To appear on the Proceedings of the Symposium Year on
Material Instabilities and Continuum Mechanics

An Approximation Lemma for $W^{1,p}$ Functions

by Emilio Acerbi & Nicola Fusco

In this paper we give a direct proof of the following approximation result:

Theorem. *Let $\Omega \subset \mathbf{R}^n$ be a regular open set, and $p \geq 1$. There exists a constant c such that, for every $u \in W^{1,p}(\Omega)$ and every $K > 0$ there exists $v \in W^{1,\infty}(\Omega)$ satisfying*

$$\|v\|_{1,\infty} \leq K$$

$$\text{meas}\{x\colon\ u(x) \neq v(x)\} \leq c\,\frac{\|u\|_{1,p}^p}{K^p}\ .$$

Unlike several results of Luzin type available ([3],[4]), in this theorem we do not look for an approximating function v which is close to u in the $W^{1,p}$ norm, but we want a precise control on the gradient of v. Thus, when the result is applied to a bounded sequence in $W^{1,p}$ one obtains a sequence of approximating functions which is bounded in $W^{1,\infty}$.
This theorem was successfully applied by the authors in two different contexts: in a weaker form ([1]) when dealing with the semicontinuity of quasiconvex integrals, and in the general form ([2]) to obtain a regularity theorem for the same functionals.
The proof was not written explicitly in [2], where the reader was referred to the weaker result of [1], so we decided to present the details here, also thinking that this Lemma might be of interest by itself.

For every $f \in L^1_{loc}(\mathbf{R}^n)$ we set

$$Mf(x) = \sup_{r>0} \fint_{B_r(0)} |f(x+y)|\, dy \ ,$$

and if $u \in W^{1,1}_{loc}(\mathbf{R}^n)$ we write

$$M'u(x) = M(|Du|)(x) \ .$$

We now show that through the maximal function M' we may control the difference quotient of u outside a small set. We set, for any $u \in W^{1,1}_{loc}(\mathbf{R}^n)$ and for any $\lambda > 0$

$$H_\lambda = \{x\colon\ M'u(x) < \lambda\} \ .$$

Lemma 1. *There exists a constant $c_1 = c_1(n)$ such that, for every $u \in C_0^\infty(\mathbf{R}^n)$, we have*

$$\frac{|u(x) - u(y)|}{|x-y|} \le c_1\, \lambda$$

for all $x, y \in H_\lambda$.

PROOF. Fix $\lambda > 0$ and $x \in H_\lambda$, and set

$$S_{k,r}(x) = \{y \in B_r(x)\colon\ \frac{|u(x) - u(y)|}{|x-y|} \ge k\lambda\} \ .$$

We prove that, as k increases, $S_{k,r}(x)$ occupies a smaller and smaller portion of $B_r(x)$, independently of the particular function u: indeed,

$$k\lambda \frac{\mathrm{meas} S_{k,r}(x)}{\mathrm{meas} B_r(x)} \le \fint_{B_r(0)} \frac{|u(x+y) - u(x)|}{|y|} dy = \fint_{B_r} \left| \int_0^1 \frac{1}{|y|} \langle Du(x+ty), y\rangle dt \right| dy$$

$$\le \fint_{B_r} \int_0^1 |Du(x+ty)|\, dt\, dy = \int_0^1 \fint_{B_{tr}} |Du(x+y)|\, dy\, dt \le \int_0^1 M'(x)\, dt < \lambda \ ,$$

so that

(1) $$\mathrm{meas} S_{k,r}(x) \le \frac{1}{k}\ \mathrm{meas} B_r \ .$$

Now fix $x, y \in H_\lambda$, and set $r = |x-y|$. Denote by γ_n the measure of the intersection of two balls of radius 1 in $\mathbf{R}^n$, whose centers are also at distance 1 from each other. If $\omega_n = \mathrm{meas} B_1$, it is clear from (1) that for $k = 3\omega_n/\gamma_n$ the measure of $S_{k,r}(x) \cup S_{k,r}(y)$ is less than the measure of $B_r(x) \cap B_r(y)$, so that we may choose

$$z \in [B_r(x) \cap B_r(y)] \setminus [S_{k,r}(x) \cup S_{k,r}(y)] \ .$$

Then

$$|u(x) - u(z)| < k\lambda r$$

$$|u(y) - u(z)| < k\lambda r \ ,$$

whence

$$|u(x) - u(y)| < 2k\lambda r = \frac{6\omega_n}{\gamma_n} |x-y| \ ,$$

and the Lemma is proved.

The following result may be found in [5].

Lemma 2. *For every $p \geq 1$ there exists $c_2 = c_2(n,p)$ such that*

$$\text{meas}\{x\colon\ Mf(x) \geq \lambda\} \leq c_2 \frac{\|f\|_p^p}{\lambda^p}$$

for every $f \in L^p(\mathbf{R}^n)$ and $\lambda > 0$.

We may now extend Lemma 1 to the case of $W^{1,p}$ functions.

Lemma 3. *For every $u \in W^{1,p}(\mathbf{R}^n)$ there exists $E \subset \mathbf{R}^n$ with* meas$E = 0$ *such that*

$$\frac{|u(x) - u(y)|}{|x - y|} \leq c_1\, \lambda$$

for every $\lambda > 0$ and every $x, y \in H_\lambda \setminus E$.

PROOF. Let $(u_h) \subset C_0^\infty(\mathbf{R}^n)$ be a sequence converging to u strongly in $W^{1,p}(\mathbf{R}^n)$ and a.e. in $\mathbf{R}^n$. We remark that

$$|Mf(x) - Mg(x)| \leq M(f-g)(x)\ ,$$

so that, using Lemma 2, we have for all $\epsilon > 0$,

$$\lim_h \text{meas}\{x\colon\ |M'u(x) - M'u_h(x)| \geq \epsilon\} \leq \lim_h \text{meas}\{x\colon\ M'(u - u_h)(x) \geq \epsilon\}$$

$$\leq c_2 \epsilon^{-p} \lim_h \|Du - Du_h\|_p^p = 0\ ,$$

i.e., $M'u_h \to M'u$ in measure. We may then suppose that, at least for a subsequence,

$$M'u_h \to M'u \ \text{ a.e. in } \mathbf{R}^n\ .$$

Set

$$\mathbf{R}^n \setminus E = \{x\colon\ u_h(x) \to u(x),\ M'u_h(x) \to M'u(x)\}\ ;$$

then meas$E = 0$, and for every $x, y \in H_\lambda \setminus E$, and h sufficiently large, we have

$$M'u_h(x) < \lambda\ , \quad M'u_h(y) < \lambda\ ,$$

so that by Lemma 1

$$\frac{|u_h(x) - u_h(y)|}{|x - y|} \leq c_1\, \lambda\ ,$$

and the assertion follows by taking the limit as $h \to \infty$.

The following extension result is well known.

Lemma 4. *Let X be a metric space, $E \subseteq X$, and let $f\colon E \to \mathbf{R}$ be a K-lipschitz function. There exists a function $g : X \to \mathbf{R}$ which is K-lipschitz and satisfies*

$$g \equiv f \ \text{ in } E\ , \quad \sup_X |g| = \sup_E |f|\ .$$

Now we have all the necessary ingredients needed to prove the main result.

PROOF OF THE THEOREM. Fix $u \in W^{1,p}(\Omega)$; since Ω is regular, we may assume that $u \in W^{1,p}(\mathbf{R}^n)$, with

$$\|u\|_{1,p} \leq c_3\, \|u\|_{1,p,\Omega}$$

and c_3 independent of u. For $K > 0$ let $\lambda = K/2c_1$, then consider the set

$$H'_\lambda = \{x\colon\ Mu(x) + M'u(x) < \lambda\}\ .$$

By Lemma 2, we have

$$\mathrm{meas}(\mathbf{R}^n \setminus H'_\lambda) \le \mathrm{meas}\{x\colon\ Mu(x) \ge \frac{\lambda}{2}\} + \mathrm{meas}\{x\colon\ M'u(x) \ge \frac{\lambda}{2}\} \le c_4 \frac{\|u\|^p_{1,p,\Omega}}{K^p}\ .$$

On the other hand, by Lemma 3, we have

$$|u(x)| \le Mu(x) < \lambda \le \frac{K}{2}$$

$$\frac{|u(x) - u(y)|}{|x - y|} < c_1\,\lambda = \frac{K}{2}$$

a.e. on H'_λ. Applying Lemma 4 we obtain a function v with lipschitz constant $K/2$, and L^∞ norm not greater than $K/2$, so that

$$\|v\|_{1,\infty} \le K\ .$$

Since $v = u$ a.e. in H'_λ the theorem is proved.

As an example, an easy consequence of the above theorem is the following:

Corollary 5. *Let Ω be a regular open subset of $\mathbf{R}^n$ and $p \ge 1$, and let $(u_h) \subset W^{1,p}(\Omega)$ converge strongly to u in $W^{1,p}$. Then for every $\epsilon > 0$ there exist a subset A_ϵ of Ω, with $\mathrm{meas}A_\epsilon < \epsilon$, a subsequence (h_k) and a sequence $(w_k) \subset W^{1,\infty}(\Omega)$ such that*

$$w_k \equiv u_{h_k} \quad \text{in } \Omega \setminus A_\epsilon$$

$$w_k \to w \quad \text{strongly in } W^{1,\infty}(\Omega)$$

and

$$w \equiv u \quad \text{in } \Omega \setminus A_\epsilon\ .$$

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Emilio Acerbi – Scuola Normale Superiore – P. dei Cavalieri, 7 – 56100 PISA

Nicola Fusco – Dipartimento di Matematica – V. Mezzocannone, 8 – 80134 NAPOLI