^1-t_m}{t-t_m} D_2 v_h(t,x^2) \;+\; \tfrac{x^1-t}{t_m-t} D_2 w_h(t_m,x^2)\,. \tag{6.15}$$

By (6.14) we have

$$\begin{aligned} \int_t^{t_m}\!\!\int_B |D_1 z_h(x^1,x^2)|\, dx^2 dx^1 \;&\le\; \int_B \big|w_h(t_m,x^2) - u(t_m,x^2)\big|\, dx^2 + \\ &+ \int_B \big|u(t_m,x^2) - u(t,x^2)\big|\, dx^2 \;+\; \int_B \big|u(t,x^2) - v_h(t,x^2)\big|\, dx^2\,. \end{aligned}$$

By (6.8) and (6.9) the first and the last integral in the right hand side of the previous formula tend to 0 as h tends to $+\infty$. Therefore,

$$\limsup_{h\to\infty} \int_t^{t_m}\!\!\int_B |D_1 z_h(x^1,x^2)|\, dx^2 dx^1 \;\le\; \int_B \big|u(t_m,x^2) - u(t,x^2)\big|\, dx^2 \;=\; \varepsilon_m^1\,, \tag{6.16}$$

and (ε_m^1) tends to 0 as m tends to $+\infty$ by (6.4).

From (6.15) we obtain

$$\int_t^{t_m}\!\!\int_B |D_2 z_h(x^1,x^2)|\,dx^2\,dx^1 \;\le\; \delta_m \int_B |D_2 v_h(t,x^2)|\,dx^2 \;+\; \delta_m \int_B |D_2 w_h(t_m,x^2)|\,dx^2\,,$$

so that (6.10) and (6.11) give

$$\begin{aligned}\limsup_{h\to\infty} \int_t^{t_m}\!\!\int_B |D_2 z_h(x^1,x^2)|\,dx^2 dx^1 \;\le \\ \le\; \delta_m \int_B |D_2 u(t,x^2)|\,dx^2 \;+\; \delta_m \int_B |D_2 u(t_m,x^2)|\,dx^2 \;=\; \varepsilon_m^2\,,\end{aligned} \tag{6.17}$$

and (ε_m^2) tends to 0 as m tends to $+\infty$ by (6.5) and (6.6).

Let us estimate now the components of $\mathcal{M}(\nabla z_h)$ of the form $\mathcal{M}^{0\beta}(\nabla z_h)$ with $|\beta| = 2$. By (1.3) we have

$$\left|\mathcal{M}^{0\beta}(\nabla z_h(x))\right| \;=\; \left|\det\begin{pmatrix} D_1 z_h^{\beta_1}(x) & D_2 z_h^{\beta_1}(x) \\ D_1 z_h^{\beta_2}(x) & D_2 z_h^{\beta_2}(x)\end{pmatrix}\right|.$$

By (6.8), (6.9), (6.14) the first column of this matrix is bounded in $L^\infty(]t,t_m[\times B;\mathbf{R}^2)$ and converges in $L^1(]t,t_m[\times B;\mathbf{R}^2)$, as h tends to $+\infty$, to the vector

$$\delta_m^{-1}\begin{pmatrix} u^{\beta_1}(t_m,x^2) - u^{\beta_1}(t,x^2) \\ u^{\beta_2}(t_m,x^2) - u^{\beta_2}(t,x^2)\end{pmatrix}.$$

By (6.10), (6.11), (6.15) the second column converges in $L^1(]t,t_m[\times B;\mathbf{R}^2)$ to

$$\begin{pmatrix} \frac{x^1-t_m}{t-t_m} D_2 u^{\beta_1}(t,x^2) \;+\; \frac{x^1-t}{t_m-t} D_2 u^{\beta_1}(t_m,x^2) \\ \frac{x^1-t_m}{t-t_m} D_2 u^{\beta_2}(t,x^2) \;+\; \frac{x^1-t}{t_m-t} D_2 u^{\beta_2}(t_m,x^2)\end{pmatrix}.$$

Therefore

$$\lim_{h\to\infty} \int_t^{t_m}\!\!\int_B \left|\mathcal{M}^{0\beta}(\nabla z_h(x^1,x^2))\right| dx^2 dx^1 \;\le\; \varepsilon_m^3\,, \tag{6.18}$$

where

$$\varepsilon_m^3 \;=\; \int_B \left|u(t_m,x^2) - u(t,x^2)\right|\Big(\left|D_2 u(t,x^2)\right| + \left|D_2 u(t_m,x^2)\right|\Big)\,dx^2\,.$$

Since u is bounded, (ε_m^3) converges to 0 as m tends to $+\infty$ by (6.4), (6.5), (6.6). As

$$|\mathcal{M}(\nabla z_h)| \le 1 + |D_1 z_h| + |D_2 z_h| + \sum_{|\beta|=2} |\mathcal{M}^{0\beta}(\nabla z_h)|\,,$$

from (6.16), (6.17), (6.18) we obtain that $\varepsilon_m \le \varepsilon_m^0 + \varepsilon_m^1 + \varepsilon_m^2 + \binom{k}{2}\varepsilon_m^3$, where $\varepsilon_m^0 = \delta_m \mathcal{L}^1(B)$. This shows that (ε_m) tends to 0 and concludes the proof of the proposition. $\square$

To prove Theorem 6.1 we need the following decomposition lemma for polyrectangles.

Lemma 6.5. *Let A be a polyrectangle in $\mathbf{R}^2$. Then for every $\varepsilon > 0$ there exist two families of polyrectangles $(A_i)_{1\le i\le i_0}$ and $(A_{i,j})_{1\le i\le i_0,\,1\le j\le j_0}$ with the following properties:*

$$\operatorname{diam}(A_{i,j}) \le \varepsilon\,; \tag{6.19}$$

$$A = \bigcup_{1\le i\le i_0} A_i \quad \textit{and} \quad A_i = \bigcup_{1\le j\le j_0} A_{i,j}\,; \tag{6.20}$$

$$\textit{for } i < i_0 \textit{ the sets } B_i = \bigcup_{1\le \kappa\le i} A_\kappa \textit{ and } A_{i+1} \textit{ have a nice intersection;} \tag{6.21}$$

$$\textit{for } j < j_0 \textit{ the sets } B_{i,j} = \bigcup_{1\le \kappa\le j} A_{i,\kappa} \textit{ and } A_{i,j+1} \textit{ have a nice intersection.} \tag{6.22}$$

Proof. Let us fix $\varepsilon > 0$. Let E^1 (resp. E^2) be the projection on the x^1-axis (resp. on the x^2-axis) of the union of all one-dimensional faces of A perpendicular to the x^1-axis (resp. to the x^2-axis). As E^1 and E^2 are finite sets, we can find a finite number of open intervals $I_1 =]a_1, b_1[$, $I_2 =]a_2, b_2[$, $\ldots$, $I_{i_0} =]a_{i_0}, b_{i_0}[$, and $J_1 =]c_1, d_1[$, $J_2 =]c_2, d_2[$, $\ldots$, $J_{j_0} =]c_{i_0}, d_{i_0}[$, with $a_i < a_{i+1} < b_i < b_{i+1}$ for every $i < i_0$, with $c_j < c_{j+1} < d_j < d_{j+1}$ for every $j < j_0$, and with diameter less than $\varepsilon/\sqrt{2}$, such that the intervals $I_i \cap I_{i+1} =]a_{i+1}, b_i[$ do not intersect E^1, the intervals $J_j \cap J_{j+1} =]c_{j+1}, d_j[$ do not intersect E^2, while the union $I_1 \cup \cdots \cup I_{i_0} =]a_1, b_{i_0}[$ contains the projection of A on the x^1-axis, and the union $J = J_1 \cup \cdots \cup J_{j_0} =]c_1, d_{j_0}[$ contains the projection of A on the x^2-axis.

For every $1 \le i \le i_0$ and for every $1 \le j \le j_0$ we define $A_i = A \cap \big(I_i \times J\big)$ and $A_{i,j} = A \cap \big(I_i \times J_j\big)$. Conditions (6.19) and (6.20) are clearly satisfied. In order to prove (6.21) for $i < i_0$, we note that $B_i = A \cap \big(]a_1, b_i[\times J\big)$. Let us consider the strip $S = \{x \in \mathbf{R}^2 : a_{i+1} < x^1 < b_i\}$. Then $A_{i+1} \subseteq \{x \in \mathbf{R}^2 : a_{i+1} < x^1\}$, $B_i \subseteq \{x \in \mathbf{R}^2 : x^1 < b_i\}$, and

$$S \cap B_i = S \cap A_{i+1} = A \cap \big(]a_{i+1}, b_i[\times J\big) = A \cap S\,.$$

Since the strip S does not meet any one-dimensional face of A perpendicular to the x^1-axis, the intersection $A \cap S$ can be written as a product $]a_{i+1}, b_i[\times C$, where C is a suitable open subset of $\mathbf{R}$. This concludes the proof of (6.21). The proof of (6.22) is analogous. $\square$

Proof of Theorem 6.1. Given an open set Ω with $\Omega \subset\subset \Omega_1 \cup \Omega_2$, there exists a polyrectangle A such that $\Omega \subseteq A \subset\subset \Omega_1 \cup \Omega_2$. By the Lebesgue Covering Lemma there exists $\varepsilon > 0$ such that any open ball with radius ε about each point of A is contained either in Ω_1 or in Ω_2. By Lemma 6.5 there exist two families of polyrectangles $(A_i)_{1\le i\le i_0}$ and $(A_{i,j})_{1\le i\le i_0,\,1\le j\le j_0}$ which satisfy conditions (6.19)–(6.22). By (6.19) and by the choice of ε each polyrectangle $A_{i,j}$ is contained either in Ω_1 or in Ω_2. Therefore, Remark 5.7 implies that all polyrectangles $A_{i,j}$ belong to the family $\mathcal{E}(u, \Omega_1 \cup \Omega_2)$ defined by (6.1). From Proposition 6.3 and from (6.22) we obtain by induction that the sets $B_{i,j}$ belong to $\mathcal{E}(u, \Omega_1 \cup \Omega_2)$ for every $1 \le i \le i_0$ and for every $1 \le j \le j_0$. In particular, taking (6.20) into account, for $j = j_0$ we get $A_i \in \mathcal{E}(u, \Omega_1 \cup \Omega_2)$ for every $1 \le i \le i_0$. From Proposition 6.3 and from (6.21) we obtain by induction that the sets B_i belong to $\mathcal{E}(u, \Omega_1 \cup \Omega_2)$ for every $1 \le i \le i_0$. In particular, taking (6.20) into account, for $i = i_0$ we get $A \in \mathcal{E}(u, \Omega_1 \cup \Omega_2)$. As $\Omega \subseteq A$, the conclusion follows from Remark 5.7. □

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