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A transmission problem in the calculus of variations

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Abstract. We study hölder regularity of minimizers of the functional $\int_\Omega |Du|^{p(x)}dx$, where $p(x)$ takes only two values and jumps across a Lipschitz surface. No restriction on the two values is imposed.

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1 Introduction

Many classical results concern the regularity of minimizers u of

$$\int_\Omega F(x, Du)\,dx,$$

where F satisfies the standard growth conditions

$$|z|^p \le F(x,z) \le c(1+|z|^p) \quad \text{for some } p > 1. \tag{1}$$

Following an example by Giaquinta [5], see also [7],[9], several papers appeared in which (1) is replaced by

$$|z|^p \le F(x,z) \le c(1+|z|^q) \quad \text{for some } q > p > 1.$$

Most of these deal with anisotropic (but essentially homogeneous) situations, as e.g.

$$F(x,z) = \sum_{i=1}^N a_i(x)|z_i|^{p_i}$$

with $a_i \ge c > 0$ and $1 < p_1 \le \cdots \le p_N$, so that the growth with respect to z is the same for all x: see e.g. [1], [10], [3].

However, in some physical situations (as e.g. electrostatic fields in which conductivity depends on the intensity of the field, or thermal equilibrium in composite nonlinearly conductive materials) it is natural to consider energies of the form

$$\int_\Omega a(x, |Du|)|Du|^2\, dx$$

in which the growth exponent of $a(x, z)$ with respect to z depends on the position x. The simplest model is the functional

$$\int_\Omega |Du|^{p(x)}\, dx, \quad \text{with } 1 < p_0 \leq p(x) \leq q_0, \tag{2}$$

which was investigated by Zhikov [13] in the context of homogenization.

We study some regularity properties for minimizers of a class of functionals which includes (2) in the case when $p(x)$ has only two possible values (this corresponds to the case of a conductor made of two different homogeneous materials). To fix the ideas, take an open set Ω which is split by a Lipschitz surface Σ in two parts, Ω^- and Ω^+, and take

$$\mathscr{F}(u) = \int_{\Omega^-} |Du|^p\, dx + \int_{\Omega^+} |Du|^q\, dx \tag{3}$$

with $1 < p < q$. One of the results we prove (Theorem 2.3) implies that any local minimizer u of $\mathscr{F}$ in Ω is Hölder continuous. We remark that we only deal with the scalar case ($u : \Omega \to \mathbb{R}$) but, unlike any previous result (see [1], [11], [12]) we do not impose any restriction on q with respect to p. It is well known that any minimizer to (3) is smooth inside Ω^- and Ω^+, thus the point here (as in classical transmission problems) is to provide regularity across Σ. Trying to follow the general lines of [2] and [8], one is led to an unbalanced Caccioppoli estimate (6) in the balls intersecting Σ, which is still enough to prove boundedness of the minimizers u, which is done in Sect. 3, but is not useful to bound the oscillation of u; this difficulty is overcome in Sect. 4.

2 Notation and statement of the results

Let Ω be a connected open subset of $\mathbb{R}^n$, and let Σ be a compact lipschitz continuous $(n-1)$-dimensional manifold in $\mathbb{R}^n$: by this we mean that for every $x_0 \in \Sigma$ there exist a neighbourhood U of x_0, and a bilipschitz mapping from U to $\mathbb{R}^n$ such that the image of $\Sigma \cap U$ lies in a hyperplane; due to the compactness of Σ, we denote by L the greatest of the lipschitz constants of the mappings needed to cover Σ and their inverses.

We denote by Ω^+, Ω^- two open subsets of Ω such that Ω is the disjoint union of Ω^+, Ω^- and $\Omega \cap \Sigma$, and for every $x_0 \in \Omega \cap \Sigma$ and every neighbourhood U of x_0 there exists a neighbourhood V of x_0, contained in U, and such that both $\Omega^+ \cap V$ and $\Omega^- \cap V$ are connected (i.e., Σ locally separates Ω^+ and Ω^-).

Let $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Caratheodory function (i.e., measurable with respect to $x \in \Omega$, continuous with respect to $z \in \mathbb{R}^n$) satisfying

$$\begin{aligned} &|z|^q \leq F(x, z) \leq L(1 + |z|^q) \qquad \text{for a.e. } x \in \Omega^+ \\ &|z|^p \leq F(x, z) \leq L(1 + |z|^p) \qquad \text{for a.e. } x \in \Omega^- \end{aligned} \tag{4}$$

with $1 \le p \le q$; note that we used the same letter L as above. We will sometimes use the notation

$$p(x) = \begin{cases} q & \text{if } x \in \Omega^+ \\ p & \text{if } x \in \Omega^- . \end{cases}$$

We introduce a space of weakly differentiable functions in Ω by setting for every open set A

$$W^{1,(q^+,p^-)}(A) = W^{1,p}(A) \cap W^{1,q}(A \cap \Omega^+)$$

and

$$W^{1,(q^+,p^-)}_{\text{loc}}(\Omega) = \{u : u \in W^{1,(q^+,p^-)}(\Omega') \ \forall \Omega' \subset\subset \Omega\} .$$

In the sequel we study local minimizers of the functional

$$\mathscr{F}(v, A) = \int_A F(x, Dv)\, dx ,$$

i.e., functions $u \in W^{1,(q^+,p^-)}_{\text{loc}}(\Omega)$ such that

$$\mathscr{F}(u, \operatorname{spt} \varphi) \le \mathscr{F}(u + \varphi, \operatorname{spt} \varphi) \qquad \forall \varphi \in C^1_0(\Omega) .$$

First we give a higher integrability and a local boundedness result:

Theorem 2.1 *Assume condition* (4) *holds; for every* $1 < p \le q$ *there exists* $\varepsilon > 0$, *depending on* (n, p, q, L), *such that if* $u \in W^{1,(q^+,p^-)}_{\text{loc}}(\Omega)$ *is a local minimizer of* $\mathscr{F}$ *then*

$$|Du|^{p(x)} \in L^{1+\varepsilon}_{\text{loc}}(\Omega) ,$$

and for every ball $B_R(x_0) \subset \Omega$

$$\left[-\!\!\!\!\!\!\int_{B_{R/2}(x_0)} |Du|^{(1+\varepsilon)p(x)}\, dx \right]^{1/(1+\varepsilon)} \le c -\!\!\!\!\!\!\int_{B_R(x_0)} (1 + |Du|^{p(x)})\, dx ,$$

with c independent of u. Moreover if $1 < p \le q \le n$ *then* $\varepsilon \ge \varepsilon_0(n, p, L)$.

Theorem 2.2 *Assume condition* (4) *holds; for any* $1 \le p \le q$, *every local minimizer* $u \in W^{1,(q^+,p^-)}_{\text{loc}}(\Omega)$ *of* $\mathscr{F}$ *is locally bounded in* Ω.

To obtain the following Hölder continuity result we need another assumption:

(5) $\qquad F$ is convex with respect to z.

Then we have:

Theorem 2.3 *Assume conditions* (4),(5) *hold, with* $1 < p \le q$. *If* $u \in W^{1,(q^+,p^-)}_{\text{loc}}(\Omega)$ *is a local minimizer of* $\mathscr{F}$ *satisfying*

$$\sup_{\Omega} |u| \le M$$

there exists $\alpha > 0$, *depending on* (n, p, q, L, M), *such that* $u \in C^{0,\alpha}(\overline{\Omega})$.

However,

Proposition 2.4 *For every $p > 1$ there exists $q_0 < n$ such that if $q > q_0$ the convexity assumption* (5) *is no longer needed in Theorem* 2.3.

Remark 2.5 Due to Theorem 2.2, one may remove the assumption $\sup_\Omega |u| \leq M$ in Theorem 2.3 and Proposition 2.4, thus obtaining that for every $\Omega' \subset\subset \Omega$ one has $u \in C^{0,\alpha}(\overline{\Omega'})$ with α depending on $\sup_{\Omega'} |u|$.

Remark 2.6 Our results are local, hence we will always assume that $u \in W^{1,(q^+,p^-)}(\Omega)$, remarking that all the relevant constants will not be affected by this restriction.

Remark 2.7 In a subject which is already burdened with technicalities, we preferred not to introduce any explicit dependence of F on u; the most interested readers, although, could quite easily deal with this general case.

Remark 2.8 It is not restrictive to suppose in the sequel that Σ lies on a hyperplane. Indeed, fix $x_0 \in \Sigma \cap \Omega$: there exist a neighbourhood U of x_0 in $\mathbb{R}^n_x$, a neighbourhood U' of 0 in $\mathbb{R}^n_{x'}$, and a bilipschitz homeomorphism $\Phi : U' \to U$ such that $\Sigma \cap U = \Phi(\{x' \in U' : x'_n = 0\})$. If F satisfies (4), and eventually (5), the function

$$G(x', z) = |\det D\Phi(x')|\, F\Big(\Phi(x'), z\, \big[D\Phi(x')\big]^{-1}\Big)$$

satisfies (4), and eventually (5), although with different constants, depending on L and n but not on x_0. Moreover if u is a local minimizer of $\mathcal{F}$ in U, then $u'(x') = u(\Phi(x'))$ is a local minimizer of $\int_{U'} G(x', Dv)\, dx'$. Finally, the summability and Hölder continuity exponents of u and u' are the same.

We will often make use of cubes: all will be supposed to have a face parallel to Σ, and to be contained in Ω. The symbol Q_R (or occasionally $Q_R(x_0)$ to stress its center) will denote a cube with side R, whereas $Q_{\lambda R}$ will be for any positive λ a cube concentric with Q_R. Also, if the center of Q_R belongs to Σ we set $Q_R^+ = Q_R \cap \Omega^+$, and $Q_R^- = Q_R \cap \Omega^-$.

If $u : Q_R \to \mathbb{R}$ we set for any real K

$$Q_{R,K} = \{x \in Q_R : u(x) > K\}\ ,$$

and similarly for $Q^+_{R,K}$ and $Q^-_{R,K}$.

Finally, the symbol $⨍_A$ denotes the average over a set A.

3 Higher integrability and local boundedness

As in the standard case $p = q$, the tools needed in order to prove higher integrability for the gradient Du, and local boundedness of a minimizer u, are Gehring theorem, and suitable versions of Caccioppoli (Proposition 3.4) and Sobolev - Poincaré (Proposition 3.2) inequalities.

In the sequel, we shall usually not remark any dependence on n and L, which is shared by virtually all constants.

Let Q_R be a cube centered on Σ, set $\Sigma_R = Q_R \cap \Sigma$ and denote by $\mathrm{Tr}\,(u)$ the trace of u on Σ, whenever it exists.

Proposition 3.1 *If $u \in W^{1,s}(Q_R^+)$ with $1 \le s \le s_0 < n$ then*

$$\left(\fint_{Q_R^+}\left|\frac{u-\lambda}{R}\right|^{s^*} dx\right)^{1/s^*} \le c\left(\fint_{Q_R^+}|Du|^s\,dx\right)^{1/s},$$

where

$$\lambda = \fint_{\Sigma_R} \mathrm{Tr}\,(u)\,d\mathscr{H}_{n-1}(\sigma)$$

and $c = c(s_0)$.

Proof Assume the statement is false. Then we may find a sequence $(s_h)_h$ with $1 \le s_h \le s_0$, and a sequence $(u_h)_h$, with $u_h \in W^{1,s_h}(Q_R^+)$, such that

$$\fint_{\Sigma_R} \mathrm{Tr}\,(u_h)\,d\mathscr{H}_{n-1}(\sigma) = 0\,,$$

$$\fint_{Q_R^+}|Du_h|^{s_h}\,dx \to 0\,,$$

$$\fint_{Q_R^+}\left|\frac{u_h}{R}\right|^{s_h^*} dx = 1\,.$$

Hence $u_h \to 0$ strongly in $W^{1,1}$. Remarking that

$$\left(\frac{s_h^*}{1^*}-1\right)\frac{s_h}{s_h-1} = s_h^*\,,$$

if we set $v_h = |u_h|^{s_h^*/1^*}$ then

$$\|Dv_h\|_1 \le c\|Du_h\|_{s_h}\,\|u_h\|_{s_h^*} \to 0$$

and, interpolating between 1 and s^*,

$$\|v_h\|_1 \to 0\,,$$

since $\|u_h\|_1 \to 0$. By Rellich theorem, $\|v_h\|_{1^*} = \|u_h\|_{s_h^*}^{s_h^*} \to 0$, thus achieving a contradiction.

As we did for p and q, we set

$$r(x) = \begin{cases} s & \text{if } x \in Q_R^+ \\ r & \text{if } x \in Q_R^-. \end{cases}$$

If $u \in W^{1,(s^+,r^-)}(Q_R)$, in particular $u \in W^{1,r}(Q_R)$, therefore the traces of u on both sides of Σ_R agree $\mathscr{H}_{n-1}$-a.e. on Σ_R, and their averages on Σ_R are the same. Therefore we have the following Sobolev-Poincaré inequality:

Proposition 3.2 *If $1 \le r \le s \le s_0 < n$ and $u \in W^{1,(s^+,r^-)}(Q_R)$, then*

$$\fint_{Q_R}\left(\left|\frac{u-\lambda}{R}\right|^{r(x)}\right)^{n/(n-r)} dx \le c\left(\fint_{Q_R}|Du|^{r(x)}\,dx\right)^{n/(n-r)},$$

where

$$\lambda = \fint_{\Sigma_R} \mathrm{Tr}\,(u)\, d\mathscr{H}_{n-1}(\sigma)$$

and $c = c(s_0)$. *The result still holds with both exponents* $n/(n-r)$ *replaced by any number between* 1 *and* $n/(n-r)$. *Moreover, if* $u \in W_0^{1,r}(Q_R)$ *the inequality above holds even if we take* $\lambda = 0$ *instead of the average of the trace.*

Proof Remarking that

$$\left[\fint_{Q_R^+} \left(\left|\frac{u-\lambda}{R}\right|^s\right)^{n/(n-r)} dx\right]^{(n-r)/n} \le \left[\fint_{Q_R^+} \left(\left|\frac{u-\lambda}{R}\right|^s\right)^{n/(n-s)} dx\right]^{(n-s)/n},$$

we get by Proposition 3.1 (which clearly holds also with r and Q_R^-)

$$\fint_{Q_R^-} \left(\left|\frac{u-\lambda}{R}\right|^r\right)^{n/(n-r)} dx \le c \left(\fint_{Q_R^-} |Du|^r\, dx\right)^{n/(n-r)}$$

$$\fint_{Q_R^+} \left(\left|\frac{u-\lambda}{R}\right|^s\right)^{n/(n-r)} dx \le c \left(\fint_{Q_R^+} |Du|^s\, dx\right)^{n/(n-r)},$$

and the result follows. The final remarks are easy.

The following algebraic lemma is similar to [4] Sect. 5, Lemma 3.1.

Lemma 3.3 *Let* $f : [R, 2R] \to [0, +\infty)$ *be a bounded function satisfying*

$$f(t_1) \le \vartheta f(t_2) + \frac{\alpha}{(t_2 - t_1)^p} + \frac{\beta}{(t_2 - t_1)^q} + \gamma$$

for some $0 < \vartheta < 1$ *and all* $R \le t_1 < t_2 \le 2R$, *with* $\alpha, \beta, \gamma \ge 0$ *and* $1 \le p \le q$. *Then there exists a constant c such that*

$$f(R) \le c\left(\frac{\alpha}{R^p} + \frac{\beta}{R^q} + \gamma\right).$$

The constant $c = c(\vartheta, q)$ *is increasing with respect to q.*

Proof Let $\tau = \big((1+\vartheta)/2\big)^{1/q} < 1$, and set $t_k = 2R(1 - \tau^k/2)$. Then

$$t_0 = R\,, \qquad \lim_k t_k = 2R\,, \qquad t_{k+1} - t_k = R(1-\tau)\tau^k\,,$$

and for all k we get from the assumption on f

$$f(t_k) \le \vartheta f(t_{k+1}) + \frac{1}{(1-\tau)^q \tau^{kq}}\left(\frac{\alpha}{R^p} + \frac{\beta}{R^q} + \gamma\right),$$

whence by induction, remarking that $\vartheta/\tau < 1$, we obtain

$$f(R) \le \frac{\tau^q}{(1-\tau)^q\,(\tau^q - \vartheta)}\left(\frac{\alpha}{R^p} + \frac{\beta}{R^q} + \gamma\right),$$

hence the result with $c = \dfrac{2}{1-\vartheta}\left[\left(\dfrac{2}{1+\vartheta}\right)^{1/q} - 1\right]^{-q}$.

This result enables us to prove a Caccioppoli inequality for the minimizers.

Proposition 3.4 *There exists a constant $c = c(n,p,q)$ such that for every minimizer $u \in W^{1,(q^+,p^-)}(\Omega)$ of $\mathscr{F}$ and any Q_R such that $Q_{2R} \subseteq \Omega$*

$$\fint_{Q_R} |Du|^{p(x)}\,dx \le c\left(1 + \fint_{Q_{2R}} \left|\frac{u-\lambda}{R}\right|^{p(x)} dx\right)$$

for every $\lambda \in \mathbb{R}$. Moreover if $1 \le p \le q \le n$ we have $c \le c(n)$.

Proof Fix λ, and take Q_R centered on Σ. With a standard choice of test functions, using the hole-filling technique (see e.g. [4] p. 160) we get for any $R \le t < s \le 2R$

$$\begin{aligned}\int_{Q_t} |Du|^{p(x)}\,dx \le \vartheta \int_{Q_s} |Du|^{p(x)}\,dx &\\ + \frac{c}{(s-t)^p} \int_{Q_{2R}^-} |u-\lambda|^p\,dx &\\ + \frac{c}{(s-t)^q} \int_{Q_{2R}^+} |u-\lambda|^q\,dx + cR^n &\end{aligned}$$

where $\vartheta < 1$ and c depend on p,q,n and if $1 \le p \le q \le n$

$$0 < \vartheta_1(n) \le \vartheta \le \vartheta_2(n) < 1\,, \qquad c \le c(n)\,.$$

For this class of cubes the result then follows from Lemma 3.3. If Q_{2R} is entirely contained in Ω^+ or in Ω^-, the proof is even easier. Finally, if $Q_{2R}(x_0) \cap \Sigma \ne \emptyset$, there exists $x_0' \in \Sigma$ such that

$$Q_R(x_0) \subset Q_{3R}(x_0') \subset Q_{6R}(x_0') \subset Q_{8R}(x_0)\,;$$

since the result is true on $Q_{3R}(x_0')$, we deduce in the case $Q_{8R}(x_0) \subset \Omega$

$$\fint_{Q_R(x_0)} |Du|^{p(x)}\,dx \le c\left(1 + \fint_{Q_{8R}(x_0)} \left|\frac{u-\lambda}{R}\right|^{p(x)} dx\right),$$

and the inequality with Q_R and Q_{2R} follows by a (finite) covering argument.

The Sobolev - Poincaré and Caccioppoli inequalities just proved are the key tools to prove the higher integrability result.

Lemma 3.5 *There exists a constant $c = c(n,p,q)$ such that for every minimizer $u \in W^{1,(q^+,p^-)}(\Omega)$ of $\mathscr{F}$ and for every Q_R such that $Q_{2R} \subset \Omega$*

$$\fint_{Q_R} |Du|^{p(x)}\,dx \le c\left[1 + \left(\fint_{Q_{2R(x_0)}} |Du|^{p(x)/(1+\sigma)}\,dx\right)^{1+\sigma}\right],$$

where $\sigma = p/n$ if $p \ge n/(n-1)$, and $\sigma = p-1$ if $1 \le p < n/(n-1)$. Moreover if $1 \le p \le q \le n$ we have $c \le c(n)$.

Proof We deal first with the case $p \geq n/(n-1)$. If Q_{2R} is centered on Σ, we choose

$$\lambda = \fint_{\Sigma_{2R}} \mathrm{Tr}\,(u)\, d\mathscr{H}_{n-1}(\sigma)$$

in Proposition 3.4, then we apply Proposition 3.2 on Q_{2R} with

$$r = \frac{n}{n+p}\, p\,, \qquad s = \frac{n}{n+p}\, q\,,$$

so that

$$p(x) = \frac{n}{n-r}\, r(x)\,,$$

and we immediately get the required inequality. We remark that if $p \leq q \leq n$ we have $r, s \leq n^2/(n+1) < n$.

If $Q_{2R} \subset \Omega^-$ the result follows by Proposition 3.4 and the classical Sobolev - Poincaré inequality, with

$$\lambda = \fint_{Q_{2R}} u\, dx\,.$$

The same argument, if $Q_{2R} \subset \Omega^+$, leads to

$$\begin{aligned}\fint_{Q_R} |Du|^q\, dx &\leq c\left[1 + \left(\fint_{Q_{2R}} \left(|Du|^q\right)^{n/(n+q)} dx\right)^{(n+q)/n}\right]\\ &\leq c\left[1 + \left(\fint_{Q_{2R}} \left(|Du|^q\right)^{n/(n+p)} dx\right)^{(n+p)/n}\right].\end{aligned}$$

The generic case $Q_{2R} \cap \Sigma \neq \emptyset$ is dealt with as in the proof of Proposition 3.4.

The case $1 \leq p < n/(n-1)$ is treated as above, but with the choice $r = 1$, $s = q/p$ in Proposition 3.2, applied with exponent p instead of $n/(n-1)$.

The higher integrability result (Theorem 2.1) follows easily from Lemma 3.5 and Gehring lemma ([4] Sect. 5, Proposition 1.1).

We now prove the local boundedness result (Theorem 2.2).

Proof Since by [6] we already know that $u \in L^\infty_{\mathrm{loc}}(\Omega^+ \cup \Omega^-)$, we only have to show that $u \in L^\infty(Q_R)$ for every cube Q_R centered on Σ with $Q_{2R} \subset \Omega$. The starting point is the following Caccioppol inequality for $(u-K)^+$, which can be obtained as in Proposition 3.4 by a suitable choice of test functions: for every $\varrho' < \varrho < 2R$ there exists $c = c(p,q)$ such that for every K and every minimizer u

$$\int_{Q_{\varrho',K}} |Du|^{p(x)}\, dx \leq c\left(\int_{Q_{\varrho,K}} \left|\frac{(u-K)^+}{\varrho - \varrho'}\right|^{p(x)} dx + |Q_{\varrho,K}|\right). \tag{6}$$

The rest of the proof is (simpler than) the proof of [8] Lemma 5.4, p. 76.

4 Hölder continuity

In the standard case $p = q$, inequality (6) is enough to get the hölder continuity of a minimizer u, but if $p \neq q$ it does not allow to bound the oscillation of u on a ball intersecting Σ. Thus, assuming henceforth $1 < p < q$, we first prove that u is locally hölder continuous on $\Omega^+ \cup \Sigma$, then we use the following result, which states that a minimizer of $\mathscr{F}$ with hölder continuous datum on a part of the boundary is hölder continuous up to that part of the boundary.

Theorem 4.1 *Assume* (4) *holds; if* $u \in W^{1,p}(\Omega^-) \cap L^\infty(\Omega^-)$ *is a minimizer of* $\mathscr{F}$ *in its Dirichlet class, and if the trace of* u *on* Σ *is of class* $C^{0,\beta}$*, then*

$$u \in C^{0,\alpha}(\Omega^- \cup \Sigma)\,,$$

where $\alpha = \alpha(p, \beta, \sup u)$.

Proof Take any cube Q_ϱ intersecting Ω^- and such that $\overline{Q_\varrho}$ does not intersect $\partial\Omega^- \setminus \Sigma$.

If $Q_\varrho \cap \Sigma \neq \emptyset$, consider for every $K \geq \sup_{Q_\varrho \cap \Sigma} u$ and every ζ with compact support in Q_ϱ the function $(u-K)^+\zeta$: its trace vanishes on the boundary of $Q_\varrho \cap \Omega^-$, thus it may be used as a test function to get

$$\int_{Q_{\varrho',K}\cap\Omega^-} |Du|^p\,dx \leq \gamma\left(\int_{Q_{\varrho,K}\cap\Omega^-} \left|\frac{(u-K)^+}{\varrho-\varrho'}\right|^p dx + \operatorname{meas}(Q_{\varrho,K}\cap\Omega^-)\right)$$

for every $\varrho' < \varrho$, with $\gamma = \gamma(p)$ constant.

Otherwise, if $Q_\varrho \subset \Omega^-$, we already have (6) which may be written as the inequality above, with no restriction on K. In conclusion, referring to [8] p. 90, u belongs to $\mathscr{B}_p(\Omega^- \cup \Sigma, \sup|u|, \gamma, \infty, 0)$, and the result comes from [8] Theorem 7.1, p. 91.

Thus, we only have to prove regularity up to Σ_R in $(\Omega')^+$ with $\Omega' \subset\subset \Omega$. A first result in this direction (Proposition 2.4) is an easy consequence of the higher integrability:

Proof Referring to Theorem 2.1, if $p > 1$ is fixed and $q > q_0 = n/\big(1 + \varepsilon_0(p)\big)$ then $u \in W^{1,n+\delta}_{\mathrm{loc}}(\Omega^+ \cup \Sigma)$ with $\delta = \delta(p, q) > 0$, thus $u \in C^{0,\beta}(\Omega^+ \cup \Sigma)$; since $u \in L^\infty(\Omega)$, we have $u \in C^{0,\alpha}(\Omega)$ by Theorem 4.1.

According to this result, we may confine ourselves in the sequel to the case $1 < p < q < n$. Also, all cubes chosen below will be contained in Ω, without mentioning it any further. By assumptions (4),(5) one easily gets

$$|F(x, z+w) - F(x, z)| \leq L\,(1 + \varepsilon^{p(x)}|z|^{p(x)} + \varepsilon^{-p(x)/(p(x)-1)}|w|^{p(x)})\,. \tag{7}$$

The next two lemmas will enable us to get the proper energy estimate which we will use instead of (6) to prove the hölder continuity following the general lines of [2], see also [8] Chap. 2, Sect. 6. Hereafter, if $A \subset \Omega^+$ and $f : A \to \mathbb{R}$ we denote by $\tilde{A}$ the symmetric of A with respect to Σ, and by $\tilde{f}(x)$ the function f evaluated at the symmetric of the point $x \in \tilde{A}$. For typographical reasons, we employ the notation $\tilde{Q}^+_{\varrho,K}$ instead of putting a wide tilde over the whole symbol.

Lemma 4.2 *For every* $\varrho' < \varrho$ *and* $K' > K \geq \sup_{Q^+_\varrho} u - 1$ *we have*

$$(K'-K)^p \int_{\tilde{Q}^+_{\varrho',K'}} |Du|^p\,dx \leq c \int_{Q^+_{\varrho,K}} \left(|Du|^p + \left|\frac{(u-K)^+}{\varrho-\varrho'}\right|^p + 1\right) dx\,,$$

with $c = c(M, p)$.

Proof Let $\zeta \in C_0^\infty(Q_\varrho)$ satisfy: $0 \le \zeta \le 1$, $\zeta = 1$ on $Q_{\varrho'}$, $|D\zeta| \le 2/(\varrho - \varrho')$ and set

$$\varphi = |(\tilde{u} - K)^+|^p (u - \tilde{u})\, \zeta^p \,;$$

then φ vanishes outside $\tilde{Q}^+_{\varrho,K}$. Remarking that $|(\tilde{u} - K)^+|^p \zeta^p \le 1$, and by the convexity of F,

$$\begin{aligned}
&\mathscr{F}(u, \tilde{Q}^+_{\varrho,K}) \\
&\quad \le \mathscr{F}(u - \varphi, \tilde{Q}^+_{\varrho,K}) \\
&\quad = \int_{\tilde{Q}^+_{\varrho,K}} F\Big[x, (1 - |(\tilde{u} - K)^+|^p \zeta^p) Du \\
&\qquad\quad + |(\tilde{u} - K)^+|^p \zeta^p \Big(D\tilde{u} - p\frac{(u - \tilde{u})D\tilde{u}}{(\tilde{u} - K)^+} - p\frac{(u - \tilde{u})D\zeta}{\zeta}\Big)\Big]\, dx \\
&\quad \le \int_{\tilde{Q}^+_{\varrho,K}} (1 - |(\tilde{u} - K)^+|^p \zeta^p) F(x, Du)\, dx \\
&\qquad\quad + \int_{\tilde{Q}^+_{\varrho,K}} |(\tilde{u} - K)^+|^p \zeta^p F\Big(x, D\tilde{u} - p\frac{(u - \tilde{u})D\tilde{u}}{(\tilde{u} - K)^+} - p\frac{(u - \tilde{u})D\zeta}{\zeta}\Big)\, dx
\end{aligned}$$

so that

$$\int_{\tilde{Q}^+_{\varrho,K}} |(\tilde{u} - K)^+|^p \zeta^p |Du|^p\, dx \le c(M, p) \int_{Q^+_{\varrho,K}} (1 + |Du|^p + |(u - K)^+||D\zeta|^p)\, dx$$

and the result follows.

Lemma 4.3 *For every $\varrho' < \varrho$ and every $K \in \mathbb{R}$ and $\varepsilon > 0$*

$$\begin{aligned}
\int_{Q^+_{\varrho',K}} |Du|^q\, dx \le \varepsilon^p &\int_{\tilde{Q}^+_{\varrho,K}} |Du|^p\, dx \\
&+ c \int_{Q^+_{\varrho,K}} \Big(\Big| \frac{(u - K)^+}{\varrho - \varrho'} \Big|^q + \varepsilon^{-pq/(p-1)(q-p)} \Big)\, dx
\end{aligned}$$

with $c = c(p, q)$.

Proof Let ζ be as in Lemma 4.2; the function

$$\varphi(x) = \begin{cases} (u - K)^+\, \zeta^q & \text{if } x \in Q^+_\varrho \\ (\tilde{u} - K)^+\, \tilde{\zeta}^q & \text{if } x \in Q^-_\varrho \end{cases}$$

has compact support, thus

$$\begin{aligned}
&\int_{Q^+_{\varrho,K} \cup \tilde{Q}^+_{\varrho,K}} F(x, Du)\, dx \\
&\qquad \le \int_{Q^+_{\varrho,K}} F\Big[x, (1 - \zeta^q) Du + \zeta^q \Big(-q\, \frac{(u - K)^+}{\zeta}\, D\zeta \Big)\Big]\, dx \\
&\qquad\quad + \int_{\tilde{Q}^+_{\varrho,K}} F(x, Du - \tilde{\zeta}^q D\tilde{u} - q\tilde{\zeta}^{q-1} (\tilde{u} - K)^+ D\tilde{\zeta})\, dx
\end{aligned}$$

and using again the convexity of F and (7)

$$\begin{aligned}\int_{Q^+_{\varrho,K}} & \zeta^q\, F(x,Du)\,dx \\ & \le c\int_{Q^+_{\varrho,K}} |(u-K)^+|^q\,|D\zeta|^q\,dx \\ & \quad + c\int_{\tilde Q^+_{\varrho,K}} [1+\varepsilon^p|Du|^p + \varepsilon^{-p/(p-1)}(|\tilde\zeta^q D\tilde u|^p + |\tilde\zeta^{q-1}(\tilde u-K)^+\,D\tilde\zeta|^p)]\,dx\,.\end{aligned}$$

By Young inequality we deduce

$$\begin{aligned}\int_{Q^+_{\varrho,K}} & \zeta^q\, F(x,Du)\,dx \\ & \le c\int_{Q^+_{\varrho,K}}\left(\left|\frac{(u-K)^+}{\varrho-\varrho'}\right|^q + \varepsilon^{-pq/(p-1)(q-p)}\right)dx \\ & \quad + \frac12\int_{Q^+_{\varrho,K}} \zeta^q|Du|^q\,dx \\ & \quad + c\varepsilon^p\int_{\tilde Q^+_{\varrho,K}} |Du|^p\,dx\,,\end{aligned}$$

and the result follows.

The following lemma gives the appropriate energy estimate on Q^+: set $r = n[1+(p-1)(q-p)/p]$. Then we have:

Lemma 4.4 *There exist $c = c(M,p,q) > 0$ and $\varrho_0 < 1$ such that the inequality*

$$\begin{aligned}\int_{Q^+_{\varrho',K'}} |Du|^q\,dx \le c\Bigg[& [\mathrm{meas}\,(Q^+_{\varrho,K})]^{1-q/r} \\ & + \int_{Q^+_{\varrho,K}}\left((K'-K)^{-pq/(p-1)(q-p)} + \left|\frac{(u-K)^+}{\varrho-\varrho'}\right|^q\right)dx\Bigg]\end{aligned}$$

holds for every $\varrho' < \varrho < \varrho_0$ and every K,K' with $\sup_{Q^+_\varrho} u - 1 < K < K' < \sup_{Q^+_\varrho} u$.

Proof Let N be a positive integer such that $N \ge (r-q)p/(q-p)q$, so that

$$\frac{q-p}{p}\,\frac{q}{r}\,N \ge 1 - \frac{q}{r}\,,$$

and divide both $[\varrho',\varrho]$ and $[K,K']$ into N subintervals by setting for $i = 0,\dots,N$

$$\varrho_i = \varrho' + i\frac{\varrho-\varrho'}{N}\,, \qquad K_i = K' = i\frac{K'-K}{N}\,;$$

moreover set

$$\sigma_i = \frac{\varrho_i + \varrho_{i+1}}{2}\,.$$

We apply Lemma 4.3 with ϱ' replaced by ϱ_i ,ϱ by σ_i, K by K_i and ε by $K_i - K_{i+1}$, and we majorize the last integral thus getting

$$\begin{aligned}
&\int_{Q^+_{\varrho_i,K_i}} |Du|^q \, dx \\
&\qquad \le (K_i - K_{i+1})^p \int_{\tilde{Q}^+_{\sigma_i,K_i}} |Du|^p \, dx \\
&\qquad\quad + c(p,q) \int_{Q^+_{\varrho,K}} \left[\left(\frac{(u-K)^+}{\varrho - \varrho'} \right)^q + (K'-K)^{-pq/(p-1)(q-p)} \right] dx \, .
\end{aligned}$$

Lemma 4.2 applied to the first term appearing at the right-hand side yields

$$\begin{aligned}
&(K_i - K_{i+1})^p \int_{\tilde{Q}^+_{\sigma_i,K_i}} |Du|^p \, dx \\
&\qquad \le c(M,p) \int_{Q^+_{\varrho_{i+1},K_{i+1}}} \left[|Du|^p + \left(\frac{(u-K_{i+1})^+}{\varrho_{i+1} - \sigma_i} \right)^p + 1 \right] dx \\
&\qquad \le \hat{c} \int_{Q^+_{\varrho_{i+1},K_{i+1}}} |Du|^p \, dx \\
&\qquad\quad + c(p,q,M) \int_{Q^+_{\varrho,K}} \left[\left(\frac{(u-K)^+}{\varrho - \varrho'} \right)^q + (K'-K)^{-pq/(p-1)(q-p)} \right] dx
\end{aligned}$$

by Young inequality, where we used the fact that $K' - K \le 1$. As for the integral of $|Du|^p$, we apply Hölder and Young inequality to get

$$\begin{aligned}
\hat{c} \int_{Q^+_{\varrho_{i+1},K_{i+1}}} |Du|^p \, dx \le{}& [\operatorname{meas}(Q^+_{\varrho,K})]^{\frac{q-p}{p}\frac{q}{r}} \int_{Q^+_{\varrho_{i+1},K_{i+1}}} |Du|^q \, dx \\
&+ c(M,p,q)[\operatorname{meas}(Q^+_{\varrho,K})]^{1-\frac{q}{r}} \, ,
\end{aligned}$$

so that finally

$$\begin{aligned}
&\int_{Q^+_{\varrho_i,K_i}} |Du|^q \, dx \\
&\qquad \le [\operatorname{meas}(Q^+_{\varrho,K})]^{\frac{q-p}{p}\frac{q}{r}} \int_{Q^+_{\varrho_{i+1},K_{i+1}}} |Du|^q \, dx \\
&\qquad\quad + c(M,p,q) \int_{Q^+_{\varrho,K}} \left[\left(\frac{(u-K)^+}{\varrho - \varrho'} \right)^q + (K'-K)^{-pq/(p-1)(q-p)} \right] dx \\
&\qquad\quad + c(M,p,q)[\operatorname{meas}(Q^+_{\varrho,K})]^{1-\frac{q}{r}} \, .
\end{aligned}$$

We choose ϱ_0 such that $\operatorname{meas}(Q^+_{\varrho_0}) \le 1$ and $\int_{Q^+_{\varrho_0}} |Du|^q \, dx \le 1$ for all cubes; letting $i = 0, \dots, N-1$ in the previous estimate yields

$$\begin{aligned}
&\int_{Q^+_{\varrho',K'}} |Du|^q\,dx \\
&\quad \le [\mathrm{meas}\,(Q^+_{\varrho,K})]^{\frac{q-p}{p}\frac{q}{r}} \int_{Q^+_{\varrho,K}} |Du|^q\,dx \\
&\qquad + c(M,p,q)\int_{Q^+_{\varrho,K}} \left[\left(\frac{(u-K)^+}{\varrho-\varrho'}\right)^q + (K'-K)^{-pq/(p-1)(q-p)}\right] dx \\
&\qquad + c(M,p,q)[\mathrm{meas}\,(Q^+_{\varrho,K})]^{1-\frac{q}{r}} \\
&\quad \le c(M,p,q)\int_{Q^+_{\varrho,K}} \left[\left(\frac{(u-K)^+}{\varrho-\varrho'}\right)^q + (K'-K)^{-pq/(p-1)(q-p)}\right] dx \\
&\qquad + c(M,p,q)[\mathrm{meas}\,(Q^+_{\varrho,K})]^{1-\frac{q}{r}}
\end{aligned}$$

by our choice of N.

We remark that the choice of ϱ_0 is the only point where we used the condition $Du \in L^q(\Omega^+)$.

The following is a rephrasing of the result just proved.

Remark 4.5 With the notation of Lemma 4.4, if $K' \ge K + \lambda\varrho^{1-n/r}$ for some $\lambda \le 1$, the inequality

$$\int_{Q^+_{\varrho',K'}} |Du|^q\,dx \le c\left[\int_{Q^+_{\varrho,K}} \left(\frac{(u-K)^+}{\varrho-\varrho'}\right)^q dx + \lambda^{-pq/(p-1)(q-p)}[\mathrm{meas}\,(Q^+_{\varrho,K})]^{1-\frac{q}{r}}\right]$$

holds, with $c = c(M,p,q)$.

The following lemma is a slight modification of [2] Lemma 2, see also [8] Lemma 3.5, p. 55, whereas the subsequent one is technical (see [8] Lemma 4.7, p. 66).

Lemma 4.6 *There exists $c = c(n)$ such that for every $u \in W^{1,1}(Q^+_\varrho)$ and every $K'' > K'$*

$$(K''-K')[\mathrm{meas}\,(Q^+_{\varrho,K''})]^{1-1/n} \le \frac{c\varrho^n}{\mathrm{meas}\,(Q^+_\varrho \setminus Q^+_{\varrho,K'})} \int_{Q^+_{\varrho,K'}\setminus Q^+_{\varrho,K''}} |Du|\,dx\,.$$

Lemma 4.7 *Let $\{a_i\}$ be a sequence of nonnegative real numbers, satisfying for some positive constants c,ε, and $b > 1$*

$$a_{i+1} \le cb^i a_i^{1+\varepsilon}\,.$$

There exists $\vartheta_0 = \vartheta_0(c,b,\varepsilon) > 0$ such that

$$a_0 \le \vartheta_0 \quad \Rightarrow \quad \lim_i a_i = 0\,.$$

Using the inequality stated in Lemma 4.4, it is possible to prove the equivalent of the results of [8] Chap. 2, Sect. 6.

Lemma 4.8 *With the notation of Lemma 4.4, there exists $\vartheta = \vartheta(M,p,q)$ such that for any $\varrho < \varrho_0$ and any $K \ge \sup_{Q^+_\varrho} u - 1$, the inequality*

$$\operatorname{meas}(Q^+_{\varrho,K}) \le \vartheta \varrho^n$$

implies

$$\operatorname{meas}(Q^+_{\frac{\varrho}{2},K+\frac{H}{2}}) = 0$$

provided $H = \sup_{Q^+_\varrho} u - K \ge \varrho^{1-n/r}$.

Proof We set

$$\varrho_i = \frac{\varrho}{2}(1+2^{-i}), \qquad K_i = K + \frac{H}{2}(1-2^{-i}), \qquad K_i' = \frac{K_i + K_{i+1}}{2},$$

so that

$$\varrho_i - \varrho_{i+1} = 2^{-i-2}\varrho, \qquad K_i' = K_i + 2^{-i-3}H \ge K_i + 2^{-i-3}\varrho^{1-n/r},$$

and we may apply Remark 4.5 to obtain

$$\begin{aligned} &\int_{Q^+_{\varrho_{i+1},K_i'}} |Du|^q\,dx \\ &\le c\left(\frac{2^{iq}}{\varrho^q}\int_{Q^+_{\varrho_i,K_i}} |(u-K_i)^+|^q\,dx + 2^{ipq/(p-1)(q-p)}[\operatorname{meas}(Q^+_{\varrho_i,K_i})]^{1-q/r}\right) \\ &\le c\,2^{iaq}(H^q\varrho^{\frac{nq}{r}-q}+1)[\operatorname{meas}(Q^+_{\varrho_i,K_i})]^{1-q/r}, \end{aligned} \tag{8}$$

where $a = \max\{1, p/(p-1)(q-p)\}$. Next, from Lemma 4.6 we have

$$\begin{aligned} &(K_{i+1}-K_i')[\operatorname{meas}(Q^+_{\varrho_{i+1},K_{i+1}})]^{1-1/n} \\ &\qquad\le \frac{c\varrho^n}{\operatorname{meas}(Q^+_{\varrho_{i+1}} \setminus Q^+_{\varrho_{i+1},K_i'})}\int_{Q^+_{\varrho_{i+1},K_i'}} |Du|\,dx. \end{aligned} \tag{9}$$

Now,

$$\operatorname{meas}(Q^+_{\varrho_{i+1},K_i'}) \le \operatorname{meas}(Q^+_{\varrho,K}) \le \vartheta\varrho^n,$$

and we will choose ϑ such that

$$\vartheta\varrho^n \le \frac{1}{2}\operatorname{meas}(Q^+_{\varrho/2}) \le \frac{1}{2}\operatorname{meas}(Q^+_{\varrho_{i+1}}), \tag{10}$$

therefore

$$\frac{H^q}{2^{(i+3)q}}[\operatorname{meas}(Q^+_{\varrho_{i+1},K_{i+1}})]^{q-q/n} \le c[\operatorname{meas}(Q^+_{\varrho_{i+1},K_i'})]^{q-1}\int_{Q^+_{\varrho_{i+1},K_i'}} |Du|^q\,dx,$$

and (8) implies

$$[\operatorname{meas}(Q^+_{\varrho_{i+1},K_{i+1}})]^{q-q/n} \le c\,2^{i(a+1)q}(\varrho^{\frac{nq}{r}-q} + H^{-q})[\operatorname{meas}(Q^+_{\varrho_i,K_i})]^{q-q/r}.$$

Using the assumption on H,

$$[\operatorname{meas}(Q^+_{\varrho_{i+1},K_{i+1}})]^{1-1/n} \le c\,2^{i(a+1)}\varrho^{\frac{n}{r}-1}[\operatorname{meas}(Q^+_{\varrho_i,K_i})]^{1-1/r},$$

that is

$$\left(\frac{\operatorname{meas}(Q^+_{\varrho_{i+1},K_{i+1}})}{\varrho^n}\right)^{1-1/n} \leq c\,2^{i(a+1)}\left(\frac{\operatorname{meas}(Q^+_{\varrho_i,K_i})}{\varrho^n}\right)^{1-1/r},$$

Besides (10), we impose on ϑ the condition that $\vartheta \leq \vartheta_0$, and the conclusion follows from Lemma 4.7.

The difference between this proof and [8] Lemma 6.1, which is essentially [2] Lemma 4, is that the energy estimate (Lemma 4.4) involves different levels K,K' of u, thus we must introduce the level K_i', average of K_i and K_{i+1}, whereas in [8] our inequalities (8) and (9) appear with $K_i' = K_i$.

Now we refer to [8] Lemma 6.2, p. 85: if a similar average is taken of the levels $\mu_i - \omega/2^t$ and $\mu_i - \omega/2^{t+1}$ to get the versions valid in our case of (6.15) and of the formula at the bottom of p. 87, one may repeat line by line the proof of Lemma 6.2, with some simplifications ($\delta = 1$ in our case) and the obvious modifications (one works on Q^+_ϱ, not on Q_ϱ), to obtain:

Lemma 4.9 *There exists a constant $s = s(M,p,q)$ such that for every $\varrho < \varrho_0/4$ at least one of the inequalities*

$$\operatorname{osc}[u, Q^+_\varrho] \leq 2^s \varrho^{1-n/r}$$

$$\operatorname{osc}[u, Q^+_\varrho] \leq (1 - 2^{1-s})\operatorname{osc}[u, Q^+_{4\varrho}]$$

holds.

This result, together with the next ([8] Lemma 4.8, p. 66), will enable us to prove hölder continuity.

Lemma 4.10 *Assume that for each $\varrho < \varrho_0/4$ at least one of the following inequalities holds:*

$$\operatorname{osc}[u, Q^+_\varrho] \leq c_1 \varrho^\varepsilon$$

$$\operatorname{osc}[u, Q^+_\varrho] \leq \vartheta \operatorname{osc}[u, Q^+_{4\varrho}]$$

for suitable constants c_1, $\varepsilon \leq 1$ and $\vartheta < 1$. Then for all $\varrho < \varrho_0$

$$\operatorname{osc}[u, Q^+_\varrho] \leq c_2(\varrho/\varrho_0)^\beta, \tag{11}$$

where

$$\beta = \min\{-\log\vartheta/\log 4, \varepsilon\}$$

and c_2 is independent of ϱ.

We may now prove Theorem 2.3:

Proof Inequality (11), together with the well known interior estimates in Ω^+, implies that $u \in C^{0,\beta}(\Omega^+ \cup \Sigma)$. The result then follows by Theorem 4.1, remarking that the hölder exponent depends on Ω only through $M = \sup_\Omega |u|$.

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