

Semicontinuity Problems in the Calculus of Variations

EMILIO ACERBI & NICOLA FUSCO

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Introduction

Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a twice continuously differentiable function, and for $u \in W^{1,1}(a, b)$ set

$$F(u; a, b) = \int_a^b f(x, u(x), u'(x)) dx.$$

In a paper of TONELLI [17] it is proved that the functional F is lower semicontinuous (lsc) in the topology of $L^\infty(a, b)$ if and only if the function f is convex in the last variable. Later, several authors generalized this result: among the many theorems obtained, in which x is allowed to belong to \mathbb{R}^n and considerably less regularity on f is required, we recall particularly Theorem 12 of SERRIN [15], in which for the first time differentiability conditions on f are dropped, and the following result due to MARCELLINI & SBORDONE [11]:

If $f: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies:

- (i) f is measurable in x , and continuous in (s, ξ) , and
- (ii) $0 \leq f(x, s, \xi) \leq g(x, |s|, |\xi|)$,

where g is increasing in $|s|$ and $|\xi|$, and is locally summable in x , then the functional

$$(0.1) \quad F(u, \Omega) = \int_{\Omega} f(x, u(x), Du(x)) dx$$

is sequentially weakly lower semicontinuous* on $W^{1,p}(\Omega)$, with $1 \leq p \leq +\infty$, if and only if f is convex in ξ .

(See also EKELAND and TEMAM [8] for the case in which f does not depend on u).

* That is, $F(u, \Omega) \leq \liminf F(u_n, \Omega)$ whenever $u_n \rightarrow u$ in the weak topology of $W^{1,p}(\Omega)$. When $p = \infty$, weak convergence should be replaced by weak* convergence. In what follows we shall use the abbreviation *swlsc* for “sequential weak lower semicontinuity”, or *sw* lsc* when “weak” is replaced by “weak*”.

On the other hand, if we allow the function u to be vector-valued, i.e. $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, then the convexity hypothesis turns out to be sufficient, but too strong to be necessary, for F to be lsc: MORREY proved in [13] that, under strong regularity assumptions, F is *sw* lsc* on $W^{1,\infty}(\Omega; \mathbb{R}^m)$ if and only if f is quasi-convex, that is for every $s \in \mathbb{R}^m$ and for almost every $x \in \mathbb{R}^n$ the function $\xi \mapsto f(x, s, \xi)$ satisfies the condition

$$(0.2) \quad f(x, s, \xi) \cdot \text{meas}(\Omega) \leq \int_{\Omega} f(x, s, \xi + Dw(y)) \, dy$$

for every $\xi \in \mathbb{R}^m$, for every bounded open set $\Omega \subset \mathbb{R}^n$, and for every $w \in C_0^\infty(\Omega; \mathbb{R}^m)$. Although it is technically not easy to handle, this condition arises in a natural way in many problems (especially in elastostatics); moreover, it is equivalent to convexity in ξ in the case $m = 1$.

The theorem of MORREY was extended by MEYERS [12] to the semicontinuity (on $W^{k,p}(\Omega; \mathbb{R}^m)$) of functionals of the type

$$(0.3) \quad \int_{\Omega} f(x, u(x), \dots, D^k u(x)) \, dx,$$

always under strong continuity hypotheses.

In section II, by means of techniques basically relying on a recent theorem of LIU [10], which allows us to deduce semicontinuity on $W^{1,p}(\Omega; \mathbb{R}^m)$ from semicontinuity on $W^{1,\infty}(\Omega; \mathbb{R}^m)$, we prove the following main result (theorem [II.4]):

If $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ satisfies

(i) *f is measurable in x , continuous in (s, ξ) , and*

(ii) $0 \leq f(x, s, \xi) \leq a(x) + C(|s|^p + |\xi|^p)$,

where $p \geq 1$, a is a non-negative locally summable function, and C is a non-negative constant,

then the functional (0.1) is swlsc on $W^{1,p}(\Omega; \mathbb{R}^m)$ if and only if $\xi \mapsto f(x, s, \xi)$ is quasi-convex, for every s and almost every x .

Counterexamples valid even in the convex case show that these hypotheses are almost the minimal ones necessary to obtain a theorem of this kind.

As a particular case of our result, we deduce the weak semicontinuity on $W^{1,n}(\Omega; \mathbb{R}^n)$ of the functional

$$\int_{\Omega} b(x) | \det Du(x) | \, dx$$

(where b is non-negative, and bounded on bounded sets of \mathbb{R}^n). The integrand satisfies a stronger hypothesis than quasi-convexity (namely *polyconvexity*, a condition introduced and studied by BALL in [2], [3], [4], [5]), but the result does not seem to be previously known.

In the last section we prove a representation theorem for the greatest swlsc functional which is less than or equal to $\int_{\Omega} f(x, u(x), Du(x)) \, dx$, where f is not necessarily quasi-convex. We show that, under reasonable continuity assumptions on f , this functional has the form $\int_{\Omega} \phi(x, u(x), Du(x)) \, dx$, where ϕ is the greatest

quasi-convex function which is less than or equal to f . A similar result has been proved by DACOROGNA [6], if f is a polyconvex function.

We remark that, using more complicated notations as in [12], [5], our results can be extended to the case of functionals of the type (0.3). For other results and additional bibliography on quasi-convexity, see the many important papers by BALL, MEYERS, and MORREY.

I. Notation and Preliminary Lemmas

If $a \in \mathbb{R}^n$, then $|a|$ is its euclidean norm; if ξ is an $m \times n$ matrix, $|\xi|$ is the norm of ξ when regarded as a vector in \mathbb{R}^{mn} . The Lebesgue measure of a measurable subset S of \mathbb{R}^n will be denoted by $\text{meas}(S)$.

Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \leq p \leq +\infty$, $m \geq 1$; we define $L^p(\Omega; \mathbb{R}^m)$ to be the collection of all m -tuples $(f^{(1)}, \dots, f^{(m)})$ of real functions in $L^p(\Omega)$. Analogously, we say that $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ if u belongs to $L^p(\Omega; \mathbb{R}^m)$ together with its distribution derivatives $\frac{\partial u^{(i)}}{\partial x_j}$, $1 \leq i \leq m$, $1 \leq j \leq n$. The $m \times n$ matrix of these derivatives will be denoted by the symbol Du ; $W^{1,p}(\Omega; \mathbb{R}^m)$ becomes a Banach space if it is endowed with the norm

$$\|u\|_{W^{1,p}(\Omega; \mathbb{R}^m)} = \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)},$$

where

$$|u|(x) = |u(x)|, \quad |Du|(x) = |Du(x)|.$$

Finally, $u \in C_0^1(\Omega; \mathbb{R}^m)$ if each $u^{(i)}$ is a C^1 function on Ω with compact support, while $W_0^{1,p}(\Omega; \mathbb{R}^m)$ is the closure of $C_0^1(\Omega; \mathbb{R}^m)$ in the topology of $W^{1,p}(\Omega; \mathbb{R}^m)$.

Definition [I.1]. $f: \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is weakly quasi-convex if for every $\tilde{\xi} \in \mathbb{R}^{nm}$, $\tilde{\eta} \in \mathbb{R}^m$ and $\tilde{\lambda} \in \mathbb{R}^n$ the functions

$$\lambda \mapsto f(\tilde{\xi} + \lambda \otimes \tilde{\eta}), \quad \eta \mapsto f(\tilde{\xi} + \tilde{\lambda} \otimes \eta)$$

are convex, where $(\lambda \otimes \eta)_{ij} = \lambda_i \eta_j$.

Definition [I.2]. A continuous function $f: \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is quasi-convex if for every $\tilde{\xi} \in \mathbb{R}^{nm}$, for every open subset Ω of \mathbb{R}^n , and every function $z \in C_0^1(\Omega; \mathbb{R}^m)$ we have

$$(I.1) \quad \text{meas}(\Omega) \cdot f(\tilde{\xi}) \leq \int_{\Omega} f(\tilde{\xi} + Dz(x)) \, dx.$$

In what follows, if f is a real function defined in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm}$, we will say that f is quasi-convex in ξ if there exists a set $I \subset \mathbb{R}^n$, with $\text{meas}(I) = 0$, such that for every $\tilde{x} \in \mathbb{R}^n \setminus I$ and $\tilde{s} \in \mathbb{R}^m$ the function $\xi \mapsto f(\tilde{x}, \tilde{s}, \xi)$ is quasi-convex.

To prove that a function f is quasi-convex, note that it is enough to verify (I.1) for one open set Ω , and for every $z \in C_0^\infty(\Omega; \mathbb{R}^m)$; moreover if f is quasi-convex then (I.1) holds for every $z \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$.

Quasi-convexity implies weak quasi-convexity, which in turn implies that the function locally satisfies a Lipschitz condition. If $m = 1$ or $n = 1$, then quasi-convexity is equivalent to convexity; in general, a function $f \in C^2(\mathbb{R}^{nm})$ is weakly quasi-convex if and only if for every $\xi \in \mathbb{R}^{nm}$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^m$ there holds

$$\sum_{i,j=1}^n \sum_{h,k=1}^m \frac{\partial^2 f}{\partial \xi_{ih} \partial \xi_{jk}}(\xi) \lambda_i \lambda_j \eta_h \eta_k \geq 0$$

(Legendre-Hadamard condition).

The proofs of the previous remarks may be found in [2], [12], [14]. The following result ([12], Lemma 1) will be useful to disengage Definition [I.1] from the boundary condition on z .

Lemma [I.3]. *Let $f: \mathbb{R}^{nm} \rightarrow \mathbb{R}$ be quasi-convex. For every bounded open set $\Omega \subset \mathbb{R}^n$ and every sequence $(z_k) \subset W^{1,\infty}(\Omega; \mathbb{R}^m)$ which is weakly* convergent to zero, we have*

$$\text{meas}(\Omega) \cdot f(\xi) \leq \liminf_{k \rightarrow \infty} \int_{\Omega} f(\xi + Dz(x)) dx$$

for every $\xi \in \mathbb{R}^{nm}$.

Definition [I.4]. *$f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is a Carathéodory function if the following conditions are satisfied:*

for every $(s, \xi) \in \mathbb{R}^m \times \mathbb{R}^{nm}$, $x \mapsto f(x, s, \xi)$ is measurable;

for almost all $x \in \mathbb{R}^n$, $(s, \xi) \mapsto f(x, s, \xi)$ is continuous.

The following result of SCORZA-DRAGONI ([8], page 235) characterizes the class of Carathéodory functions.

Lemma [I.5]. *A mapping $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ is a Carathéodory function if and only if for every compact set $K \subset \mathbb{R}^n$ and $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset K$, with $\text{meas}(K \setminus K_\varepsilon) < \varepsilon$, such that the restriction of f to $K_\varepsilon \times \mathbb{R}^m \times \mathbb{R}^{nm}$ is continuous.*

The next lemma may be found in [7].

Lemma [I.6]. *Let $G \subset \mathbb{R}^n$ be measurable, with $\text{meas}(G) < \infty$. Assume (M_k) is a sequence of measurable subsets of G such that, for some $\varepsilon > 0$, the following estimate holds:*

$$\text{meas}(M_k) \geq \varepsilon \text{ for all } k \in \mathbb{N}.$$

Then a subsequence (M_{k_h}) can be selected such that $\bigcap_{h \in \mathbb{N}} M_{k_h} \neq \emptyset$.

Lemma [I.7]. *Let (ϕ_k) be a bounded sequence in $L^1(\mathbb{R}^n)$. Then to each $\varepsilon > 0$ there exists a triple $(A_\varepsilon, \delta, S)$, where A_ε is measurable and $\text{meas}(A_\varepsilon) < \varepsilon$, $\delta > 0$,*

and S is an infinite subset of \mathbb{N} , such that for all $k \in S$

$$\int_B |\phi_k(x)| dx < \varepsilon$$

whenever B and A_ε are disjoint and $\text{meas}(B) < \delta$.

Proof. We reason by contradiction. Hence we suppose that there exists a $\varepsilon > 0$ such that, for every $(A_\varepsilon, \delta, S)$ as in the statement of the theorem, we may choose a measurable set B , with $B \cap A_\varepsilon = \emptyset$ and $\text{meas}(B) < \delta$, and an index $k \in S$ such that

$$\int_B |\phi_k(x)| dx \geq \varepsilon.$$

This implies that for every set A , with $\text{meas}(A) < \varepsilon$, and every infinite set $S \subset \mathbb{N}$, there exists both a set C , with $C \cap A = \emptyset$ and $\text{meas}(A \cup C) < \varepsilon$, and an infinite subset T of S such that

$$\int_C |\phi_k(x)| dx \geq \varepsilon \quad \text{for all } k \in T.$$

This will be proved later; now we show that we are led to a contradiction.

Set $A = \emptyset$, $S = \mathbb{N}$, and let C_1 and T_1 be as above. Starting from $A = C_1$ and $S = T_1$, we pass to C_2 and T_2 , where $C_1 \cap C_2 = \emptyset$ and

$$\int_{C_1 \cup C_2} |\phi_k(x)| dx = \int_{C_1} |\phi_k(x)| dx + \int_{C_2} |\phi_k(x)| dx \geq 2\varepsilon$$

for all $k \in T_2$. Since $\text{meas}(C_1 \cup C_2) < \varepsilon$, we may set $A = C_1 \cup C_2$, $S = T_2$, and continue with the same argument. If

$$N > \varepsilon^{-1} \sup_{k \in \mathbb{N}} \|\phi_k\|_{L^1(\mathbb{R}^n)},$$

then after N iterations we obtain the contradiction.

We return to the interrupted proof: let A and S be as stated, set $S_1 = S$ and take $\delta_1 < (\varepsilon - \text{meas}(A))/2$. There exist a set B_1 disjoint from A , with $\text{meas}(B_1) < \delta_1$, and an index $k_1 \in S_1$, such that

$$\int_{B_1} |\phi_{k_1}(x)| dx \geq \varepsilon.$$

Applying induction, put

$$\delta_n = \frac{1}{2} \delta_{n-1}, S_n = \{k \in S_{n-1} : k > k_{n-1}\},$$

and set

$$C = \bigcup_{h \in \mathbb{N}} B_h, T = \{k_h : h \in \mathbb{N}\},$$

then C and T satisfy our requirements: $C \cap A = \emptyset$, $\text{meas}(C) + \text{meas}(A) < \varepsilon$, $T \subset S$ is infinite, and

$$\int_C |\phi_k(x)| dx \geq \varepsilon$$

for all $k \in T$. This completes the proof.

If $r > 0$ and $x \in \mathbb{R}^n$, set $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, and $\text{meas}(B_r(x)) = \omega_n r^n$.

Definition [I.8]. Let $u \in C_0^\infty(\mathbb{R}^n)$. We define

$$(M^*u)(x) = (Mu)(x) + \sum_{i=1}^n (MD_iu)(x),$$

where we set

$$(Mf)(x) = \sup_{r>0} \frac{1}{\omega_n r^n} \int_{B_r(x)} |f(y)| dy$$

for every locally summable f .

Lemma [I.9]. If $u \in C_0^\infty(\mathbb{R}^n)$ then $M^*u \in C^0(\mathbb{R}^n)$ and

$$|u(x)| + \sum_{i=1}^n |D_iu(x)| \leq (M^*u)(x)$$

for all $x \in \mathbb{R}^n$. Moreover (see [16]) if $p > 1$ then

$$\|M^*u\|_{L^p(\mathbb{R}^n)} \leq c(n, p) \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

and if $p = 1$ then

$$\text{meas}\{x \in \mathbb{R}^n : (M^*u) \geq \lambda\} \leq \frac{c(n)}{\lambda} \|u\|_{W^{1,1}(\mathbb{R}^n)}$$

for all $\lambda > 0$.

Lemma [I.10]. Let $u \in C_0^\infty(\mathbb{R}^n)$, and put

$$U(x, y) = \frac{|u(y) - u(x) - \sum_{i=1}^n D_iu(x)(y_i - x_i)|}{|y - x|}.$$

Then for every $x \in \mathbb{R}^n$ and $r > 0$

$$\int_{B_r(x)} U(x, y) dy \leq 2\omega_n r^n (M^*u)(x).$$

The proof is contained in [10], Lemma 2. By modifying another demonstration of [10], we are also able to prove

Lemma [I.11]. Let $u \in C_0^\infty(\mathbb{R}^n)$ and $\lambda > 0$, and set

$$H^\lambda = \{x \in \mathbb{R}^n : (M^*u)(x) < \lambda\}.$$

Then for every $x, y \in H^\lambda$ we have

$$\frac{|u(y) - u(x)|}{|y - x|} \leq c(n) \lambda.$$

Proof. Let $c'(n)$ be such that for every $x, y \in \mathbb{R}^n$ with $|x - y| = r$, we have

$$(I.2) \quad \text{meas} (B_r(x) \cap B_r(y)) > \frac{2}{c'(n)} \omega_n r^n.$$

For $\delta > 0$ set

$$W_r(x, \delta) = \{y \in B_r(x) : U(x, y) < \delta\},$$

whence by Lemma [I.10]

$$\text{meas} (B_r(x) \setminus W_r(x, \delta)) \leq \frac{2}{\delta} \omega_n r^n (M^*u)(x).$$

If $z \in H^\lambda$, then

$$(I.3) \quad \text{meas} (B_r(z) \setminus W_r(z, 2c'(n)\lambda)) \leq \frac{2\omega_n r^n}{2c'(n)\lambda} (M^*u)(z) < \frac{\omega_n r^n}{c'(n)}.$$

Let $x, y \in H^\lambda$ with $r = |x - y|$. By (I.2) and (I.3)

$$W_r(x, 2c'(n)\lambda) \cap W_r(y, 2c'(n)\lambda) \neq \emptyset.$$

Choose \tilde{z} in this intersection, so that $|\tilde{z} - x| < r$, $|\tilde{z} - y| < r$. Then

$$\begin{aligned} \frac{|u(y) - u(x)|}{|y - x|} &\leq U(y, \tilde{z}) + \sum_{i=1}^n [|D_i u(x)| + |D_i u(y)|] \\ &\leq (4c'(n) + 2)\lambda, \end{aligned}$$

as required.

Lemma [I.12]. *Let X be a metric space, E a subspace of X , and k a positive real number. Then any k -Lipschitz mapping from E into \mathbb{R} can be extended by a k -Lipschitz mapping from X into \mathbb{R} .*

For the proof see [8], page 298. We conclude this preliminary section by defining

$$G^\nu = \{2^{-\nu}(x + Y) : x \in \mathbb{Z}^n\}, \quad \nu \in \mathbb{N},$$

where $Y = (0, 1)^n = \{y \in \mathbb{R}^n : 0 < y_i < 1 \text{ for } 1 \leq i \leq n\}$.

II. Semicontinuity Theorems

If f is a real function defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm}$, and if the left hand side of (II.1) makes sense, then we define (for every measurable set $S \subset \mathbb{R}^n$)

$$(II.1) \quad \int_S f(x, u(x), Du(x)) dx = F(u, S).$$

Theorem [II.1]. Let $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ satisfy:

(II.2) f is a Carathéodory function;

(II.3) f is quasi-convex in ξ ;

(II.4) $0 \leq f(x, s, \xi) \leq a(x) + b(s, \xi)$ for every $x \in \mathbb{R}^n$, $s \in \mathbb{R}^m$, and $\xi \in \mathbb{R}^{nm}$, where a is a non-negative locally summable function on \mathbb{R}^n , and $b \geq 0$ is locally bounded on $\mathbb{R}^m \times \mathbb{R}^{nm}$.

Then for every open set Ω in \mathbb{R}^n the functional $u \mapsto F(u, \Omega)$ is sw^* -lsc on $W^{1,\infty}(\Omega; \mathbb{R}^m)$.

Proof. Let us suppose first that $\Omega = (0, 1)^n$. Fix $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ and $(z_k) \subset W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $z_k \rightharpoonup 0$ (weak* convergence) in $W^{1,\infty}(\Omega; \mathbb{R}^m)$; we must prove that

$$F(u, \Omega) \leq \liminf_{k \rightarrow \infty} F(u + z_k, \Omega).$$

Without loss of generality we may suppose $a(x) < +\infty$ for every x . Put

$$\lambda = \|u\|_{W^{1,\infty}(\Omega; \mathbb{R}^m)} + \sup_{k \in \mathbb{N}} \|z_k\|_{W^{1,\infty}(\Omega; \mathbb{R}^m)}$$

$$M = \sup \{b(s, \xi) : |s| \leq \lambda, |\xi| \leq \lambda\}.$$

Now take $\varepsilon > 0$, and let $\alpha \geq 1$ be so large that if

$$E = \{x \in \Omega : a(x) \leq \alpha\} \setminus I$$

then

$$\text{meas}(\Omega \setminus E) < \frac{\varepsilon}{M}, \quad \int_{\Omega \setminus E} a(x) dx < \varepsilon.$$

By Lemma [I.5] there exists a compact set $K \subset \Omega$ such that f is continuous on $K \times \mathbb{R}^m \times \mathbb{R}^{nm}$ and

$$\text{meas}(\Omega \setminus K) < \frac{\varepsilon}{\alpha + M}.$$

If we neglect sets of measure zero, then for all $\nu \in \mathbb{N}$ we can write

$$\Omega = \bigcup_{h=1}^{2^{n\nu}} Q_h^\nu$$

with $Q_h^\nu \in G^\nu$. The range $1 \leq h \leq 2^{n\nu}$ will be assumed henceforth, and we shall also write \sum_h and \bigcup_h when h ranges from 1 to $2^{n\nu}$. Define

$$(u)_h^\nu = 2^{-n\nu} \int_{Q_h^\nu} u(y) dy, \quad (u)^\nu(x) = \sum_h (u)_h^\nu \chi_{Q_h^\nu}(x)$$

$$(Du)_h^\nu = 2^{-n\nu} \int_{Q_h^\nu} Du(y) dy, \quad (Du)^\nu(x) = \sum_h (Du)_h^\nu \chi_{Q_h^\nu}(x).$$

Note that

$$\|(u)^v\|_{L^\infty(\Omega; \mathbb{R}^m)} + \|(Du)^v\|_{L^\infty(\Omega; \mathbb{R}^{nm})} \leq \|u\|_{W^{1,\infty}(\Omega; \mathbb{R}^m)},$$

and that the sequences $((u)^v)$ and $((Du)^v)$ converge pointwise a.e. to u and Du respectively.

For every v and h fix $x_h^v \in Q_h^v \cap K \cap E$, if this set is not empty. Then

$$F(u + z_k, \Omega) \geq F(u + z_k, K \cap E) = a_k + b_k^v + c_k^v + d^v + e,$$

where we put

$$\begin{aligned} a_k &= \int_{K \cap E} [f(x, (u + z_k)(x), (Du + Dz_k)(x)) - f(x, u(x), (Du + Dz_k)(x))] dx; \\ b_k^v &= \sum_h \int_{Q_h^v \cap K \cap E} [f(x, u(x), (Du + Dz_k)(x)) - f(x_h^v, (u)_h^v, (Du)_h^v + Dz_k(x))] dx; \\ c_k^v &= \sum_h \int_{Q_h^v \cap K \cap E} [f(x_h^v, (u)_h^v, (Du)_h^v + Dz_k(x)) - f(x_h^v, (u)_h^v, (Du)_h^v)] dx; \\ d^v &= \sum_h \int_{Q_h^v \cap K \cap E} [f(x_h^v, (u)_h^v, (Du)_h^v) - f(x, u(x), Du(x))] dx. \end{aligned}$$

By the uniform continuity of f on the bounded sets of $K \times \mathbb{R}^n \times \mathbb{R}^{nm}$ we have $\lim_{k \rightarrow \infty} a_k = 0$. Similarly the uniform continuity of f and the pointwise convergence of $((u)^v)$ and $((Du)^v)$ imply that

$$\lim_{v \rightarrow \infty} d^v = 0, \quad \lim_{v \rightarrow \infty} b_k^v = 0 \text{ uniformly with respect to } k.$$

Hence we may suppose that v is large enough to ensure that $|b_k^v| + |d^v| < \varepsilon$ for all k .

Now note that

$$\begin{aligned} & \left| \sum_h \int_{Q_h^v \setminus (K \cap E)} [f(x_h^v, (u)_h^v, (Du)_h^v + Dz_k(x)) - f(x_h^v, (u)_h^v, (Du)_h^v)] dx \right| \\ & \leq 2 \sum_h \int_{Q_h^v \setminus (K \cap E)} [a(x_h^v) + M] dx \\ & \leq 2 \left[(\alpha + M) \text{meas}(\Omega \setminus K) + M \text{meas}(\Omega \setminus E) + \int_{\Omega \setminus E} \alpha dx \right] \\ & \leq 4\varepsilon + 2 \int_{\Omega \setminus E} a(x) dx \leq 6\varepsilon. \end{aligned}$$

Applying Lemma [I.3] to each Q_h^v , we find that

$$\liminf_{k \rightarrow \infty} c_k^v \geq -6\varepsilon.$$

Finally,

$$e = F(u, K \cap E) \geq F(u, \Omega) - 3\varepsilon.$$

As $k \rightarrow \infty$, the foregoing estimates yield

$$\liminf_{k \rightarrow \infty} F(u + z_k, \Omega) \geq F(u, \Omega) - 10\varepsilon,$$

Since ε was arbitrary, this proves our result for the special choice of Ω noted at the beginning.

It is easy to see that the same argument applies to every hypercube Ω with edges parallel to the coordinate axes; the assertion for a generic Ω follows from the fact that the supremum of a family of lsc functions is lsc. This completes the proof.

A slight modification of the proof yields the same theorem even if f satisfies (II.2) and (II.3), and $|f|$ satisfies (II.4) (see [12], [9]). Note that if f is defined on $\Omega \times \mathbb{R}^m \times \{\xi \in \mathbb{R}^{nm} : |\xi| < r\}$ for some $r > 0$ and the hypotheses of theorem [II.1] hold, then the functional $u \mapsto F(u, \Omega)$ is sw*lsc on the space of functions $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ such that $\|Du\|_{L^\infty(\Omega; \mathbb{R}^{nm})} < r$.

The inverse to theorem [II.1] is given by

Theorem [II.2]. *Let $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ satisfy (II.2) and (II.4). Assume the functional $u \mapsto F(u, \Omega)$ to be sw*lsc on $W^{1,\infty}(\Omega; \mathbb{R}^m)$ for every open set $\Omega \subset \mathbb{R}^n$. Then f is quasi-convex in ξ .*

Proof. We have to show that, if we fix an open set $\Omega \subset \mathbb{R}^n$, then there exists a set $I \subset \Omega$, with $\text{meas}(I) = 0$, such that $\xi \mapsto f(x, s, \xi)$ is quasi-convex for every $x \in \Omega \setminus I$ and $s \in \mathbb{R}^m$. To this end, we will use only the fact that $u \mapsto F(u, \Omega)$ is lsc for that particular Ω .

By Lemma [I.5] we can choose a nondecreasing sequence (K_i) of compact sets, with $\text{meas}(\Omega \setminus K_i) < \frac{1}{i}$, such that f is continuous on each $K_i \times \mathbb{R}^m \times \mathbb{R}^{nm}$.

Define I in the following way: $x \in \Omega \setminus I$ if the following conditions are satisfied:

$$(II.5) \quad \begin{aligned} x &\in \bigcup_{i \in \mathbb{N}} K_i; \\ a(x) &< +\infty; \end{aligned}$$

x is a Lebesgue point for χ_{K_i} , for every i ;*

x is a Lebesgue point for $a \cdot \chi_{\Omega \setminus K_i}$, for every i .

Fix $\tilde{x} \in \Omega \setminus I$, $\tilde{s} \in \mathbb{R}^m$, $\tilde{\xi} \in \mathbb{R}^{nm}$, where clearly we may suppose $\tilde{x} = 0$, and also set $u(x) = \tilde{s} + \tilde{\xi} \cdot x$, where $\tilde{\xi}$ is regarded as an $m \times n$ matrix. Let $z \in C_0^\infty(Y; \mathbb{R}^m)$, and put

$$\lambda = \|u\|_{W^{1,\infty}(\Omega; \mathbb{R}^m)} + \|z\|_{W^{1,\infty}(Y; \mathbb{R}^m)}.$$

Define z periodically on \mathbb{R}^n , setting $z(x) = z(x + y)$ for every $y \in \mathbb{Z}^n$. Let \tilde{k} be so large that $2^{-\tilde{k}}Y \subset \Omega$; for $k \geq \tilde{k}$ and $v \in \mathbb{N}$ define

$$z_k^v(x) = \begin{cases} 2^{-kv} z(2^{kv}x) & \text{if } x \in 2^{-k}Y \\ 0 & \text{otherwise,} \end{cases}$$

* This means that $\lim_{r \rightarrow 0} [\text{meas}(B_r(x))]^{-1} \int_{B_r(x)} \chi_{K_i}(y) dy = 1$.

so that $\|z_k^v\|_{W^{1,\infty}(\Omega;\mathbb{R}^m)} \leq \lambda$. For every k , $z_k^v \rightarrow 0$ (weak*) in $W^{1,\infty}(\Omega;\mathbb{R}^m)$ as $v \rightarrow +\infty$, hence $z_k^v \rightarrow 0$ strongly in $L^\infty(\Omega;\mathbb{R}^m)$. Also for fixed k if we neglect sets of measure 0, then

$$2^{-k}Y = \bigcup_h Q_h^{kv}$$

with $Q_h^{kv} \in G^{kv}$ for $1 \leq h \leq 2^{nv}$. We denote by x_h^v the corner of Q_h^{kv} nearest to the origin, so that $Q_h^{kv} = x_h^v + 2^{-kv}Y$.

By (II.5), we may suppose that $0 \in K_i$ for all i . Choose $\varepsilon > 0$. Then there exists \tilde{i} such that for $i \geq \tilde{i}$ we have

$$\int_{\Omega \setminus K_i} [a(x) + M] dx < \varepsilon,$$

where

$$M = \sup \{b(s, \xi) : |s| + |\xi| \leq 2\lambda\}.$$

Let \tilde{f}_i be a continuous function on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm}$, coinciding with f on $K_i \times \{(s, \xi) : |s| + |\xi| \leq 2\lambda\} = K_i \times B_{2\lambda}$. We may also suppose that \tilde{f}_i satisfies $0 \leq \tilde{f}_i \leq \max_{K_i \times B_{2\lambda}} f$. Choose a function $\psi \in C_0^0(\Omega)$ so that

$$0 \leq \psi(x) \leq 1 \text{ for all } x \in \Omega,$$

$$\psi(x) = 1 \text{ for all } x \in K_i,$$

$$\int_{\Omega \setminus K_i} \psi(x) dx < \varepsilon / \max_{K_i \times B_{2\lambda}} f.$$

The function $f_i = \psi \tilde{f}_i$ is another continuous extension of f outside $K_i \times B_{2\lambda}$. We can split the functional $F(u + z_k^v, 2^{-k}Y)$ as follows:

$$F(u + z_k^v, 2^{-k}Y) = a^v + b^v + c^v,$$

where we set

$$\begin{aligned} a^v &= \int_{2^{-k}Y} [f(x, (u + z_k^v)(x), (Du + Dz_k^v)(x)) \\ &\quad - f_i(x, (u + z_k^v)(x), (Du + Dz_k^v)(x))] dx; \\ b^v &= \sum_h \int_{Q_h^{kv}} [f_i(x, (u + z_k^v)(x), (Du + Dz_k^v)(x)) \\ &\quad - f_i(x_h^v, u(x_h^v), Du(x_h^v) + Dz_k^v(x))] dx; \\ c^v &= \sum_h \int_{Q_h^{kv}} f_i(x_h^v, u(x_h^v), Du(x_h^v) + Dz(2^{kv}x)) dx \\ &= \sum_h 2^{-nk^v} \int_Y f_i(x_h^v, u(x_h^v), Du(x_h^v) + Dz(y)) dy. \end{aligned}$$

Our choice of f_i yields $|a^v| < 2\varepsilon$ for every v and $i \geq \tilde{i}$. Moreover since $u \in C^1(\bar{\Omega}; \mathbb{R}^m)$ and f_i is uniformly continuous, we have $\lim_{v \rightarrow \infty} b^v = 0$. Finally c^v has

the form of a Cauchy sum, over the cube $2^{-k}Y$, of the continuous function

$$x \mapsto \int_Y f_i(x, u(x), Du(x) + Dz(y)) dy.$$

Hence it is convergent as $v \rightarrow \infty$, with

$$\lim_{v \rightarrow \infty} c^v = \int_{2^{-k}Y} \left[\int_Y f_i(x, u(x), Du(x) + Dz(y)) dy \right] dx.$$

Combining the above three lines we have

$$\limsup_{v \rightarrow \infty} F(u + z_k^v, 2^{-k}Y) \leq 2\varepsilon + \int_{2^{-k}Y} \left[\int_Y f_i(x, u(x), Du(x) + Dz(y)) dy \right] dx.$$

Let ψ tend to χ_{K_i} . Since $f = f_i$ on $K_i \times B_{2\lambda}$, it follows from the dominated convergence theorem that

$$\limsup_{v \rightarrow \infty} F(u + z_k^v, 2^{-k}Y) \leq 2\varepsilon + \int_{K_i \cap 2^{-k}Y} \left[\int_Y f(x, u(x), Du(x) + Dz(y)) dy \right] dx.$$

By the semicontinuity of $u \mapsto F(u, \Omega)$ and the fact that $z_k^v \equiv 0$ on $\Omega \setminus 2^{-k}Y$

$$\begin{aligned} F(u, \Omega) &= F(u, 2^{-k}Y) + F(u, \Omega \setminus 2^{-k}Y) \\ &\leq \liminf_{v \rightarrow \infty} [F(u + z_k^v, 2^{-k}Y) + F(u, \Omega \setminus 2^{-k}Y)]. \end{aligned}$$

Hence for $i \geq \tilde{i}$

$$F(u, 2^{-k}Y) \leq 2\varepsilon + \int_{2^{-k}Y \times Y} \chi_{K_i}(x) f(x, u(x), Du(x) + Dz(y)) dx dy.$$

Letting $i \rightarrow +\infty$, and using the fact that ε is arbitrary, we get

$$F(u, 2^{-k}Y) \leq \int_{2^{-k}Y} \left[\int_Y f(x, u(x), Du(x) + Dz(y)) dy \right] dx,$$

so that

$$2^{nk} \int_{2^{-k}Y} \left[f(x, u(x), Du(x)) - \int_Y f(x, x, u(x), Du(x) + Dz(y)) dy \right] dx \leq 0.$$

Call $\mu(x; u, z)$ the integrand in the square brackets; our hypotheses on the set I , and the continuity of f on $K_{\tilde{i}} \times B_{2\lambda}$, then yield

$$\begin{aligned} &\lim_{k \rightarrow \infty} 2^{nk} \int_{2^{-k}Y \cap K_{\tilde{i}}} \mu(x; u, z) dx \\ &= \lim_{k \rightarrow \infty} \left(2^{nk} \int_{2^{-k}Y} \chi_{K_{\tilde{i}}}(x) dx \right) \left([\text{meas}(2^{-k}Y \cap K_{\tilde{i}})]^{-1} \int_{2^{-k}Y \cap K_{\tilde{i}}} \mu(x; u, z) dx \right) \\ &= f(0, \tilde{s}, \tilde{\xi}) - \int_Y f(0, \tilde{s}, \tilde{\xi} + Dz(y)) dy. \end{aligned}$$

On the other hand the integral of μ on $2^{-k}Y \setminus K_{\tilde{r}}$ is small because

$$\begin{aligned} \left| 2^{nk} \int_{2^{-k}Y \setminus K_{\tilde{r}}} \mu(x; u, z) dx \right| &\leq 2^{nk} \int_{2^{-k}Y \setminus K_{\tilde{r}}} [a(x) + M] dx \\ &= 2^{nk} \int_{2^{-k}Y} [a(x) + M] \chi_{\Omega \setminus K_{\tilde{r}}}(x) dx, \end{aligned}$$

which tends to zero as $k \rightarrow \infty$. These estimates show that (I.1) is satisfied on the open set Y , hence on every open set $\Omega \subset \mathbb{R}^n$. \square

Note that the proof remains almost unchanged if we suppose that (II.2) holds, that $|f|$ satisfies (II.4), and that the functional $u \mapsto F(u, \Omega)$ is sw*lsc on each Dirichlet class $\tilde{u} + W_0^{1,\infty}(\Omega; \mathbb{R}^m)$, with \tilde{u} a polynomial of degree one.

Remark [II.3]. Let f satisfy (II.2) and (II.4). Assume that the functional $u \mapsto F(u, \Omega)$ is sw*lsc on the space of functions u in $W^{1,\infty}(\Omega; \mathbb{R}^m)$ such that $\|Du\|_{L^\infty(\Omega; \mathbb{R}^{nm})} < r$ (where $r > 0$). Then there exists a set $I \subset \Omega$, with $\text{meas}(I) = 0$, such that for every $\tilde{x} \in \Omega \setminus I$, $\tilde{s} \in \mathbb{R}^m$, and $\tilde{\xi} \in B_r(0) \subset \mathbb{R}^{nm}$, and for every $z \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$ such that $\|\tilde{\xi} + Dz\|_{L^\infty(\Omega; \mathbb{R}^{nm})} < r$, we have

$$\text{meas}(\Omega) \cdot f(\tilde{x}, \tilde{s}, \tilde{\xi}) \leq \int_{\Omega} f(\tilde{x}, \tilde{s}, \tilde{\xi} + Dz(x)) dx.$$

Theorems [II.1] and [II.2] generalize results contained in [12], [9]. Our next theorem deals with semicontinuity in $W^{1,p}$, $p \geq 1$.

Theorem [II.4]. Let $1 \leq p < +\infty$, and assume that $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ satisfies (II.2), (II.3) and

$$(II.6) \quad 0 \leq f(x, s, \xi) \leq a(x) + C(|s|^p + |\xi|^p) \text{ for every } x \in \mathbb{R}^n, s \in \mathbb{R}^m, \xi \in \mathbb{R}^{nm},$$

where C is a non-negative constant and a is a non-negative locally summable function on \mathbb{R}^n .

Then for every open set $\Omega \subset \mathbb{R}^n$ the functional $u \mapsto F(u, \Omega)$ is swlsc on $W^{1,p}(\Omega; \mathbb{R}^m)$.

Proof. As in theorem [II.1] we may confine ourselves to a particular set Ω , say a ball. Take $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $(z_k) \subset W^{1,p}(\Omega; \mathbb{R}^m)$ with $z_k \rightarrow 0$ (weakly) in $W^{1,p}(\Omega; \mathbb{R}^m)$. We may suppose

$$\liminf_{k \rightarrow \infty} F(u + z_k, \Omega) = \lim_{k \rightarrow \infty} F(u + z_k, \Omega).$$

This will allow us to select subsequences without altering $\liminf_{k \rightarrow \infty} F(u + z_k, \Omega)$; hence we need not indicate subsequences, denoting all of them with the same index k .

By an extension theorem ([1], Theorem 4.26) we may assume each z_k to be defined on \mathbb{R}^n , with $\|z_k\|_{W^{1,p}(\mathbb{R}^n; \mathbb{R}^m)}$ bounded uniformly with respect to k . Since $C_0^\infty(\mathbb{R}^n; \mathbb{R}^m)$ is dense in $W^{1,p}(\mathbb{R}^n; \mathbb{R}^m)$ and $u \mapsto F(u, \Omega)$ is continuous in the strong topology of $W^{1,p}(\Omega; \mathbb{R}^m)$, there exists a sequence $(w_k) \subset C_0^\infty(\mathbb{R}^n; \mathbb{R}^m)$ such that

$$\|w_k - z_k\|_{W^{1,p}(\mathbb{R}^n; \mathbb{R}^m)} < \frac{1}{k}, \quad |F(u + w_k, \Omega) - F(u + z_k, \Omega)| < \frac{1}{k}.$$

Hence we may assume the sequence (z_k) to be in $C_0^\infty(\mathbb{R}^n; \mathbb{R}^m)$, and to be bounded in $W^{1,p}(\mathbb{R}^n; \mathbb{R}^m)$.

Let $\eta: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous increasing function, with $\eta(0) = 0$, such that for every measurable set $B \subset \Omega$

$$\int_B [a(x) + C(|u(x)|^p + |Du(x)|^p)] dx < \eta(\text{meas}(B)).$$

Fix $\varepsilon > 0$, and apply Lemma [I.7] to each of the m sequences $((M^*z_k^{(i)})^p)$, $1 \leq i \leq m$. This gives a subsequence (z_k) , a set $A_\varepsilon \subset \Omega$, with $\text{meas}(A_\varepsilon) < \varepsilon$, and a real number $\delta > 0$ such that

$$\int_B [(M^*z_k^{(i)})(x)]^p dx < \varepsilon$$

for all k , for $1 \leq i \leq m$, and for every $B \subset \Omega \setminus A_\varepsilon$ with $\text{meas}(B) < \delta$. By Lemma [I.9] we may take $\lambda > 0$ so large that for all i, k

$$(II.7) \quad \text{meas} \{x \in \mathbb{R}^n : (M^*z_k^{(i)})(x) \geq \lambda\} < \min(\varepsilon, \delta).$$

For all i, k set

$$H_{i,k}^\lambda = \{x \in \mathbb{R}^n : (M^*z_k^{(i)})(x) < \lambda\}, \quad H_k^\lambda = \bigcap_{i=1}^m H_{i,k}^\lambda.$$

Lemma [I.11] ensures that, for all $x, y \in H_k^\lambda$ and $1 \leq i \leq m$,

$$\frac{|z_k^{(i)}(y) - z_k^{(i)}(x)|}{|y - x|} \leq c(n) \lambda.$$

Let $g_k^{(i)}$ be a Lipschitz function extending $z_k^{(i)}$ outside H_k^λ , with Lipschitz constant not greater than $c(n) \lambda$ (Lemma [I.12]). Since H_k^λ is an open set we have

$$g_k^{(i)}(x) = z_k^{(i)}(x), \quad Dg_k^{(i)}(x) = Dz_k^{(i)}(x)$$

for all $x \in H_k^\lambda$, and

$$\|Dg_k^{(i)}\|_{L^\infty(\mathbb{R}^n)} \leq c(n) \lambda.$$

We may also assume

$$\|g_k^{(i)}\|_{L^\infty(\mathbb{R}^n)} \leq \|z_k^{(i)}\|_{L^\infty(H_k^\lambda)} \leq \lambda.$$

We may suppose that, at least for a subsequence,

$$g_k^{(i)} \rightharpoonup v^{(i)} \text{ (weak* in } W^{1,\infty}(\Omega))$$

for $1 \leq i \leq m$. Put $(g_k^{(1)}, \dots, g_k^{(m)}) = g_k$, $(v^{(1)}, \dots, v^{(m)}) = v$; we have

$$\begin{aligned} F(u + z_k, \Omega) &\geq F(u + g_k, (\Omega \setminus A_\varepsilon) \cap H_k^\lambda) \\ &= F(u + g_k, \Omega \setminus A_\varepsilon) - F(u + g_k, (\Omega \setminus A_\varepsilon) \setminus H_k^\lambda). \end{aligned}$$

Since

$$\text{meas} [(\Omega \setminus A_\varepsilon) \setminus H_k^\lambda] \leq \sum_{i=1}^m \text{meas} [(\Omega \setminus A_\varepsilon) \setminus H_{i,k}^\lambda] < m \min(\varepsilon, \delta)$$

by (II.6) and by our choice of A_ε we obtain

$$\begin{aligned} F(u + g_k, (\Omega \setminus A_\varepsilon) \setminus H_k^\lambda) &\leq 2^{p-1} \{ \eta(m\varepsilon) + c(n, \Omega) \lambda^p \text{meas} [(\Omega \setminus A_\varepsilon) \setminus H_k^\lambda] \} \\ &\leq 2^{p-1} \{ \eta(m\varepsilon) + c(n, \Omega) \sum_{i=1}^m \int_{(\Omega \setminus A_\varepsilon) \setminus H_{i,k}^\lambda} [(M^* z_k^{(i)})(x)]^p dx \} \\ &\leq 2^{p-1} \{ \eta(m\varepsilon) + mc(n, \Omega) \varepsilon \} = O(\varepsilon). \end{aligned}$$

Thus

$$F(u + z_k, \Omega) \geq F(u + g_k, \Omega \setminus A_\varepsilon) - O(\varepsilon).$$

Choose an open set $\Omega' \subset \Omega$ containing $\Omega \setminus A_\varepsilon$ and such that

$$|F(u + g_k, \Omega') - F(u + g_k, \Omega \setminus A_\varepsilon)| < \varepsilon$$

(this is possible since the functions g_k are uniformly bounded in $W^{1,\infty}(\Omega; \mathbb{R}^m)$).

Applying Theorem [II.1] to the functional

$$I(w, S) = \int_S \gamma(x, w(x), Dw(x)) dx,$$

where

$$\gamma(x, s, \xi) = f(x, u(x) + s, Du(x) + \xi),$$

we are led to

$$\begin{aligned} \lim_{k \rightarrow \infty} F(u + z_k, \Omega) &\geq \liminf_{k \rightarrow \infty} F(u + g_k, \Omega') - \varepsilon - O(\varepsilon) \\ &\geq F(u + v, \Omega') - \varepsilon - O(\varepsilon). \end{aligned}$$

At least for a subsequence we may suppose that $z_k(x) \rightarrow 0$ for almost all $x \in \Omega$. Set

$$G = \{x \in \Omega : v(x) \neq 0\}$$

and

$$\tilde{G} = G \cap \{x \in \Omega : z_k(x) \rightarrow 0\},$$

so that $\text{meas}(G) = \text{meas}(\tilde{G})$. Since the functions g_k are continuous and converge to v in L^∞ , we have

$$g_k(x) \rightarrow v(x)$$

for all $x \in \Omega$, hence for all $x \in G$. If we now suppose

$$\text{meas}(G) > (m + 1) \varepsilon$$

we obtain a contradiction. Indeed by (II.7)

$$\text{meas}(\tilde{G} \cap H_k^\lambda) > \varepsilon \text{ for all } k,$$

and by Lemma [I.6], for a subsequence,

$$\left(\bigcap_{h \in \mathbb{N}} H_{k_h}^\lambda \right) \cap \tilde{G} \neq \emptyset.$$

If \tilde{x} belongs to this set, then

$$v(\tilde{x}) = \lim_{h \rightarrow \infty} g_{k_h}(\tilde{x}) = \lim_{h \rightarrow \infty} z_{k_h}(\tilde{x}) = 0,$$

contrary to the definition of G .

We may thus write, by the positivity of f ,

$$\begin{aligned} \lim_{k \rightarrow \infty} F(u + z_k, \Omega) &\geq F(u, \Omega' \setminus G) - O(\varepsilon) - \varepsilon \\ &\geq F(u, \Omega) - O(\varepsilon) - \varepsilon - \eta[(m+2)\varepsilon], \end{aligned}$$

which concludes the proof since ε is arbitrary.

In this proof the role played by the hypothesis $f \geq 0$ is fundamental. Indeed if (II.6) is changed to

$$|f(x, s, \xi)| \leq a(x) + C(|s|^p + |\xi|^p),$$

and (II.2), (II.3) are satisfied, then Theorem [II.4] is false, at least for $n > 2$, but one can prove that for all $\varepsilon > 0$ the functional $u \mapsto F(u, \Omega)$ is swlsc on $W^{1,p+\varepsilon}(\Omega; \mathbb{R}^m)$ (see [9]).

Since semicontinuity on $W^{1,p}$ implies semicontinuity on $W^{1,\infty}$, we may summarize the results of this section as follows:

Statement [II.5]. *Let $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ be a Carathéodory function which satisfies (II.6) for some $p \geq 1$ [or alternately satisfies (II.4)]. Then the functional $u \mapsto F(u, \Omega)$ is swlsc on $W^{1,p}(\Omega; \mathbb{R}^m)$ [or is sw*lsc on $W^{1,\infty}(\Omega; \mathbb{R}^m)$] if and only if f is quasi-convex in ξ .*

III. A Representation Theorem

In this section, given a functional of the type (II.1) with f not necessarily quasi-convex in ξ , we deal with the problem of finding its lsc envelope on $W^{1,p}(\Omega; \mathbb{R})$, i.e. the greatest functional less than or equal to F which is swlsc on $W^{1,p}(\Omega; \mathbb{R}^m)$. As a consequence of statement [II.5], it will suffice to treat the case $p = +\infty$.

Let $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying (II.4). For every $r > 0$ and for every Ω bounded open set of \mathbb{R}^n , if $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\|Du\|_{L^\infty(\Omega; \mathbb{R}^{nm})} \leq r$, we define

$$F(r, u, \Omega) = \inf \{ \liminf_{k \rightarrow \infty} F(u_k, \Omega) : u_k \rightharpoonup u \text{ (weak*) in } W^{1,\infty}(\Omega; \mathbb{R}^m) \}$$

$$\text{and } \|Du_k\|_{L^\infty(\Omega; \mathbb{R}^{nm})} \leq r \}$$

$$F_0(r, u, \Omega) = \inf \{ \liminf_{k \rightarrow \infty} F(u_k, \Omega) : (u_k - u) \rightarrow 0 \text{ (weak*) in } W_0^{1,\infty}(\Omega; \mathbb{R}^m) \}$$

$$\text{and } \|Du_k\|_{L^\infty(\Omega; \mathbb{R}^{nm})} \leq r \},$$

where $F(u, \Omega)$ is defined by (II.1). The argument employed in [11], Lemmas 3.3 and 4.5, leads us to the following results.

Lemma [III.1]. *If f satisfies the foregoing hypotheses, then for every $r > 0$ and every $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\|Du\|_{L^\infty(\Omega; \mathbb{R}^{nm})} < r$ there exists a function $h_u \in L^1(\Omega)$ such that*

$$F(r, u, \Omega') = F_0(r, u, \Omega') = \int_{\Omega'} h_u(x) dx$$

for every open set $\Omega' \subset \Omega$.

Lemma [III.2]. *Let $u_1, u_2 \in W^{1,\infty}(\Omega; \mathbb{R}^m)$, with $\|Du_i\|_{L^\infty(\Omega; \mathbb{R}^{nm})} < r$ and $\|u_i\|_{L^\infty(\Omega; \mathbb{R}^m)} < d$, $i = 1, 2$. Then for every open set $\Omega' \subset \Omega$ we have*

$$|F(r, u_1, \Omega') - F(r, u_2, \Omega')| \leq \int_{\Omega'} \omega(x, d, 3r, \|u_1 - u_2\|_{W^{1,\infty}(\Omega; \mathbb{R}^m)}) dx,$$

where

$$\omega(x, d, r, \delta) = \sup \{|f(x, s_1, \xi_1) - f(x, s_2, \xi_2)| : |s_i| < d, |\xi_i| < r \text{ for } i = 1, 2, \\ \text{and } |s_1 - s_2| + |\xi_1 - \xi_2| < \delta\}.$$

We now use these results to prove

Lemma [III.3]. *To each $r > 0$ there exists a Carathéodory function ϕ_r , defined on $\Omega \times \mathbb{R}^m \times \{\xi \in \mathbb{R}^{nm} : |\xi| < r\}$ such that for every $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\|Du\|_{L^\infty(\Omega; \mathbb{R}^{nm})} < r$ we have*

$$\phi_r(x, u(x), Du(x)) = h_u(x) \text{ for almost every } x \in \Omega.$$

Proof. Let \mathcal{A}_r be the class of all affine functions on \mathbb{R}^n with rational coefficients and with gradient less than r in norm. Also let L be the set of the points in \mathbb{R}^n which are Lebesgue points for every function h_u , with $u \in \mathcal{A}_r$. For $x \in L$, $s \in \mathbb{Q}^m$, $\xi \in \mathbb{Q}^{nm}$, with $|\xi| < r$, put

$$\phi_r(x, s, \xi) = h_u(x),$$

where $u \in \mathcal{A}_r$, $u(x) = s$, $Du = \xi$. Lemma [III.2] implies that ϕ_r is continuous in (s, ξ) for almost every $x \in L$. Since $L \times \mathbb{Q}^m \times \mathbb{Q}^{nm}$ is dense in $L \times \mathbb{R}^m \times \mathbb{R}^{nm}$, we may therefore extend the definition of ϕ_r to $\Omega \times \mathbb{R}^m \times \{\xi \in \mathbb{R}^{nm} : |\xi| < r\}$, obtaining

$$|\phi_r(x, s_1, \xi_1) - \phi_r(x, s_2, \xi_2)| \leq \omega(x, d, 3r, \delta)$$

or almost every $x \in \Omega$ and for $|s_i| < d$, $|\xi_i| < r$ ($i = 1, 2$), whenever $|s_1 - s_2| + |\xi_1 - \xi_2| < \delta$. This inequality yields

$$h_u(x) = \phi_r(x, u(x), Du(x))$$

for every $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\|Du\|_{L^\infty(\Omega; \mathbb{R}^{nm})} < r$ and for almost every $x \in \Omega$.

It remains to be proved that for all (s, ξ) the function $x \mapsto \phi_r(x, s, \xi)$ is measurable. Let $s_1 \in \mathbb{R}$ and let u be affine with $u(0) = s_1$, $Du = \xi$. For almost every $x \in \Omega$ we have

$$\phi_r(x, s_1 + \xi \cdot x, \xi) = h_u(x),$$

hence this function is measurable. If ψ is a simple function, i.e. $\psi(x) = \sum_{i=1}^k s_i \chi_{E_i}(x)$, with each E_i measurable and $E_i \cap E_j = \emptyset$ if $i \neq j$, then

$$\phi_r(x, \psi(x) + \xi \cdot x, \xi) = \sum_{i=1}^k \phi_r(x, s_i + \xi \cdot x, \xi) \chi_{E_i}(x).$$

Therefore by an approximation argument we can prove that $x \mapsto \phi_r(x, \theta(x) + \xi \cdot x, \xi)$ is measurable, for $\theta \in L^1(\Omega)$. This happens in particular if $\theta(x) = s - \xi \cdot x$, and the proof is complete.

The above lemma, together with the semicontinuity of $F(r, u, \Omega)$ and Remark [II.3], implies

Remark [III.4]. For every $(x, s, \xi) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm}$ set

$$\phi(x, s, \xi) = \lim_{r \rightarrow \infty} \phi_r(x, s, \xi) = \inf_{r > |\xi|} \phi_r(x, s, \xi).$$

The function ϕ is measurable in x , upper semi-continuous in s , continuous in ξ , and quasi-convex in ξ .

Let $\tilde{x} \in \Omega$, $\tilde{s} \in \mathbb{R}^m$. Lemma [III.3] implies that for all $r > 0$ there exists a function $g_r^{(\tilde{x}, \tilde{s})}$ such that

$$\int_{\Omega} g_r^{(\tilde{x}, \tilde{s})}(Du(x)) \, dx = \inf \left\{ \liminf_{k \rightarrow \infty} \int_{\Omega} f(\tilde{x}, \tilde{s}, Du_k(x)) \, dx : u_k \rightarrow u \right.$$

$$\left. \text{(weak*) in } W^{1,\infty}(\Omega; \mathbb{R}^m) \text{ and } \|Du_k\|_{L^\infty(\Omega; \mathbb{R}^{nm})} \leq r \right\}$$

$$\text{for all } u \in W^{1,\infty}(\Omega; \mathbb{R}^m) \text{ with } \|Du\|_{L^\infty(\Omega; \mathbb{R}^{nm})} < r.$$

Put

$$g^{(\tilde{x}, \tilde{s})}(\xi) = \lim_{r \rightarrow \infty} g_r^{(\tilde{x}, \tilde{s})}(\xi).$$

Theorem [III.5]. For almost every $x \in \Omega$ and every $s \in \mathbb{R}^m$ the function $\xi \mapsto \phi(x, s, \xi)$ is the greatest quasi-convex function less than or equal to $\xi \mapsto f(x, s, \xi)$.

Proof. Let $K \subset \Omega$ be a compact set such that f is continuous on $K \times \mathbb{R}^m \times \mathbb{R}^{nm}$. For $x \in K$, $s \in \mathbb{R}^m$, set

$$g_r(x, s, \xi) = g_r^{(x,s)}(\xi).$$

By the uniform continuity of f on bounded subsets of $K \times \mathbb{R}^m \times \mathbb{R}^{nm}$, g_r is continuous on $K \times \mathbb{R}^m \times \{\xi \in \mathbb{R}^{nm} : |\xi| < r\}$. Since K is arbitrary, g_r is defined for almost every $x \in \Omega$, and every $s \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{nm}$, with $|\xi| < r$. Moreover, Lemma [I.5] implies that g_r is a Carathéodory function, and it is quasi-convex in ξ since the same holds for all $g_r^{(x,s)}$. As we remarked after Theorem [II.1], the functional

$$G_r(u, \Omega) = \int_{\Omega} g_r(x, u(x), Du(x)) \, dx$$

is sw*lsc on $\{u \in W^{1,\infty}(\Omega; \mathbb{R}^m) : \|Du\|_{L^\infty(\Omega; \mathbb{R}^{nm})} < r\}$.

If we set, for all x, s, ξ ,

$$g(x, s, \xi) = g^{(x,s)}(\xi),$$

then the functional

$$u \mapsto \int_{\Omega} g(\tilde{x}, \tilde{s}, Du(x)) \, dx$$

is the lsc envelope on $W^{1,\infty}(\Omega; \mathbb{R}^m)$ of the functional

$$u \mapsto \int_{\Omega} f(\tilde{x}, \tilde{s}, Du(x)) \, dx.$$

Hence $\xi \mapsto g(\tilde{x}, \tilde{s}, \xi)$ is the greatest quasi-convex function not greater than $\xi \mapsto f(\tilde{x}, \tilde{s}, \xi)$. This implies $g \geq \phi$. For every $r > 0$, G_r is semicontinuous, hence $G_r(u, \Omega) \leq F(r, u, \Omega)$, and $g_r \leq \phi_r$, whence $g \leq \phi$. \square

Note that the function ϕ does not necessarily represent the lsc envelope of $u \mapsto F(u, \Omega)$. Indeed, if ϕ is not a Carathéodory function, there is a counterexample even if f is convex in ξ (example 3.11 in [11]).

We give here some conditions which ensure that ϕ is a Carathéodory function.

Theorem [III.6]. *If either of the conditions*

(III.1) $f = f(x, \xi)$, or

(III.2) $|f(x, s_1, \xi) - f(x, s_2, \xi)| < \omega(x, |s_1 - s_2|) \beta(|\xi|)$,
 where $\omega: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^+$ is a Carathéodory function, $\omega(x, 0) = 0$, and β is increasing and non-negative,

is satisfied, then ϕ is a Carathéodory function.

Proof. If (III.1) holds, the result follows from Remark [III.4]. Next assume that (III.2) holds. We note (see [8], Corollary 2.4) that if $\psi: \mathbb{R}^q \rightarrow \mathbb{R}^+$ is a convex function and if we set $M = \max_{|y| \leq R} \psi(y)$, then for all $r < R$ and $y_1, y_2 \in B_r(0)$ there holds

$$|\psi(y_1) - \psi(y_2)| \leq \frac{M}{R-r} |y_1 - y_2|.$$

Since quasi-convexity implies weak quasi-convexity, this estimate shows that if $\psi: \mathbb{R}^{nm} \rightarrow \mathbb{R}^+$ is quasi-convex then for all $r < R$ and $\xi_1, \xi_2 \in B_r(0)$ we have

$$|\psi(\xi_1) - \psi(\xi_2)| \leq \frac{M \sqrt{\min(m, n)}}{R-r} |\xi_1 - \xi_2|,$$

where we have put $M = \max_{|\xi| \leq R} \psi(\xi)$.

Note that ϕ almost everywhere satisfies the inequality

$$|\phi(x, s_1, \xi) - \phi(x, s_2, \xi)| < \omega(x, |s_1 - s_2|) \beta(|\xi|),$$

as one can see by proving the same estimate for the function g , and then using the equality $g = \phi$.

Choose $R > 0$, and for all $x \in \Omega$ put

$$M(x) = a(x) + \sup \{b(s, \xi) : |s| \leq R, |\xi| \leq R\} \\ \geq \sup \{\phi(x, s, \xi) : |s| \leq R, |\xi| \leq R\}.$$

For almost all $x \in \Omega$ one has, for all $r < R$, $s_1, s_2 \in B_r(0) \subset \mathbb{R}^m$, and $\xi_1, \xi_2 \in B_r(0) \subset \mathbb{R}^{nm}$,

$$|\phi(x, s_1, \xi_1) - \phi(x, s_2, \xi_2)| \leq \frac{M(x) \sqrt{\min(m, n)}}{R - r} |\xi_1 - \xi_2| + \\ + \omega(x, |s_1 - s_2|) \beta(R). \quad \square$$

We summarize the results of section III as follows.

Statement [III.7]. Let $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}$ be a Carathéodory function which satisfies (II.6) for some $p \geq 1$ [or alternately satisfies (II.4)], and let either one of the conditions (III.1), (III.2) hold. Then the lsc envelope on $W^{1,p}(\Omega; \mathbb{R}^m)$ [on $W^{1,\infty}(\Omega; \mathbb{R}^m)$] of $u \mapsto F(u, \Omega)$ is the functional

$$u \mapsto \int_{\Omega} \phi(x, u(x), Du(x)) dx,$$

where for almost all $x \in \Omega$ and for all $s \in \mathbb{R}^m$ the function

$$\xi \mapsto \phi(x, s, \xi)$$

is the greatest quasi-convex function which is less than or equal to $\xi \mapsto f(x, s, \xi)$.

This theorem provides an extension of the results of [6] to the case in which f depends on u as well as on x and ξ .

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Scuola Normale Superiore

Pisa

and

Istituto di Matematica “R. Caccioppoli”
Università di Napoli

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