Partial Regularity under Anisotropic (p,q) Growth Conditions

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1 . Introduction

In this paper we give a contribution to the study of the regularity of minimizers of integral functionals of the kind $\int_\Omega f(Du)\,dx$ under the assumption

$$|\xi|^p \le f(\xi) \le c(1+|\xi|^q). \tag{1.1}$$

Until recently it was customary to take

$$q = p,$$

but the question of whether regularity in the general case (1.1) could be obtained remained open. In 1987 some examples were produced (see [7],[11]) that showed that the answer to this question is in general negative. The example, as modified by [9], shows that in the particular case of

$$\int_\Omega [|Du|^2 + |D_1 u|^{p_1}]\,dx,$$

with $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, if p_1 is too far from 2 (depending on n) there exist minimizers which are not only discontinuous, but even unbounded.

More in general, consider the model functional

$$\int_\Omega [|Du|^p + \sum_{\alpha=1}^{k} |D_\alpha u|^{p_\alpha}]\,dx, \tag{1.2}$$

where $1 \le k \le n$, and $2 \le p < p_\alpha$ for $\alpha = 1,\dots,k$, and set

$$\frac{1}{\bar p} = \frac{1}{n}\Big[\frac{n-k}{p} + \sum_{\alpha=1}^{k} \frac{1}{p_\alpha}\Big];$$

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assume for simplicity $\bar{p} < n$ and define $\bar{p}^* = n\bar{p}/(n-\bar{p})$. In the examples [7],[9],[11] one has

$$\max\{p_\alpha : \alpha = 1, \dots, k\} > \bar{p}^*;$$

in the general case (1.2), a sufficient condition for the minimizers to be bounded is that

$$\max\{p_\alpha : \alpha = 1, \dots, k\} \le \bar{p}^* \tag{1.3}$$

(see [3],[5]). This condition means essentially that the exponents p_α may not be too dispersed, nor (if $k < n$) too far from p. The presence of the harmonic mean in condition (1.3) depends (see [14] and Lemma 2.1 below) on the fact that if $Du \in L^p$ and $D_\alpha u \in L^{p_\alpha}$ for $\alpha = 1, \dots, k$ then $u \in L^{\bar{p}^*}$.

Higher regulatity, such as boundedness or Hölder continuity of Du, has been studied in two papers by Marcellini [12],[13] where, however, the more restrictive condition

$$\max\{p_\alpha : \alpha = 1, \dots, k\} < \frac{np}{n-2} \tag{1.4}$$

is needed.

All these results deal with scalar minimizers. In this paper we prove a theorem concerning the vector-valued case, i.e. when

$$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N.$$

As is to be expected for systems, the regularity we prove is only partial. Precisely, we have (as a consequence of the more general Theorem 2.3) the following

Theorem . *If $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ is a minimizer of (1.2), with $D_\alpha u \in L^{p_\alpha}$ for $\alpha = 1, \dots, k$, and if*

$$\max\{p_\alpha : \alpha = 1, \dots, k\} < \bar{p}^* \tag{1.5}$$

then Du is Hölder continuous in an open set Ω_0 such that $\operatorname{meas}(\Omega \setminus \Omega_0) = 0$.

It is to be remarked that the assumption (1.5) on the exponents p_α is close to condition (1.3), and indeed it would be interesting to replace (1.5) by (1.3).

Another question arising naturally from this statement is whether, in the scalar case, one could improve the results of [12],[13] by deducing from it a global regularity result ($\Omega_0 = \Omega$). It seems to us that an important step to show that the singular set is empty lies in proving that Du is bounded, and this is exactly the point where condition (1.4) is used in [12],[13]: it would therefore be interesting to prove boundedness of Du in the scalar case under a less restrictive condition than (1.4), possibly condition (1.5).

We give in the next section the notation needed, and we state the main theorem, while section 3 is devoted to its proof; a fourth section of remarks is also present.

2 . Notations and preliminary lemmas

By ω_h we shall denote any sequence converging to 0 as $h \to \infty$, and by c any positive constant; both ω_h and c may sometimes vary from line to line.

A cut-off function η between two open sets $A \subset\subset B$ is a smooth function with compact support in B, values between 0 and 1, value 1 in A, and gradient less than $2/\text{dist}\,(A, \partial B)$.

If $p_\alpha \geq 1$ for all $\alpha = 1, \dots, n$, we define for any open subset Ω of $\mathbb{R}^n$ the spaces

$$W^{1,(p_\alpha)}(\Omega; \mathbb{R}^N) = \{u \in W^{1,1}(\Omega; \mathbb{R}^N) : D_\alpha u \in L^{p_\alpha}(\Omega; \mathbb{R}^N) \text{ for all } \alpha\}$$

$$W_0^{1,(p_\alpha)}(\Omega; \mathbb{R}^N) = \{u \in W_0^{1,1}(\Omega; \mathbb{R}^N) : D_\alpha u \in L^{p_\alpha}(\Omega; \mathbb{R}^N) \text{ for all } \alpha\},$$

with the natural norm $\|u\|_1 + \sum_\alpha \|D_\alpha u\|_{p_\alpha}$. If no confusion is possible, we shall omit $\mathbb{R}^N$ when mentioning these spaces.

Since the index α will always take its values in the set $\{1, \dots, n\}$, we shall henceforth omit any explicit reference to this range.

If the harmonic mean of $\{p_\alpha\}$ is $\bar{p}$, and if $\bar{p} < n$, then we write

$$\bar{p}^* = \frac{n\bar{p}}{n - \bar{p}},$$

the Sobolev exponent; if $\bar{p} \geq n$, we shall denote by $\bar{p}^*$ any number strictly larger than the maximum of $\{p_\alpha\}$: we make this choice for future convenience; remark that in the case $\bar{p} < n$ it is not guaranteed that $\bar{p}^* > \max\{p_\alpha\}$.

Finally, for any integrable function g on a set E, we denote its average by

$$(g)_E = \fint_E g\,dx = [\text{meas}\,(E)]^{-1} \int_E g\,dx;$$

if E is a ball $B_r(x_0)$, instead of $(g)_{B_r(x_0)}$ we shall simply write $(g)_{x_0,r}$ or even $(g)_r$.

The following lemma is essential when dealing with anisotropic functionals of the type (1.2).

Lemma 2.1 . *Let $Q \subset \mathbb{R}^n$ be a cube with edges parallel to the coordinate axes, and if $\bar{p} < n$ then assume that $p_\alpha < \bar{p}^*$ for all α (otherwise no restriction on $\{p_\alpha\}$ is needed). Then*

$$\|u\|_{\bar{p}^*} \leq c\Big(\|u\|_1 + \sum_\alpha \|D_\alpha u\|_{p_\alpha}\Big) \tag{2.1}$$

for all $u \in W^{1,(p_\alpha)}(Q)$. If $(u)_Q = 0$, then (2.1) *holds without $\|u\|_1$; moreover if $u \in W_0^{1,(p_\alpha)}(\Omega)$ then* (2.1) *holds, without $\|u\|_1$, for the generic bounded open set Ω, not only for a cube.*

PROOF . By Theorem 1.2 of [14] we have

$$\|u\|_{\bar{p}^*} \leq c\Big(\prod_\alpha \|D_\alpha u\|_{p_\alpha}\Big)^{1/n} \tag{2.2}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, where the constant c depends on n, $\{p_\alpha\}$ and (only in the case $\bar{p} \geq n$) also on $\bar{p}^*$ and the measure of the support of u. It is easy to see that $C_0^\infty(\mathbb{R}^n)$ is dense in $W_0^{1,(p_\alpha)}(\Omega)$, thus giving (2.1) without $\|u\|_1$ by the geometric mean - arithmetic mean inequality. Take any cube Q with edges parallel to the coordinate axes, and denote by $3Q$ the cube with the same center as Q, and three times the side, and let η be a cut-off function between Q and $3Q$. If $u \in W^{1,(p_\alpha)}(Q)$, we may extend it by reflections to a function $\tilde{u} \in W^{1,(p_\alpha)}(3Q)$; set $v = \eta\tilde{u}$: since $v \in W_0^{1,(p_\alpha)}(3Q)$, in particular $v \in L_{\bar{p}^*}(3Q)$, thus $u = v_{|Q} \in L^{\bar{p}^*}(Q)$. By (2.2),

$$\begin{aligned}\|u\|_{L^{\bar{p}^*}(Q)} \le \|v\|_{L^{\bar{p}^*}(3Q)} &\le c\prod_\alpha\Big(\|D_\alpha\tilde{u}\|_{L^{p_\alpha}(3Q)} + \|\tilde{u}\|_{L^{p_\alpha}(3Q)}\Big)^{1/n}\\ &\le c\sum_\alpha\Big(\|D_\alpha u\|_{L^{p_\alpha}(Q)} + \|u\|_{L^{p_\alpha}(Q)}\Big)\\ &\le c\Big(\sum_\alpha \|D_\alpha u\|_{L^{p_\alpha}(Q)} + \varepsilon\|u\|_{L^{\bar{p}^*}(Q)} + C_\varepsilon\|u\|_{L^1(Q)}\Big),\end{aligned}$$

and the result follows. The case $(u)_Q = 0$ may then be deduced easily by the Sobolev-Poincaré inequality. ■

The following lemma is just a technicality.

Lemma 2.2 . *If $\gamma > -1$ and $a, b \in \mathbb{R}^k$ we have*

$$c_1 \le \frac{\int_0^1 |a+sb|^\gamma\, ds}{(|a|^2+|b|^2)^{\gamma/2}} \le c_2$$

with $c_1, c_2 > 0$ depending only on γ and k.

The proof may be found in [1] Lemma 2.1 for $\gamma \le 0$, and e.g. in [4] Lemma 8.1 in the case $\gamma > 0$. We remark that the same lemma is true if the integral is replaced by $\int_0^1 (1-s)|a+sb|^\gamma\, ds$.

Let $p_\alpha \ge p \ge 2$, and let $f : \mathbb{R}^{nN} \to \mathbb{R}$, $f_\alpha : \mathbb{R}^N \to \mathbb{R}$ be functions of class C^2 satisfying for some positive c,ν,L the following assumptions: first, some growth and coercivity conditions:

$$\frac{1}{c}|\xi|^p \le f(\xi) \le c|\xi|^p, \qquad \frac{1}{c}|\xi_\alpha|^{p_\alpha} \le f_\alpha(\xi_\alpha) \le c|\xi_\alpha|^{p_\alpha} \tag{2.3}$$

$$|D^2 f(\xi)| \le c|\xi|^{p-2}, \qquad |D^2 f_\alpha(\xi_\alpha)| \le c|\xi_\alpha|^{p_\alpha-2} \tag{2.4}$$

$$D^2 f(\xi)\eta\eta \ge \nu|\xi|^{p-2}|\eta|^2, \qquad D^2 f_\alpha(\xi_\alpha)\eta_\alpha\eta_\alpha \ge \nu|\xi_\alpha|^{p_\alpha-2}|\eta_\alpha|^2; \tag{2.5}$$

then, Hölder continuity of the second derivatives: for some $0 < \delta < \min\{1, p-2\}$

$$\begin{aligned}&|D^2 f(\xi) - D^2 f(\eta)| \le c(|\xi|^{p-2-\delta} + |\eta|^{p-2-\delta})|\xi-\eta|^\delta\\ &|D^2 f_\alpha(\xi_\alpha) - D^2 f_\alpha(\eta_\alpha)| \le c(|\xi_\alpha|^{p_\alpha-2-\delta} + |\eta_\alpha|^{p_\alpha-2-\delta})|\xi_\alpha-\eta_\alpha|^\delta;\end{aligned} \tag{2.6}$$

finally, a uniformity condition on f which ensures that for ξ close to zero it behaves very much like $|\xi|^p$:

$$\lim_{t\to 0^+}\frac{Df(t\xi)}{t^{p-1}} = L|\xi|^{p-2}\xi. \tag{2.7}$$

Let $c_\alpha \geq 0$, and define on $W^{1,p}(\Omega)$ the functional (possibly infinite)

$$\mathcal{F}(u) = \int_\Omega [f(Du) + \sum_\alpha c_\alpha f_\alpha(D_\alpha u)]\, dx.$$

In order to avoid that f_α interferes with the leading term f, we will assume that for those f_α effectively present in the functional (i.e., those for which $c_\alpha \neq 0$) the exponent p_α is strictly larger than p:

(2.8) for each α, either $[p_\alpha > p \text{ and } c_\alpha > 0]$ or $[p_\alpha = p \text{ and } c_\alpha = 0]$;

then we have:

Theorem 2.3 . *Let $c_\alpha \geq 0$ and $p_\alpha \geq p \geq 2$ satisfy (2.8), let $u \in W^{1,(p_\alpha)}(\Omega)$ be a local minimizer of $\mathcal{F}$, and assume the growth, coercivity, Hölder continuity and uniformity conditions* (2.3), ...,(2.7) *hold. In addition, if $\bar{p} < n$, assume that $p_\alpha < \bar{p}^*$ for all α (otherwise, no further condition on p_α is needed). Then there exist a constant $\gamma > 0$, independent of u, and an open set $\Omega_0 \subset \Omega$, with* $\text{meas}\,(\Omega \setminus \Omega_0) = 0$, *such that $u \in C^{1,\gamma}(\Omega_0)$.*

This result clearly applies to the model case presented in the introduction.

3 . Proof of Theorem 2.3

We will prove Theorem 2.3 only for $c_\alpha = 1$ for all α, the case when some of the c_α vanish requiring only an obvious modification of the argument. The result will be proved via an integral characterization of Hölder continuous functions due to Campanato, asserting roughly that a function g is Hölder continuous once the integral oscillation $⨍_{B_r(x)} |g - (g)_{x,r}|^q\,dy$ decays as a power of r: thus our goal is to estimate $⨍_{B_r(x)} |Du - (Du)_{x,r}|^q\,dy$. To this aim it is useful to introduce the function

$$U(x_0, r) = ⨍_{B_r(x_0)} \Big[|(Du)_{x_0,r}|^{p-2}\,|Du - (Du)_{x_0,r}|^2 + |Du - (Du)_{x_0,r}|^p + \sum_\alpha |D_\alpha u - (D_\alpha u)_{x_0,r}|^{p_\alpha} \Big]\,dx.$$

For the minimizers of the simpler functional $\int |Du|^p\,dx$ the following result, essentially due to Uhlenbeck, holds (see [8], Theorem 3.1 modified using equation 2.4):

Lemma 3.1 . *There exist $\mu \in (0,2)$ and $\hat{c} > 0$ such that if $u \in W^{1,p}(B_1;\mathbb{R}^N)$ is a minimizer of*

$$\int_{B_1} |Du|^p\,dx$$

then for all $\varrho < 1$

$$⨍_{B_\varrho} [|(Du)_\varrho|^{p-2}|Du - (Du)_\varrho|^2 + |Du - (Du)_\varrho|^p]\,dx \le \hat{c}\varrho^\mu ⨍_{B_1} |(Du)_1|^{p-2}|Du - (Du)_1|^2 + |Du - (Du)_1|^p]\,dx.$$

We will later need this inequality as a tool to prove the following result, which is commonly called “main lemma” and is the essential ingredient to estimate the decay of U:

Lemma 3.2 . *If $u \in W^{1,(p_\alpha)}(\Omega)$ is a local minimizer of the functional $\mathcal{F}$, for every $M > 0$ there exists a constant $C(M)$ such that for every $\tau < 1$ there exists $\varepsilon > 0$ such that for any $B_r(x_0) \subset \Omega$ if*

$$U(x_0, r) < \varepsilon \quad \text{and} \quad |(Du)_{x_0,r}| < M$$

then

$$U(x_0, \tau r) \le C(M)\tau^\mu\, U(x_0, r),$$

where μ is the same exponent as in Lemma 3.1.

The proof of the main lemma is based, as usual, on a blow-up argument around a point x, however two important features should be remarked: first, if $p > 2$ the behaviour of the leading term $f(Du)$ is different depending on whether Du is “large” or “small” at x; secondly, we do not prove an *a priori* energy estimate of the Caccioppoli type, relying instead on the method of improving weak convergence of the rescaled functions v_h defined below to strong convergence.

To deal with rescaled functions, we introduce also a rescaled version of the integrand: for all $A \in \mathbb{R}^{nN}$ and $\lambda > 0$ define

$$f_{A,\lambda}(\xi) = [f(A+\lambda\xi) - f(A) - \lambda Df(A)\xi] + \sum_\alpha [f_\alpha(A_\alpha + \lambda\xi_\alpha) - f_\alpha(A_\alpha) - \lambda Df_\alpha(A_\alpha)\xi_\alpha].$$

We have:

Lemma 3.3 . *The following estimates hold:*

$$D^2 f_{A,\lambda}(\xi)\eta\eta \geq \frac{1}{c}\lambda^2\Big(p|A+\lambda\xi|^{p-2}|\eta|^2 + \sum_\alpha p_\alpha |A_\alpha + \lambda\xi_\alpha|^{p_\alpha-2}|\eta_\alpha|^2\Big)$$

$$|Df_{A,\lambda}(\xi)| \leq c\lambda^2\Big([|A|^{p-2} + |\lambda\xi|^{p-2}]|\xi| + \sum_\alpha [|A_\alpha|^{p_\alpha-2} + |\lambda\xi_\alpha|^{p_\alpha-2}]|\xi_\alpha|\Big)$$

$$|f_{A,\lambda}(\xi)| \leq c\lambda^2\Big([|A|^{p-2} + |\lambda\xi|^{p-2}]|\xi|^2 + \sum_\alpha [|A_\alpha|^{p_\alpha-2} + |\lambda\xi_\alpha|^{p_\alpha-2}]|\xi_\alpha|^2\Big),$$

with $c > 0$ independent of A,λ.

PROOF . We have

$$\Big[D^2 f_{A,\lambda}(\xi)\Big]^{ij}_{\alpha\beta} = \lambda^2\Big\{[D^2 f(A+\lambda\xi)]^{ij}_{\alpha\beta} + \sum_\gamma [D^2 f_\gamma(A_\gamma + \lambda\xi_\gamma)]^{ij}\delta_{\alpha\gamma}\delta_{\beta\gamma}\Big\}, \tag{3.1}$$

whence the required estimates follow easily. ■

PROOF OF THE MAIN LEMMA . Fix M; we shall determine $C(M)$ later. We argue by contradiction: assume there exists τ such that for every choice of ε there is a ball which violates the assert of the lemma; then, there is a sequence of balls $B_{r_h}(x_h)$ such that

$$|(Du)_{x_h,r_h}| < M, \quad U(x_h, r_h) = \lambda_h^p \to 0, \quad U(x_h, \tau r_h) > C(M)\tau^\mu\lambda_h^p.$$

We set

$$a_h = (u)_{x_h,r_h}, \quad A_h = (Du)_{x_h,r_h}, \quad \tilde{A}_h = (Du)_{x_h,\tau r_h},$$

and also for brevity $A_h^\alpha = (A_h)_\alpha$, and we define in the ball $B_1(0)$ the rescaled functions

$$v_h(z) = \frac{1}{\lambda_h r_h}[u(x_h + r_h z) - a_h - r_h A_h z],$$

so that $Dv_h(z) = \lambda_h^{-1}[Du(x_h + r_h z) - A_h]$. Then by the definition of λ_h

$$\fint_{B_1}\Big[\lambda_h^{2-p}|A_h|^{p-2}|Dv_h|^2 + |Dv_h|^p + \sum_\alpha \lambda_h^{p_\alpha - p}|D_\alpha v_h|^{p_\alpha}\Big]\,dz = 1 \tag{3.2}$$

and also

$$(v_h)_{0,1} = 0, \quad (Dv_h)_{0,1} = 0.$$

If we set

$$w_h = \Big(\frac{|A_h|}{\lambda_h}\Big)^{(p-2)/2} v_h,$$

the sequences v_h, w_h, A_h and $\lambda_h^{(p_\alpha - p)/p_\alpha} D_\alpha v_h$ are relatively compact, therefore we may suppose

$$\begin{cases} A_h \to A & \\ v_h \rightharpoonup v & \text{weakly in } W^{1,p} \\ w_h \rightharpoonup w & \text{weakly in } W^{1,2} \\ \lambda_h^{(p_\alpha - p)/p_\alpha} D_\alpha v_h \rightharpoonup 0 & \text{weakly in } L^{p_\alpha}, \end{cases} \tag{3.3}$$

the latter being an easy consequence of the fact that $p_\alpha > p$ for all α. Also, we assume that $\lim_h(|A_h|/\lambda_h)$ exists, finite or not.

The function v_h minimizes in its Dirichlet class the functional

$$\int_{B_1} [f(A_h + \lambda_h D\varphi) + \sum_\alpha f_\alpha(A_h^\alpha + \lambda_h D_\alpha \varphi)]\, dz,$$

therefore the Euler equation

$$\int_{B_1} [Df(A_h + \lambda_h Dv_h) D\varphi + \sum_\alpha Df_\alpha(A_h^\alpha + \lambda_h D_\alpha v_h) D_\alpha \varphi]\, dz = 0 \tag{3.4}$$

holds. The second variation of the functional at v_h may be written

$$\int_{B_1} [D^2 f(A_h + \lambda_h Dv_h) Dv_h D\varphi + \sum_\alpha D^2 f_\alpha(A_h^\alpha + \lambda_h D_\alpha v_h) D_\alpha v_h D_\alpha \varphi]\, dz = 0 \tag{3.5}$$

One of the crucial points in the proof is to remark, as we said, that if $p > 2$ the behaviour of the principal part $f(Du)$ in the functional $\mathcal{F}$ is different depending on whether Du is large at a point (then $|Du|^p$ is essentially quadratic) or small. In our setting, this will be reflected in the fact that for $p > 2$ two different proofs are required depending on the size of A_h (rescaled by the factor λ_h): precisely, we will find a quadratic behaviour when $|A_h|/\lambda_h \to \infty$. We assume from now on that $p > 2$; we will later make some remarks to adapt the proof to the simpler case $p = 2$.

First case : assume $\lim_h(|A_h|/\lambda_h) = +\infty$.

In particular, $A_h \neq 0$; even if $A_h \to 0$, we may assume that $\lim_h(A_h/|A_h|)$ exists, and we shall conventionally denote it by $A/|A|$.

By (3.3) we deduce that

$$v_h \rightharpoonup 0 \quad \text{weakly in } W^{1,p}; \tag{3.6}$$

dividing (3.5) by

$$|A_h|^{p-2} \left(\frac{\lambda_h}{|A_h|} \right)^{(p-2)/2}$$

we obtain

$$\int_{B_1} \int_0^1 \left[\frac{D^2 f(A_h + s\lambda_h Dv_h)}{|A_h|^{p-2}}\, Dw_h D\varphi + \sum_\alpha \frac{D^2 f_\alpha(A_h^\alpha + s\lambda_h D_\alpha v_h)}{|A_h|^{p-2}}\, D_\alpha w_h D_\alpha \varphi \right] dz = 0.$$

Using (2.6) one deduces that

$$\lim_{h\to\infty}\int_{B_1}\Big[\frac{D^2f(A_h)}{|A_h|^{p-2}}\,Dw_hD\varphi+\sum_\alpha\frac{D^2f_\alpha(A_h^\alpha)}{|A_h|^{p-2}}\,D_\alpha w_hD_\alpha\varphi\Big]\,dz=0;$$

Taking eventually a subsequence, we may assume by (2.4) that

$$\lim_{h\to\infty}\frac{D^2f(A_h)}{|A_h|^{p-2}}=C,\qquad \lim_{h\to\infty}\frac{D^2f_\alpha(A_h^\alpha)}{|A_h^\alpha|^{p-2}}=C_\alpha$$

exist, therefore w satisfies

$$\int_{B_1}[C\,Dw\,D\varphi+\sum_\alpha|A^\alpha|^{p_\alpha-p}\Big(\frac{|A^\alpha|}{|A|}\Big)^{p-2}C_\alpha\,D_\alpha w\,D_\alpha\varphi]\,dz=0, \tag{3.7}$$

which is a system with constant coefficients, elliptic by (2.5), and whose eigenvalues depend only on M and the growth conditions (2.4). Then, w satisfies for any $\tau<1$

$$\fint_{B_\tau}|Dw-(Dw)_\tau|^2\,dz\le\hat c(M)\tau^2. \tag{3.8}$$

Now

$$\begin{aligned}\lambda_h^{-p}U(x_h,\tau r_h)=\fint_{B_\tau}\Big[\Big(\frac{|\tilde A_h|}{\lambda_h}\Big)^{p-2}&|Dv_h-(Dv_h)_\tau|^2\\ &+|Dv_h-(Dv_h)_\tau|^p\\ &+\sum_\alpha\lambda_h^{p_\alpha-p}|D_\alpha v_h-(D_\alpha v_h)_\tau|^{p_\alpha}\Big]\,dz.\end{aligned} \tag{3.9}$$

We have

$$\begin{aligned}\Big(\frac{|\tilde A_h|}{\lambda_h}\Big)^{p-2}&\le c\Big[\Big(\frac{|A_h|}{\lambda_h}\Big)^{p-2}+\Big(\frac{|\tilde A_h-A_h|}{\lambda_h}\Big)^{p-2}\Big]\\ &\le c\Big(\frac{|A_h|}{\lambda_h}\Big)^{p-2}+c\Big(\fint_{B_\tau}|Dv_h|^p\,dz\Big)^{(p-2)/p},\end{aligned}$$

therefore

$$\fint_{B_\tau}\Big(\frac{|\tilde A_h|}{\lambda_h}\Big)^{p-2}|Dv_h-(Dv_h)_\tau|^2\,dz\le c\fint_{B_\tau}|Dw_h-(Dw_h)_\tau|^2\,dz+c\fint_{B_\tau}|Dv_h|^p\,dz,$$

since

$$\fint_{B_\tau}|Dv_h-(Dv_h)_\tau|^2\,dz\le\fint_{B_\tau}|Dv_h|^2\,dz\le\Big(\fint_{B_\tau}|Dv_h|^p\,dz\Big)^{2/p}.$$

Thus, we have obtained

$$\lambda_h^{-p}U(x_h,\tau r_h)\le\hat c(p)\fint_{B_\tau}\Big[|Dw_h-(Dw_h)_\tau|^2+|Dv_h|^p+\sum_\alpha\lambda_h^{p_\alpha-p}|D_\alpha v_h|^{p_\alpha}\Big]\,dz; \tag{3.10}$$

we shall later prove that

$$
\text{(3.11)}\qquad \begin{cases} w_h \to w & \text{in } W^{1,2}_{\rm loc} \\ v_h \to 0 & \text{in } W^{1,p}_{\rm loc} \\ \lambda_h^{(p_\alpha - p)/p_\alpha} D_\alpha v_h \to 0 & \text{in } L^{p_\alpha}_{\rm loc}, \end{cases}
$$

so that, taking the limit in (3.10), by (3.8) we get the contradiction, provided we chose $C(M) > \hat{c}(p)\hat{c}(M)$.

Second case : assume $\lim_h(|A_h|/\lambda_h) = l \in [0, +\infty)$.

In this case $A_h/\lambda_h \to l\bar{A}$ for some $\bar{A}$ with $|\bar{A}| = 1$, thus, dividing (3.4) by λ_h^{p-1}, we get

$$
\text{(3.12)}\qquad \int_{B_1} \Big[\frac{Df(A_h + \lambda_h Dv_h)}{\lambda_h^{p-1}}\, d\varphi + \Big\{ \sum_\alpha \lambda_h^{p_\alpha - p} \frac{Df_\alpha(A_h^\alpha + \lambda_h D_\alpha v_h)}{\lambda_h^{p_\alpha - 1}}\, D_\alpha \varphi \Big\} \Big]\, dz = 0.
$$

By (2.3),(2.4), and since $p_\alpha > p$, the integral of the quantity in curly brackets tends to zero as $h \to \infty$; we shall later prove that

$$
\text{(3.13)}\qquad v_h \to v \quad \text{in } W^{1,p}_{\rm loc}
$$

$$
\text{(3.14)}\qquad \lambda_h^{(p_\alpha - p)/p_\alpha} D_\alpha v_h \to 0 \quad \text{in } L^{p_\alpha}_{\rm loc};
$$

since by (2.4)

$$
|Df(\xi) - Df(\eta)| \le c(|\xi|^{p-2} + |\eta|^{p-2})|\xi - \eta|,
$$

using only (3.13) we remark that

$$
\lim_{h\to\infty} \int_{B_1} \frac{Df(A_h + \lambda_h Dv_h) - Df(\lambda_h l\bar{A} + \lambda_h Dv)}{\lambda_h^{p-1}}\, D\varphi\, dz = 0,
$$

thus by (2.7) we get from (3.12)

$$
\text{(3.15)}\qquad \int_{B_1} |l\bar{A} + Dv|^{p-2}(l\bar{A} + Dv) D\varphi\, dz = 0,
$$

and if we set $\hat{v}(z) = v(z) + l\bar{A}z$ then $\hat{v}$ is a solution of

$$
\int_{B_1} |D\hat{v}|^{p-2} D\hat{v}\, D\varphi\, dz = 0.
$$

By Lemma 3.1 we have the following estimate:

$$
\text{(3.16)}\qquad \begin{aligned} &\fint_{B_\tau} \Big[|(D\hat{v})_\tau|^{p-2} |D\hat{v} - (D\hat{v})_\tau|^2 + |D\hat{v} - (D\hat{v})_\tau|^p \Big]\, dz \\ &\qquad \le \hat{c}\tau^\mu \fint_{B_1} \Big[l^{p-2}|Dv|^2 + |Dv|^p \Big]\, dz \\ &\qquad \le \hat{c}\tau^\mu, \end{aligned}
$$

since by (3.3) and (3.2) the last integral is less than 1; the constant $\hat{c}$ is independent of l. Remarking that

$$\frac{\tilde{A}_h}{\lambda_h} = \frac{A_h}{\lambda_h} + (Dv_h)_\tau,$$

and using (3.13),(3.14), we may take the limit as $h \to \infty$ in (3.9), and by (3.16) we have

$$\begin{aligned}\limsup_h \lambda_h^{-p} U(x_h, \tau r_h) &= \fint_{B_\tau} \Big[|l\bar{A} + (Dv)_\tau|^{p-2} |Dv - (Dv)_\tau|^2 + |Dv - (Dv)_\tau|^p \Big]\, dz \\ &\le \hat{c}\tau^\mu,\end{aligned}$$

which gives the contradiction if we chose $C(M) > \hat{c}$.

PRELIMINARY ESTIMATES : these will be used when proving the strong convergences.

Set for brevity

$$f_h = f_{A_h, \lambda_h},$$

and for any $s < 1$ set $\mathcal{F}_h^s(\varphi) = \int_{B_s} f_h(D\varphi)\, dz$. Let ψ be any function of class C^1 on B_1, take $0 < t < s$, and take a cut-off function ζ between B_t and B_s: the function

$$\varphi_h = \zeta\psi + (1 - \zeta) v_h$$

agrees with v_h on $B_1 \setminus B_s$. We remark that v_h is also a minimizer of $\mathcal{F}_h^s$, thus we have

$$\begin{aligned}\mathcal{F}_h^s(v_h) - \mathcal{F}_h^s(\psi) &\le \mathcal{F}_h^s(\varphi_h) - \mathcal{F}_h^s(\psi) \\ &\le \int_{B_s \setminus B_t} f_h(D\varphi_h)\, dz \\ &\le c \int_{B_s \setminus B_t} [f_h(D\psi) + f_h(Dv_h)]\, dz \\ &\quad + c \int_{B_s \setminus B_t} f_h\big((\psi - v_h) \otimes D\zeta\big)\, dz.\end{aligned}$$

Fix $0 < r < 1$ and $K > 0$: we may find t,s arbitrarily close together, and satisfying $r \le t < s \le 1$, such that for infinitely many values of h (without loss of generality we assume it happens for every h)

$$\int_{B_s \setminus B_t} [f_h(D\psi) + f_h(Dv_h)]\, dz \le \frac{1}{K} \int_{B_1} [f_h(D\psi) + f_h(Dv_h)]\, dz.$$

For any such t,s we have

$$\begin{aligned}\mathcal{F}_h^s(v_h) - \mathcal{F}_h^s(\psi) &\le \frac{c}{K} \int_{B_1} [f_h(D\psi) + f_h(Dv_h)]\, dz \\ &\quad + c \int_{B_s \setminus B_t} f_h\big((\psi - v_h) \otimes D\zeta\big)\, dz.\end{aligned}$$

On the other hand, using Lemma 3.3 and Lemma 2.2

$$
\begin{aligned}
&\mathcal{F}_h^s(v_h) - \mathcal{F}_h^s(\psi)\\
&\quad = \int_{B_s} Df_h(D\psi)(Dv_h - D\psi)\,dz\\
&\qquad + \int_{B_s}\int_0^1 (1-\theta)D^2 f_h\big(D\psi + \theta(Dv_h - D\psi)\big)(Dv_h - D\psi)(Dv_h - D\psi)\,d\theta\,dz\\
&\quad \ge \int_{B_s} Df_h(D\psi)(Dv_h - D\psi)\,dz\\
&\qquad + c\int_{B_s}\Big[\lambda_h^2|A_h + \lambda_h D\psi|^{p-2}|Dv_h - D\psi|^2\\
&\qquad\qquad + \lambda_h^p|Dv_h - D\psi|^p\\
&\qquad\qquad + \sum_\alpha \lambda_h^{p_\alpha}|D_\alpha v_h - D_\alpha\psi|^{p_\alpha}\Big]\,dz,
\end{aligned}
$$

where we dropped a useless (but positive) term from the last integral. Therefore we have for any $0 < r < 1$ and $K > 0$ that there exist $r \le t < s_K \le 1$ such that for any $t < s \le s_K$

$$
\begin{aligned}
&\int_{B_r}\Big[\lambda_h^{2-p}|A_h + \lambda_h D\psi|^{p-2}|Dv_h - D\psi|^2\\
&\qquad + |Dv_h - D\psi|^p\\
&\qquad + \sum_\alpha \lambda_h^{p_\alpha - p}|D_\alpha v_h - D_\alpha\psi|^{p_\alpha}\Big]\,dz\\
&\qquad\qquad \le \frac{c}{K}\int_{B_1}\lambda_h^{-p}[f_h(D\psi) + f_h(Dv_h)]\,dz\\
&\qquad\qquad\quad + c\int_{B_s\setminus B_t}\lambda_h^{-p} f_h\big((\psi - v_h)\otimes D\zeta\big)\,dz\\
&\qquad\qquad\quad + c\int_{B_s}\lambda_h^{-p} Df_h(D\psi)(D\psi - Dv_h)\,dz,
\end{aligned}
\tag{3.17}
$$

where c depends only on p, p_α and the dimensions involved, and where we divided by λ_h^p for future convenience.

Strong convergence in the first case

Since in this case w satisfies an elliptic system with constant coefficients, it is of class C^1 in B_1; we take

$$
\psi = \Big(\frac{\lambda_h}{|A_h|}\Big)^{(p-2)/2} w
$$

in estimate (3.17), although we stick to the shorter form ψ in some places: at the left-hand side we have

$$
\begin{aligned}
&\int_{B_r}\Big[\Big|\frac{A_h}{|A_h|} + \Big\{\Big(\frac{\lambda_h}{|A_h|}\Big)^{p/2} Dw\Big\}\Big|^{p-2}|Dw_h - Dw|^2\\
&\qquad + \Big|Dv_h - \Big\{\Big(\frac{\lambda_h}{|A_h|}\Big)^{(p-2)/2} Dw\Big\}\Big|^p\\
&\qquad + \sum_\alpha \Big|\lambda_h^{(p_\alpha - p)/p_\alpha} D_\alpha v_h - \Big\{\lambda_h^{(p_\alpha - p)/p_\alpha}\Big(\frac{\lambda_h}{|A_h|}\Big)^{(p-2)/2} D_\alpha w\Big\}\Big|^{p_\alpha}\Big]\,dz.
\end{aligned}
$$

As $h \to \infty$, the quantities in curly brackets tend to zero, hence the limsup of the left-hand side of (3.17) is the same as

$$\limsup_h \int_{B_r} \Big[|Dw_h - Dw|^2 + |Dv_h|^p + \sum_\alpha |\lambda_h^{(p_\alpha - p)/p_\alpha} D_\alpha v_h|^{p_\alpha}\Big]\, dz.$$

Thus, the convergences (3.11) will be proved if we show that the three terms R_1,R_2,R_3 in the right-hand side of (3.17) tend to zero. We have

$$\begin{aligned} R_3 &= c\lambda_h^{-p} \int_{B_s} Df_h(D\psi)(D\psi - Dv_h)\, dz \\ &= c\int_{B_s} \Big\{\lambda_h^{-p}\Big(\frac{\lambda_h}{|A_h|}\Big)^{p-2} \int_0^1 D^2 f_h(\theta D\psi)\, d\theta\Big\} Dw(Dw - Dw_h)\, dz, \end{aligned}$$

but $(Dw_h - Dw) \rightharpoonup 0$ weakly in L^2, whereas using (3.1), the same argument employed to obtain (3.7) shows that the quantity in curly brackets converges strongly in L^∞ to

$$C^{ij}_{\alpha\beta} + \sum_\gamma |A_\gamma|^{p_\gamma - p}\Big(\frac{|A_\gamma|}{|A|}\Big)^{p-2} C^{ij}_\gamma\, \delta_{\alpha\gamma}\, \delta_{\beta\gamma},$$

thus R_3 goes to zero as $h \to \infty$. Now we deal with R_1: by Lemma 3.3, using (3.2), some computations yield

$$R_1 = \frac{c}{K}\,\lambda_h^{-p} \int_{B_1} [f_h(D\psi) + f_h(Dv_h)]\, dz \le \frac{c}{K}.$$

There remains only

$$R_2 = c\lambda_h^{-p} \int_{B_s \setminus B_t} f_h\big((\psi - v_h)\otimes D\zeta\big)\, dz,$$

which by Lemma 3.3 is bounded by

$$c\int_{B_1}\Big\{\frac{|w_h - w|^2}{(s-t)^2} + \frac{|v_h|^p + \omega_h}{(s-t)^p} + \sum_\alpha \frac{\omega_h}{(s-t)^{p_\alpha}}\Big\}\, dz + \sum_\alpha \frac{c}{(s-t)^{p_\alpha}} \int_{B_s} \lambda_h^{p_\alpha - p} |v_h|^{p_\alpha}\, dz.$$

The first integral vanishes as $h \to \infty$ by (3.3) and (3.6); the main difficulty in the first case is to show that the second integral vanishes too: indeed even to show that it is finite we need Lemma 2.1, since we just have $D_\alpha v_h \in L^{p_\alpha}$; moreover, we know only that

$$\lambda_h^{(p_\alpha - p)/p_\alpha} D_\alpha v_h \quad \text{is bounded in } L^{p_\alpha}.$$

Cover $\overline{B_s}$ with a finite number of cubes well contained in B_1 and with edges parallel to the coordinate axes, and let Q be any of them: we will show that

$$\int_Q \lambda_h^{p_\alpha - p} |v_h|^{p_\alpha}\, dz \to 0,$$

thus concluding the proof of (3.11). We recall that, by our choice of the meaning of $\bar{p}^*$, Lemma 2.1 implies $v_h \in L^{\bar{p}^*}(Q)$; by the assumption $p_\alpha < \bar{p}^*$, this in turn implies $v_h \in L^{p_\alpha}$, and we interpolate between p_α and $\bar{p}^*$, setting

$$\frac{1}{p_\alpha} = \frac{\theta}{p} + \frac{1-\theta}{\bar{p}^*},$$

so that

$$\theta = \frac{p(\bar{p}^* - p_\alpha)}{p_\alpha(\bar{p}^* - p)}, \quad 1-\theta = \frac{\bar{p}^*(p_\alpha - p)}{p_\alpha(\bar{p}^* - p)}.$$

Now

$$\begin{aligned}\int_Q |v_h|^{p_\alpha}\,dz &\le \Big(\int_{B_1} |v_h|^p\,dz\Big)^{\theta p_\alpha/p} \cdot \Big(\int_Q |v_h|^{\bar{p}^*}\,dz\Big)^{(1-\theta)p_\alpha/\bar{p}^*} \\ &= \omega_h \Big(\int_Q |v_h|^{\bar{p}^*}\,dz\Big)^{(1-\theta)p_\alpha/\bar{p}^*}.\end{aligned}$$

Let $p_m = \max\{p_\alpha\} < \bar{p}^*$: by Lemma 2.1 and using (3.6)

$$\begin{aligned}\Big(\int_Q |v_h|^{\bar{p}^*}\,dz\Big)^{1/\bar{p}^*} &\le c(\|v_h\|_p + \sum_\alpha \|D_\alpha v_h\|_{p_\alpha}) \\ &\le c(\omega_h + \lambda_h^{\frac{p}{p_m}-1} \sum_\alpha \|\lambda_h^{1-\frac{p}{p_\alpha}} D_\alpha v_h\|_{p_\alpha}) \le c\lambda_h^{\frac{p}{p_m}-1},\end{aligned}$$

hence

$$\lambda_h^{p_\alpha - p} \int_Q |v_h|^{p_\alpha}\,dz \le c\,\omega_h^\theta\,\lambda_h^{p_\alpha - p + (1-\theta)p_\alpha(p-p_m)/p_m}.$$

One immediately sees that the exponent of λ_h is

$$\frac{p(\bar{p}^* - p_m)(p_\alpha - p)}{p_m(\bar{p}^* - p)} > 0,$$

since $p_\alpha > p$, thus

$$\lim_h \lambda_h^{p_\alpha - p} \int_Q |v_h|^{p_\alpha}\,dz = 0.$$

Letting $h \to \infty$ we have

$$\limsup_h [R_1 + R_2 + R_3] \le c/K,$$

and letting $K \to \infty$ we obtain (3.11).

Strong convergence in the second case.

In this case the good choice in (3.17) would be $\psi = v$, but this cannot be made since it is not guaranteed that $v \in C^1$: indeed, it will be so once we prove that it solves (3.15), but this will be possible only after the proof of the strong convergence (3.13). So, we shall take instead of v an approximating function V of class C^1, then we shall let V approach v.

As $h \to \infty$ the limsup of the left-hand side of (3.17) is larger than

$$c_p \limsup_h \int_{B_r} \Big[|Dv_h - Dv|^p + \sum_\alpha \lambda_h^{p_\alpha - p}|D_\alpha v_h|^{p_\alpha}\Big]\,dz - \int_{B_r} |DV - Dv|^p\,dz.$$

As for the three terms R_1,R_2,R_3 at the right-hand side of (3.17), we deal with R_3 using the same argument employed to obtain (3.15), and we have

$$\begin{aligned}\lim_h R_3 &= \lim_h \int_{B_s} \frac{[Df(A_h + \lambda_h DV) - Df(A_h)]}{\lambda_h^{p-1}}(DV - Dv_h)\, dz \\ &= L \int_{B_s} [|l\bar{A} + DV|^{p-2}(l\bar{A} + DV) - l^{p-1}\bar{A}](DV - Dv)\, dz.\end{aligned}$$

Also, by Lemma 3.3 and by (3.2) we have

$$R_1 \leq \frac{c}{K}\Big(1 + \int_{B_s} |DV|^p\, dz + \omega_h \sum_\alpha \int_{B_s} |D_\alpha V|^{p_\alpha}\, dz\Big);$$

again as in the first case

$$\begin{aligned}R_2 \leq c \int_{B_1} \Big\{\frac{|V - v_h|^2}{(s-t)^2} + \frac{|V - v_h|^p}{(s-t)^p} + \sum_\alpha \frac{\lambda_h^{p_\alpha - p}|V|^{p_\alpha}}{(s-t)^{p_\alpha}}\Big\}\, dz \\ + c \sum_\alpha (s-t)^{-p_\alpha} \int_{B_s} \lambda_h^{p_\alpha - p}|v_h|^{p_\alpha}\, dz.\end{aligned}$$

We may deal with the second integral using the same interpolation argument as in the first case, while in the first integral we may use (3.3), thus obtaining

$$\limsup_h R_2 \leq \frac{c}{(s-t)^p}(\|V - v\|_2^2 + \|V - v\|_p^p),$$

and thus in (3.17) we may let $h \to \infty$, then $V \to v$ in $W^{1,p}$, then $K \to \infty$ to obtain (3.13) and (3.14).

Remarks for the case $p = 2$

In the case $p = 2$, the first term in the definition of U may be dropped, and there is no distinction between v_h and w_h. Using (2.6), by (3.3) one gets as $h \to \infty$

$$\int_{B_1} \Big[C\, DvD\varphi + \sum_\alpha |A^\alpha|^{p_\alpha - 2} C_\alpha\, D_\alpha v D_\alpha \varphi\Big]\, dz = 0,$$

and (3.8) follows for v. Then one has

$$\text{(3.10}'\text{)} \qquad \lambda_h^{-2} U(x_h, \tau r_h) \leq c \fint_{B_\tau} \Big[|Dv_h - (Dv_h)_\tau|^2 + \sum_\alpha \lambda_h^{p_\alpha - 2}|D_\alpha v_h|^{p_\alpha}\Big]\, dz$$

and one must prove

$$\text{(3.11}'\text{)} \qquad v_h \to v \quad \text{in } W^{1,2}_{\text{loc}}, \quad \lambda_h^{(p_\alpha - 2)/p_\alpha} D_\alpha v_h \to 0 \quad \text{in } L^{p_\alpha}_{\text{loc}}.$$

The preliminary estimates yield (3.17) with the left-hand side reduced to

$$\int_{B_r} \Big[|Dv_h - D\psi|^2 + \sum_\alpha \lambda_h^{p_\alpha - 2}|D_\alpha v_h - D_\alpha \psi|^{p_\alpha}\Big]\, dz,$$

and one chooses $\psi = v$; it is very easy to see that $R_3 \to 0$ and $R_1 \leq c/K$; also, R_2 is bounded by

$$c \int_{B_1} \Big\{\frac{|v_h - v|^2}{(s-t)^2} + \sum_\alpha \frac{\omega_h}{(s-t)^{p_\alpha}}\Big\}\, dz + \sum_\alpha \frac{c}{(s-t)^{p_\alpha}} \int_{B_s} \lambda_h^{p_\alpha - 2}|v_h|^{p_\alpha}\, dz,$$

and the rest of the proof is unchanged. ∎

The rest of the proof of Theorem 2.3 is standard (see e.g. [2], Proposition 2.7 and Proof of theorem 2.1). In particular, one sees that the Hölder exponent γ is at least equal to μ/p, and that the set of regular points can be characterized as

$$\Omega_0 = \{x \in \Omega : \limsup_{r\to 0} |(Du)_{x,r}| < +\infty,\ \lim_{r\to 0} U(x,r) = 0\}. \tag{3.18}$$

4 . Further results

This section contains some remarks and extensions of the main result.

In the case $p > 2$, the leading term $|Du|^p$ lacks ellipticity in zero; if this problem is removed, one expects that the regularity result still holds, moreover, the condition $p_\alpha > p$ is no longer necessary: indeed, let p,$p_\alpha > 2$ for $\alpha = 1, \ldots, k$, let f_α be as in Section 2, and consider the functional

$$\int_\Omega \Big[(1 + |Du|^2)^{p/2} + \sum_{\alpha=1}^{k} f_\alpha(D_\alpha u)\Big]\, dx. \tag{4.1}$$

Set

$$q_\alpha = \begin{cases} \max\{p, p_\alpha\} & \text{if } \alpha \le k \\ p & \text{otherwise,} \end{cases}$$

and define $\bar{q}^*$ from $\{q_\alpha\}$ as we did for $\bar{p}^*$ from $\{p_\alpha\}$; then we have:

Proposition 4.1 . *If $u \in W^{1,(q_\alpha)}(\Omega)$ is a local minimizer of* (4.1) *and*

$$q_\alpha < \bar{q}^* \quad \text{for all } \alpha$$

then there is an open set Ω_0, with $\operatorname{meas}(\Omega \setminus \Omega_0) = 0$, *such that $u \in C^{1,\gamma}(\Omega_0)$ for all $\gamma < 1$.*

PROOF . It is enough to follow the general lines of the proof of Lemma 3.2, with some remarks.

The function U reduces to

$$U(x_0, r) = \fint_{B_r(x_0)} \Big[|Du - (Du)_r|^2 + \sum_{\alpha=1}^{n} |D_\alpha u - (D_\alpha u)_r|^{q_\alpha}\Big]\, dx;$$

in the statement of the main lemma one has $\mu = 2$, which will give Hölder continuity for all $\gamma < 1$ instead of some γ. One sets $U(x_h, r_h) = \lambda_h^2$, and not λ_h^p, and instead of (3.2) one has

$$\fint_{B_1} \Big[|Dv_h|^2 + \sum_\alpha \lambda_h^{q_\alpha - 2} |D_\alpha v_h|^{q_\alpha}\Big]\, dz = 1,$$

so that

$$\begin{cases} A_h \to A & \\ v_h \rightharpoonup v & \text{weakly in } W^{1,2} \\ \lambda_h^{(q_\alpha - 2)/q_\alpha} Dv_h \rightharpoonup 0 & \text{weakly in } L^{q_\alpha}; \end{cases}$$

once we prove that the weak convergences are actually strong, by interpolation we also get

$$\lambda_h^{p-2} |Dv_h|^p + \sum_{\alpha=1}^{k} \lambda_h^{p_\alpha - 2} |D_\alpha v_h|^{p_\alpha} \to 0 \quad \text{in } L^1,$$

thus from the Euler equation one gets (3.7), where C may be written explicitly as

$$C = p(1 + |A|^2)^{(p-2)/2} \left(I + (p - 2) \frac{A \otimes A}{|A|^2}\right)$$

and is elliptic, with least eigenvalue greater than p and largest eigenvalue dependent on M. Therefore, v is of class C^1 and satisfies (3.8). Once the weak convergences above are proved to be strong, the lemma (and hence Proposition 4.1) follows as in Lemma 3.2 – obviously, there is no need of a "second case".

The preliminary estimates are as before, except that one knows that ψ will be just v, and (3.17) can be reduced to

$$\begin{aligned}\int_{B_r} \Big[|Dv_h - Dv|^2 + \sum_\alpha \lambda_h^{q_\alpha-2}|D_\alpha v_h - D_\alpha v|^{q_\alpha}\Big]\,dz \\ \le \frac{c}{K}\int_{B_1} \lambda_h^{-2}[f_h(Dv)+f_h(Dv_h)]\,dz \\ + c\int_{B_s\setminus B_t} \lambda_h^{-2} f_h\big((v-v_h)\otimes D\zeta\big)\,dz \\ + c\int_{B_s} \lambda_h^{-2} Df_h(Dv)(Dv-Dv_h)\,dz.\end{aligned}$$

This estimate will give us the strong convergences once we prove that the terms R_1,R_2,R_3 at the right-hand side vanish; R_1 and R_3 are dealt with as before, and R_2 is bounded by

$$c\int_{B_1}\Big[\frac{|v_h-v|^2}{(s-t)^2} + \frac{\omega_h + \lambda_h^{p-2}|v_h|^p}{(s-t)^p} + \sum_{\alpha=1}^k \frac{\omega_h + \lambda_h^{p_\alpha-2}|v_h|^{p_\alpha}}{(s-t)^{p_\alpha}}\Big]\,dz.$$

From the definition of q_α, for all ε there is C_ε such that the integral above is bounded by

$$\varepsilon + C_\varepsilon\int_{B_1}\Big[\frac{|v_h-v|^2}{(s-t)^2} + \sum_\alpha \frac{\omega_h}{(s-t)^{q_\alpha}} + \sum_\alpha \frac{\lambda_h^{q_\alpha-2}|v_h|^{q_\alpha}}{(s-t)^{q_\alpha}}\Big]\,dz.$$

Interpolating between 2 and $\bar q^*$ we get

$$\int_Q |v_h|^{q_\alpha}\,dz \le \Big(\int_{B_1}|v_h|^2\,dz\Big)^{\theta q_\alpha/2}\cdot\Big(\int_Q |v_h|^{\bar q^*}\,dz\Big)^{(1-\theta)q_\alpha/\bar q^*};$$

we estimate the last integral using Lemma 2.1 and we have

$$\lambda_h^{q_\alpha-2}\int_Q |v_h|^{q_\alpha}\,dz \le c\,\lambda_h^{2(q_\alpha-2)(\bar q^*-q_m)/[(\bar q^*-2)q_m]},$$

and the exponent is positive because $2 < q_\alpha \le q_m = \max\{q_\alpha\} < \bar q^*$. After the main lemma is proved, the conclusion is as before – in particular, the regular set Ω_0 is still given by (3.18). ∎

Remark 4.2 . *A similar result holds if the leading term $(1+|Du|^2)^{p/2}$ is replaced by a function of the same type (growth of order p at infinity, ellipticity constant bounded away from zero).*

Remark 4.3 . *If in Theorem 2.3 we assume $p = 2$ and*

$$\begin{cases} p_\alpha > 2 \quad \text{for all } \alpha & \text{if } n = 2 \\ \dfrac{2\cdot 2^*}{2+2^*-\max\{p_\beta\}} < p_\alpha < 2^* \quad \text{for all } \alpha & \text{if } n \ge 3, \end{cases} \tag{4.2}$$

then by [10] one has $u \in W^{2,2}_{\rm loc}(\Omega)$, thus as in [6] Chapter 4 we may prove that the Hausdorff dimension of the singular set $\Omega\setminus\Omega_0$ is at most $n-2$. A similar result holds in the setting of Proposition 4.1, where no explicit condition on p is needed, and the numbers q_α satisfy (4.2).

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