Regularity for minimizers of non-quadratic functionals. The case $1 < p < 2$.

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Introduction

Let f be a function defined on $\mathbf{R}^n \times \mathbf{R}^N \times \mathbf{R}^{nN}$, and set

$$I(u, A) = \int_A f\big(x, u(x), Du(x)\big)\, dx :$$

we say that u is a local minimizer for I if

$$I(u, \operatorname{spt}\varphi) \le I(u + \varphi, \operatorname{spt}\varphi) \qquad \text{for all } \varphi \in C_0^1(\mathbf{R}^n; \mathbf{R}^N).$$

In a fundamental paper, appeared in 1977, K. Uhlenbeck [10] proved everywhere $C^{1,\alpha}$ regularity for local minimizers $u \in W^{1,p}(\Omega; \mathbf{R}^N)$ of

$$\int_\Omega |Du(x)|^p\, dx,$$

with $p \ge 2$, and more in general for local minimizers of

$$\int_\Omega g\big(|Du(x)|^2\big)\, dx$$

when $g(t^2)$ behaves like t^p. This result has been generalized in two different ways: in [2],[4] dependence of the integrand on (x, u) is allowed, and in [1],[7],[8],[9] the case $1 < p < 2$ is studied. Under this assumption, regularity is proved in [7],[9] only for $N = 1$, which in the smooth case corresponds to a partial differential equation instead of a system, and in [1],[8] only for quasilinear systems.

In this paper we give a regularity theorem in the nonlinear case with $N > 1$, $1 < p < 2$ and dependence also on the variables (x, u).

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We consider first the case independent of (x, u): let $1 < p < 2$, and $f\colon \mathbf{R}^{nN} \to \mathbf{R}$ satisfy for a suitable $\mu \geq 0$ the following assumptions:

$$c_1 \left(\mu^2 + |\xi|^2\right)^{p/2} \leq f(\xi) \leq c\big(\mu^2 + |\xi|^2\big)^{p/2}; \tag{H1}$$

$$f(\xi) = g\big(|\xi|^2\big), \qquad \text{with } g \in C^2(\mathbf{R}) \text{ if } \mu > 0 \text{ or } g \in C^2(\mathbf{R} \setminus \{0\}) \text{ if } \mu = 0; \tag{H2}$$

$$|D^2 f(\xi)| \leq c \left(\mu^2 + |\xi|^2\right)^{(p-2)/2}; \tag{H3}$$

$$\langle D^2 f(\xi)\eta\,,\, \eta\rangle \geq \left(\mu^2 + |\xi|^2\right)^{(p-2)/2} |\eta|^2, \tag{H4}$$

and also, for some $\alpha \in (0, 2-p]$,

$$|D^2 f(\xi) - D^2 f(\eta)| \leq c \left(\mu^2 + |\xi|^2\right)^{(p-2)/2} \left(\mu^2 + |\eta|^2\right)^{(p-2)/2} \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{(2-p-\alpha)/2} |\xi - \eta|^\alpha. \tag{H5}$$

Then we have everywhere regularity:

Theorem 1.1 . *Let $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbf{R}^N)$ be a local minimizer of $\int f\big(Dv(x)\big)\,dx$, with f satisfying* (H1), …,(H5). *Then Du is locally λ-Hölder continuous for some $\lambda > 0$.*

For the case with (x, u) we need the following assumptions:

$$\begin{array}{l}\text{for every fixed } (x_0, u_0) \text{ the function } f(x_0, u_0, \xi) \text{ satisfies} \\ \text{(H1)}, \ldots, \text{(H5) with } \mu,\ c_1,\ c,\ \alpha \text{ independent of } (x_0, u_0);\end{array} \tag{H6}$$

$$\begin{array}{l}|f(x, u, \xi) - f(y, v, \xi)| \leq \left(\mu^2 + |\xi|^2\right)^{p/2} \omega\big(|u|, |x-y| + |u-v|\big), \text{ where} \\ \omega(s, t) = K(s) \cdot \min\{t^\gamma, L\} \text{ for some } L > 0 \text{ and } \gamma \in (0, 1], \text{ and } K \text{ is increasing.}\end{array} \tag{H7}$$

Then, denoting by $\mathcal{H}_k$ the k-dimensional Hausdorff measure, we have a partial regularity result:

Theorem 1.2 . *Let $u \in W^{1,p}_{\text{loc}}(\Omega; \mathbf{R}^N)$ be a local minimizer of $\int f\big(x, v(x), Dv(x)\big)\,dx$, with f satisfying* (H6),(H7).*Then there is an open set $\Omega_0 \subset \Omega$ such that $\mathcal{H}_{n-q}(\Omega \setminus \Omega_0) = 0$ for some $q > p$, and Du is locally λ-Hölder continuous in Ω_0 for some $\lambda > 0$.*

Our proofs follow the argument used in [4],[10] for the case $p \geq 2$, and rely heavily on the special structure (H2) of f. In our case there are some new difficulties, as for example in proving Proposition 2.6 where the simple device of adding $\varepsilon \int |Du|^2\,dx$ to the functional would not affect its lack of ellipticity at $Du = 0$.

The difference with the case $p \geq 2$ does not lie only in the technical problems involved, but also in some regularity properties of the minimizer u which come as a by-product of our estimates: precisely the function u, which is a priori only in $W^{1,p}$, comes out not only to be in $C^{1,\lambda}$, but also to have second derivatives in L^2.

Finally, we remark that it is not difficult to obtain the analogous of Theorems 1.1 and 1.2 when f has the form

$$f(x, u, \xi) = g\big(x, u, a_{\alpha\beta}(x, u)\, b_{ij}(x, u)\, \xi^i_\alpha\, \xi^j_\beta\big)$$

with a, b uniformly elliptic, bounded, symmetric and γ-Hölder continuous (see [2],[4]).

While writing this paper, we were told that also C. Hamburger [6] was working on the same subject, but using very different techniques.

Proof of Theorem I.1

To simplify the notation, the letter c will denote any constant, which may vary throughout the paper, and if no confusion is possible we omit the indication of Ω and $\mathbf{R}^k$ when writing $W^{m,p}(\Omega;\mathbf{R}^k)$. If $u \in L^p$, for any $B_R(x_0)$ we set

$$u_{x_0,R} = \frac{1}{\operatorname{meas} B_R} \int_{B_R(x_0)} u(x)\,dx = \fint_{B_R(x_0)} u(x)\,dx.$$

We will often omit the centre of the ball, thus writing only u_R and $\fint_{B_R}$.

First we give some basic inequalities:

Lemma 2.1 . *For every $\gamma \in (-1/2, 0)$ and $\mu \geq 0$ we have*

$$1 \leq \frac{\int_0^1 \big(\mu^2 + |\eta + s(\xi-\eta)|^2\big)^\gamma\,ds}{\big(\mu^2 + |\xi|^2 + |\eta|^2\big)^\gamma} \leq \frac{8}{2\gamma+1}$$

for all ξ, η in $\mathbf{R}^k$, not both zero if $\mu = 0$.

Proof . The left inequality follows from the convexity of $s \mapsto |\eta + s(\xi-\eta)|^2$, since $\gamma < 0$. In order to prove the second inequality, we may assume

$$|\xi| \leq |\eta|, \qquad \xi \neq \eta.$$

Denote by ξ_0 the point with least norm of the line through η and ξ, and set

$$s_0 = \frac{|\xi_0 - \eta|}{|\xi - \eta|};$$

in addition, for every $\lambda \in \mathbf{R}^k$ and $s \in [0,1]$ set

$$\varphi_\lambda(s) = \big(\mu^2 + |\eta + s(\lambda - \eta)|^2\big)^\gamma.$$

We remark that $s_0 \geq 1/2$; in the case $s_0 \geq 1$ we have $\varphi_\xi(s) \leq \varphi_{\xi_0}(s)$ for all s, so that

$$\int_0^1 \varphi_\xi(s)\,ds \leq \int_0^1 \varphi_{\xi_0}(s)\,ds. \tag{2.1}$$

In the case $s_0 < 1$

$$\int_0^1 \varphi_\xi(s)\,ds \leq 2\int_0^{s_0} \varphi_\xi(s)\,ds = 2s_0 \int_0^1 \varphi_{\xi_0}(s)\,ds \leq 2\int_0^1 \varphi_{\xi_0}(s)\,ds. \tag{2.2}$$

Remarking that $\varphi_{\xi_0}(s) \leq \varphi_0(s)$, from (2.1),(2.2) follows

$$\begin{aligned} \int_0^1 \big(\mu^2 + |\eta + s(\xi-\eta)|^2\big)^\gamma\,ds &\leq 2\int_0^1 \big(\mu^2 + s^2|\eta|^2\big)^\gamma\,ds \\ &\leq 2^{1-\gamma} \int_0^1 \big(\mu^2 + s^2(|\xi|^2 + |\eta|^2)\big)^\gamma\,ds \leq 4\int_0^1 \big(\mu + s\big(|\xi|^2 + |\eta|^2\big)^{1/2}\big)^{2\gamma}\,ds. \end{aligned} \tag{2.3}$$

Now if $0 \leq b \leq a$

$$\int_0^1 (a+sb)^{2\gamma}\,ds \leq a^{2\gamma} \leq 2(a^2+b^2)^\gamma$$

and if $b > a \geq 0$

$$\int_0^1 (a+sb)^{2\gamma}\,ds \leq \frac{(a+b)^{2\gamma+1}}{(2\gamma+1)b} \leq \frac{2}{2\gamma+1}(a+b)^{2\gamma} \leq \frac{2}{2\gamma+1}(a^2+b^2)^\gamma,$$

so the result follows from (2.3). ∎

Lemma 2.2 . *For every $\gamma \in (-1/2, 0)$ and $\mu \geq 0$ we have*

$$(2\gamma+1)|\xi-\eta| \leq \frac{\left|\left(\mu^2+|\xi|^2\right)^\gamma \xi - \left(\mu^2+|\eta|^2\right)^\gamma \eta\right|}{\left(\mu^2+|\xi|^2+|\eta|^2\right)^\gamma} \leq \frac{c(k)}{2\gamma+1}|\xi-\eta|$$

for every ξ, η in $\mathbf{R}^k$.

Proof . Set

$$F(\zeta) = \frac{1}{2(\gamma+1)}\left(\mu^2+|\zeta|^2\right)^{\gamma+1},$$

so that

$$DF(\zeta) = \left(\mu^2+|\zeta|^2\right)^\gamma \zeta, \qquad D^2F(\zeta) = \left(\mu^2+|\zeta|^2\right)^\gamma \Big(I + \frac{2\gamma}{\mu^2+|\zeta|^2}\,\zeta\otimes\zeta\Big);$$

in particular we have

$$\langle D^2F(\zeta)\,\lambda, \lambda\rangle \geq (2\gamma+1)\left(\mu^2+|\zeta|^2\right)^\gamma |\lambda|^2 \tag{2.4}$$

$$|D^2F(\zeta)| \leq \sqrt{k+1}\left(\mu^2+|\zeta|^2\right)^\gamma. \tag{2.5}$$

Then by (2.4) and Lemma 2.1

$$\begin{aligned}\langle DF(\xi)-DF(\eta), \xi-\eta\rangle &= \Big\langle \textstyle\int_0^1 D^2F\big(\eta+s(\xi-\eta)\big)\,ds\,(\xi-\eta), (\xi-\eta)\Big\rangle \\ &\geq (2\gamma+1)\left(\mu^2+|\xi|^2+|\eta|^2\right)^\gamma |\xi-\eta|^2,\end{aligned}$$

and the left inequality follows immediately. By (2.5) and Lemma 2.1

$$\begin{aligned}|DF(\xi)-DF(\eta)| &\leq \textstyle\int_0^1 \big|D^2F\big(\eta+s(\xi-\eta)\big)\big|\,ds\,|\xi-\eta| \\ &\leq \frac{8\sqrt{k+1}}{2\gamma+1}\left(\mu^2+|\xi|^2+|\eta|^2\right)^\gamma,\end{aligned}$$

which concludes the proof. ∎

In what follows, $u \in W^{1,p}_{\rm loc}$ is a local minimizer of $\int f(Du)\,dx$, with $\mu \geq 0$ fixed (it is not restrictive to take $\mu \leq 1$), $1 < p < 2$, and f satisfies some of the assumptions (H1), …,(H5). We set

$$H(\xi) = \left(\mu^2+|\xi|^2\right)^{p/2}$$

$$V(\xi) = \left(\mu^2+|\xi|^2\right)^{(p-2)/4}\xi$$

$$\Phi(x_0, R) = \fint_{B_R(x_0)} \left|V(Du) - \left(V(Du)\right)_{x_0,R}\right|^2 dx.$$

First we give a higher integrability result for $H(Du)$:

Proposition 2.3 . *Let f satisfy* (H1)*. There are two constants $c > 0$ and $q > 1$, both independent of μ, such that*

$$\Big(\fint_{B_{R/2}} H^q(Du)\,dx\Big)^{1/q} \leq c \fint_{B_R} H(Du)\,dx$$

for every $B_R \subset\subset \Omega$.

The proof is essentially the same as Theorem 3.1 of [3], section V.

From now on we specialize to the case $\mu > 0$, to obtain the estimates which will allow us to deal with the general case.

Proposition 2.4 . *Let f be a function of class C^2 satisfying* (H1),(H4). *Then*

$$u \in W^{2,p}_{\text{loc}}, \qquad V(Du) \in W^{1,2}_{\text{loc}}.$$

Moreover

$$\int_{B_{R/2}} \big|D\big(V(Du)\big)\big|^2\,dx \le \frac{c}{R^2}\int_{B_R} H(Du)\,dx \tag{2.6}$$

$$\int_{B_{R/2}} \big(\mu^2+|Du|^2\big)^{(p-2)/2}\,|D^2u|^2\,dx \le \frac{c}{R^2}\int_{B_R} H(Du)\,dx \tag{2.7}$$

$$\int_{B_{R/2}} |D^2u|^p\,dx \le \frac{c}{R^p}\int_{B_R} H(Du)\,dx. \tag{2.8}$$

for a suitable c independent of μ.

Proof . Since f is a convex function of class C^1, by (H1) w e have also

$$|Df(\xi)| \le c\,\big(\mu^2+|\xi|^2\big)^{(p-1)/2}. \tag{2.9}$$

Let e_s be a coordinate direction in $\mathbf{R}^n$; for every function g we define

$$\Delta_h g(x) = \frac{1}{h}\,[g(x+he_s)-g(x)].$$

For every $\varphi \in W^{1,p}$ with compact support in Ω we have

$$\int f_{\xi^i_\alpha}(Du)\,D_\alpha\varphi^i\,dx = 0,$$

so that for h small

$$\int \big[f_{\xi^i_\alpha}\big(Du(x+he_s)\big) - f_{\xi^i_\alpha}\big(Du(x)\big)\big]\,D_\alpha\varphi^i\,dx = 0.$$

Choosing $\varphi^i = \frac{1}{h}\eta^2\,\Delta_h u^i$, with $\eta \in C^2_0(B_R)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{R/2}$, $|D\eta| \le c/R$ and $|D^2\eta| \le c/R^2$, we obtain

$$\int \Delta_h\big(f_{\xi^i_\alpha}(Du)\big)\,D_\alpha\Delta_h u^i\,\eta^2\,dx = -2\int \Delta_h\big(f_{\xi^i_\alpha}(Du)\big)\Delta_h u^i\,\eta\,D_\alpha\eta\,dx. \tag{2.10}$$

But

$$\Delta_h\big(f_{\xi^i_\alpha}(Du)\big) = \int_0^1 f_{\xi^i_\alpha\,\xi^j_\beta}\big(Du + th\,D(\Delta_h u)\big)\,dt\,D_\beta(\Delta_h u^j)$$

and also

$$\Delta_h\big(f_{\xi^i_\alpha}(Du)\big) = \int_0^1 \frac{d}{dx_s}\big[f_{\xi^i_\alpha}\big(Du(x+the_s)\big)\big]\,dt;$$

then (2.10), using (H4),(2.9), implies

$$\begin{aligned}
&\int \big(\mu^2+|Du(x)|^2+|Du(x+he_s)|^2\big)^{(p-2)/2}\,|D\Delta_h u|^2\,\eta^2\,dx \\
&\quad\le 2\int\int_0^1 f_{\xi^i_\alpha}\big(Du(x+the_s)\big)\,dt\,\frac{d}{dx_s}(\Delta_h u^i\,\eta\,D_\alpha\eta)\,dx \\
&\quad\le c\int\int_0^1 \big(\mu^2+|Du(x+the_s)|^2\big)^{(p-1)/2}\,dt\,\big(|D\Delta_h u||D\eta|\eta + |\Delta_h u|(\eta|D^2\eta|+|D\eta|^2)\big)\,dx \\
&\quad\le \frac{c}{R}\int\int_0^1 \big(\mu^2+|Du(x+the_s)|^2\big)^{(p-1)/2}\,dt\,|D\Delta_h u|\eta\,dx \\
&\qquad + \frac{c}{R^2}\int_{B_R}\int_0^1 \big(\mu^2+|Du(x+the_s)|^2\big)^{(p-1)/2}\,dt\,|\Delta_h u|\,dx.
\end{aligned}$$

Applying Young inequality in the second-last line, one easily reduces to

$$\int \big(\mu^2+|Du(x)|^2+|Du(x+he_s)|^2\big)^{(p-2)/2}\,|D\Delta_h u|^2\,\eta^2\,dx$$
$$\leq \frac{c}{R^2}\int_{B_R}\int_0^1 \big(\mu^2+|Du(x+the_s)|^2\big)^{p-1}\,dt\,\big(\mu^2+|Du(x)|^2+|Du(x+he_s)|^2\big)^{(2-p)/2}\,dx$$
$$+\frac{c}{R^2}\int_{B_R}\int_0^1 \big(\mu^2+|Du(x+the_s)|^2\big)^{(p-1)/2}\,dt\,|\Delta_h u|\,dx. \tag{2.11}$$

Now by Lemma 2.2

$$\int_{B_{R/2}} \big|\Delta_h\big(V(Du)\big)\big|^2\,dx \leq \int \big(\mu^2+|Du(x)|^2+|Du(x+he_s)|^2\big)^{(p-2)/2}\,|D\Delta_h u|^2\,\eta^2\,dx; \tag{2.12}$$

joining (2.11),(2.12) and taking the limit in h we get that $V(Du)\in W^{1,2}_{\rm loc}$, together with (2.6). Also, by Lemma 2.2

$$\int_{B_{R/2}} |\Delta_h Du|^p\,dx \leq c\int_{B_{R/2}} \big|\Delta_h\big(V(Du)\big)\big|^p\big(\mu^2+|Du(x)|^2+|Du(x+he_s)|^2\big)^{p(2-p)/4}\,dx$$
$$\leq c\Big(\int_{B_{R/2}} \big|\Delta_h\big(V(Du)\big)\big|^2\,dx\Big)^{p/2}\Big(\int_{B_R} H(Du)\,dx\Big)^{(2-p)/2},$$

and this implies $u\in W^{2,p}_{\rm loc}$, together with (2.8). To conclude the proof it is now enough to revert to (2.11): taking the limit in h yields (2.7). ∎

For every $N>0$ we set

$$h_N(x)=\mu^2+\big(\min\{|Du|,N\}\big)^2.$$

We have

Lemma 2.5 . *Let f be a function of class C^2 satisfying* (H1),(H4). *Then for every $q>0$*

$$h_N^q\in W^{1,2}_{\rm loc},\qquad |Dh_N^q|\leq q\,c(N)\,|D^2u|\,\mathbf{1}_{\{|Du|\leq N\}}$$

and

$$h_N^q\,Du\in W^{1,2}_{\rm loc},\qquad D(h_N^q\,Du)=Dh_N^q\,Du+h_N^q\,D^2u.$$

Moreover

$$H(Du)\in W^{1,s}_{\rm loc},\ \textit{where } s=\tfrac{2n}{2n-p}>1.$$

If in addition f satisfies $(H3)$ we have also

$$f_{\xi^i_\alpha}(Du)\in W^{1,2}_{\rm loc},\qquad D\big(f_{\xi^i_\alpha}(Du)\big)=f_{\xi^i_\alpha\,\xi^j_\beta}(Du)\,D(D_\beta u^j).$$

Proof . A consequence of Proposition 2.4 is that $Du\in W^{1,p}_{\rm loc}$, and

$$D^2u\,\mathbf{1}_{\{|Du|\leq N\}}\in L^2_{\rm loc};$$

therefore the properties of h_N^q and $h_N^q\,Du$ are immediate, and the regularity of H is obtained by letting $N\to\infty$ in $h_N^{p/2}$, recalling (2.7). Then, approximating Du in $W^{1,p}_{\rm loc}$ with smooth functions, and using (2.7) and (H3), it is easy to prove also the assertions on $f_{\xi^i_\alpha}(Du)$. ∎

Now we use the special form (H2) of the integrand: set

$$A_{\alpha\beta}(x) = \big[g'\big(|Du|^2\big)\delta_{\alpha\beta} + 2g''\big(|Du|^2\big)\,D_\alpha u^m\,D_\beta u^m\big]\big(\mu^2+|Du|^2\big)^{(2-p)/2}.$$

We remark that if (H1), ...,(H4) hold then A is a uniformly elliptic matrix with bounded coefficients, and the ellipticity constant, the coefficients and the ratio of the greatest to the least eigenvalue are bounded independent of μ.

From now on, (H1), ...,(H4) are always assumed.

Proposition 2.6 . *There is a positive c, independent of μ, such that*

$$\int A_{\alpha\beta}\,D_\alpha\big(H(Du)\big)D_\beta\eta\,dx \le -c\int \big|D\big(V(Du)\big)\big|^2\,\eta\,dx$$

for all $\eta\in C_0^1(\Omega)$ with $\eta\ge 0$.

PROOF . In the Euler equation

$$\int f_{\xi_\alpha^i}(Du)\,D_\alpha\varphi^i\,dx = 0$$

we are allowed by Lemma 2.5 to take $\varphi = D_s(\eta h_N^q\,D_s u)$; then we have

$$\int D_s\big(f_{\xi_\alpha^i}(Du)\big)h_N^q\,D_s u^i\,D_\alpha\eta\,dx = -\int D_s\big(f_{\xi_\alpha^i}(Du)\big)\eta\,D_\alpha(h_N^q\,D_s u^i)\,dx.$$

Using (H2), the left-hand side may be written

$$\frac{2}{p}\int A_{\alpha s}\,D_s\big(H(Du)\big)\,D_\alpha\eta\,h_N^q\,dx;$$

at the right-hand side we have

$$-\int D_s\big(f_{\xi_\alpha^i}(Du)\big)(\eta\,D_s u^i\,D_\alpha h_N^q + \eta h_N^q\,D_{\alpha s}u^i)\,dx$$
$$\le q\,c(N,\eta)\int_{\{|Du|\le N\}}|D^2u|^2\,dx - c\int\big|D\big(V(Du)\big)\big|^2\,\eta h_N^q\,dx.$$

Letting $q\to 0$ we have $h_N^q\to \mathbf{1}_{\{|Du|\le N\}}$ in L^∞, so that finally

$$\int_{\{|Du|\le N\}} A_{\alpha s}\,D_s\big(H(Du)\big)D_\alpha\eta\,dx \le -c\int_{\{|Du|\le N\}}\big|D\big(V(Du)\big)\big|^2\,\eta\,dx, \tag{2.13}$$

and the result follows as $N\to\infty$. ∎

Proposition 2.7 . *There is a c independent of μ such that*

$$\sup_{B_{R/2}} H(Du) \le c \fint_{B_R} H(Du)\,dx \tag{2.14}$$

for every $B_R\subset\subset\Omega$. Moreover

$$u\in W^{2,2}_{\rm loc},\qquad H(Du)\in W^{1,2}_{\rm loc}.$$

PROOF . Fix $N>0$; we remark that by (2.13) and Lemma 2.5 the function $h_N^{p/2}$ is a $W^{1,2}_{\rm loc}$ subsolution of the elliptic operator $-D_\alpha(A_{\alpha\beta}\,D_\beta)$; then by Theorem 8.17 of [5] we have for a suitable c independent of μ

$$\sup_{B_{R/4}} h_N^{p/2} \le c\left(\fint_{B_{R/2}} h_N^{pq/2}\,dx\right)^{1/q},$$

where q is the exponent of Proposition 2.3. Taking the limit in N and using 2.3 we obtain (2.14); the regularity of u and $H(Du)$ follows then from (2.7). ∎

The proof of [4], Proposition 3.1 works also in our case, so we have

Proposition 2.8 . *There is a c independent of μ such that*

$$\Phi(x_0, R/2) \le c\,[\sup_{B_R} H(Du) - \sup_{B_{R/2}} H(Du)]$$

for every $B_R \subset\subset \Omega$.

Lemma 2.9 . *Let $B_R(x_0) \subset\subset \Omega$, and assume*

$$\sup_{B_R} |Du|^2 \le k\,(\mu^2 + |\xi|^2)$$

for some k, ξ. There are two positive constants c, δ, both dependent on k but not on μ and ξ, such that

$$\fint_{B_{R/2}} |Du - \xi|^{2+2\delta}\,dx \le c \left(\fint_{B_R} |Du - \xi|^2\,dx \right)^{1+\delta}.$$

Proof . Let $B_\varrho(y_0) \subset B_R(x_0)$ and set

$$w(x) = u(x) - u_{y_0,\varrho} - \xi(x - y_0).$$

Since for every $\varphi \in C_0^1$

$$\int f_{\xi^i_\alpha}(Du)\,D_\alpha\varphi^i\,dx = \int [f_{\xi^i_\alpha}(Du) - f_{\xi^i_\alpha}(\xi)]\,D_\alpha\varphi^i\,dx = 0,$$

we have

$$\int\int_0^1 f_{\xi^i_\alpha\,\xi^j_\beta}(\xi + s\,Dw)\,ds\,D_\beta w^j\,D_\alpha\varphi^i\,dx = 0. \tag{2.15}$$

Fix $\eta \in C_0^1(B_\varrho)$ with $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_{\varrho/2}$ and $|D\eta| \le c/\varrho$, and take $\varphi = w\eta^2$: then by (H3),(H4) and Young inequality we get from (2.15)

$$\begin{aligned}\int\int_0^1 \big(\mu^2 + |\xi + sDw|^2\big)^{(p-2)/2}\,ds\,|Dw|^2\,\eta^2\,dx \\ \le c\int\int_0^1 \big(\mu^2 + |\xi + sDw|^2\big)^{(p-2)/2}\,ds\,w^2\,|D\eta|^2\,dx;\end{aligned} \tag{2.16}$$

by Lemma 2.1 and our assumption on $\sup|Du|$

$$c(k)\big(\mu^2 + |\xi|^2\big)^{(p-2)/2} \le \int_0^1 \big(\mu^2 + |\xi + sDw|^2\big)^{(p-2)/2}\,ds \le c\,\big(\mu^2 + |\xi|^2\big)^{(p-2)/2},$$

so (2.16) becomes

$$\int_{B_{\varrho/2}} |Du - \xi|^2\,dx \le \frac{c}{\varrho^2}\int_{B_\varrho} |u - u_{y_0,\varrho} - \xi(x - y_0)|^2\,dx,$$

and the result follows by Sobolev-Poincaré inequality and Gehring lemma. ∎

From now on we use also assumption (H5). It is not restrictive to take the exponent δ in Lemma 2.9 to be less than the exponent α of (H5).

Lemma 2.10 . *There is a c, independent of μ, such that for every $\tau \in (0,1)$ there exists $\varepsilon > 0$, dependent on τ but not on μ, such that*

$$\Phi(x_0,R) \le \varepsilon \sup_{B_{R/2}} H(Du) \quad \Rightarrow \quad \Phi(x_0,\tau R) \le c\tau^2\, \Phi(x_0,R)$$

for every $B_R \subset\subset \Omega$.

Proof . We only need to prove the assertion for τ small, therefore we fix $\tau < 1/8$; we will select ε later. Take ξ such that

$$V(\xi) = \big(V(Du)\big)_{x_0,R}.$$

By Proposition 2.7

$$\sup_{B_{R/2}} H(Du) \le c \fint_{B_R} H(Du)\,dx \le c \fint_{B_R} \big(\mu^p + |V(Du)|^2\big)\,dx \le c\,\big(\mu^p + \Phi(x_0,R) + |V(\xi)|^2\big), \tag{2.17}$$

so that if $\varepsilon < 1/2c$ we deduce

$$\Phi(x_0,R) \le 2c\varepsilon\,\big(\mu^p + |V(\xi)|^2\big) \le c\varepsilon\,\big(\mu^2 + |\xi|^2\big)^{p/2}; \tag{2.18}$$

therefore, going back to (2.17),

$$\sup_{B_{R/2}} |Du|^p \le \sup_{B_{R/2}} H(Du) \le c\,\big(\mu^2 + |\xi|^2\big)^{p/2}. \tag{2.19}$$

Choose w as in Lemma 2.9, and let $v \in W^{1,2}(B_{R/4})$ be the solution of

$$\begin{cases} \int_{B_{R/4}} f_{\xi^i_\alpha\,\xi^j_\beta}(\xi)\, D_\beta v^j\, D_\alpha \varphi^i\,dx = 0 \quad \text{for all } \varphi \in W^{1,2}_0(B_{R/4}). \\ v \in w + W^{1,2}_0(B_{R/4}) \end{cases}$$

We have

$$\fint_{B_{\tau R}} |Dv - (Dv)_{\tau R}|^2\,dx \le c\tau^2 \fint_{B_{R/4}} |Dv - (Dv)_{R/4}|^2\,dx, \tag{2.20}$$

where the constant c depends only on the ratio of the eigenvalues of $D^2 f(\xi)$, and therefore is independent of μ. By (2.15) we have for all $\varphi \in W^{1,2}_0(B_{R/4})$

$$\begin{aligned} &\fint_{B_{R/4}} f_{\xi^i_\alpha\,\xi^j_\beta}(\xi)(D_\beta v^j - D_\beta w^j) D_\alpha \varphi^i\,dx \\ &\qquad = \fint_{B_{R/4}} \int_0^1 [f_{\xi^i_\alpha\,\xi^j_\beta}(\xi + sDw) - f_{\xi^i_\alpha\,\xi^j_\beta}(\xi)]\,ds\, D_\beta w^j\, D_\alpha \varphi^i\,dx; \end{aligned} \tag{2.21}$$

recalling that $\alpha < 2 - p$ we obtain by (2.19) and Lemma 2.1

$$\begin{aligned} &\int_0^1 |f_{\xi^i_\alpha\,\xi^j_\beta}(\xi + sDw) - f_{\xi^i_\alpha\,\xi^j_\beta}(\xi)|\,ds \\ &\quad \le \big(\mu^2 + |\xi|^2\big)^{(p-2)/2} \int_0^1 \big(\mu^2 + |\xi + sDw|^2\big)^{(p-2)/2} \big(\mu^2 + |\xi|^2 + |\xi + sDw|^2\big)^{(2-p-\alpha)/2} |sDw|^\alpha\,ds \\ &\quad \le c\,\big(\mu^2 + |\xi|^2\big)^{-\alpha/2} |Dw|^\alpha \int_0^1 \big(\mu^2 + |\xi + sDw|^2\big)^{(p-2)/2}\,ds \\ &\quad \le c\,\big(\mu^2 + |\xi|^2\big)^{(p-2-\alpha)/2} |Dw|^\alpha. \end{aligned}$$

Choose $\varphi = v - w$ in (2.21): using (H4) we deduce

$$\fint_{B_{R/4}} |Dv - Dw|^2\,dx \le c\left(\mu^2 + |\xi|^2\right)^{-\alpha/2} \fint_{B_{R/4}} |Dw|^{1+\alpha}\,|Dv - Dw|\,dx,$$

and using again (2.19)

$$\begin{aligned}\fint_{B_{R/4}} |Dv - Dw|^2\,dx &\le c\left(\mu^2 + |\xi|^2\right)^{-\alpha} \fint_{B_{R/4}} |Dw|^{2+2\alpha}\,dx \\ &\le c\left(\mu^2 + |\xi|^2\right)^{-\alpha} \fint_{B_{R/4}} |Dw|^{2+2\delta}\,|Dw|^{2\alpha-2\delta}\,dx \\ &\le c\left(\mu^2 + |\xi|^2\right)^{-\delta} \fint_{B_{R/4}} |Dw|^{2+2\delta}\,dx.\end{aligned}$$

By (2.19) we may apply Lemma 2.9, thus obtaining

$$\fint_{B_{R/4}} |Dv - Dw|^2\,dx \le c\left(\mu^2 + |\xi|^2\right)^{-\delta} \left(\fint_{B_{R/2}} |Du - \xi|^2\,dx\right)^{1+\delta}. \tag{2.22}$$

Now, using Lemma 2.2,

$$\begin{aligned}\Phi(x_0, \tau R) &\le \fint_{B_{\tau R}} \left|V(Du) - V\big((Du)_{\tau R}\big)\right|^2 dx \\ &\le c \fint_{B_{\tau R}} \left(\mu^2 + |Du|^2 + |(Du)_{\tau R}|^2\right)^{(p-2)/2} |Du - (Du)_{\tau R}|^2\,dx \\ &\le c\left(\mu^2 + |(Du)_{\tau R}|^2\right)^{(p-2)/2} \fint_{B_{\tau R}} |Dw - (Dw)_{\tau R}|^2\,dx.\end{aligned} \tag{2.23}$$

From (2.20) we get

$$\begin{aligned}\fint_{B_{\tau R}} |Dw - (Dw)_{\tau R}|^2\,dx &\le 2 \fint_{B_{\tau R}} \left[|Dv - (Dv)_{\tau R}|^2 + |Dv - Dw|^2\right] dx \\ &\le c\left(\tau^2 \fint_{B_{R/4}} |Dv - (Dv)_{R/4}|^2\,dx + \tau^{-n} \fint_{B_{R/4}} |Dv - Dw|^2\,dx\right) \\ &\le c\left(\tau^2 \fint_{B_{R/4}} |Dw - (Dw)_{R/4}|^2\,dx + \tau^{-n} \fint_{B_{R/4}} |Dv - Dw|^2\,dx\right) \\ &\le c\tau^2 \fint_{B_{R/2}} |Du - \xi|^2\,dx + c\tau^{-n}\left(\mu^2 + |\xi|^2\right)^{-\delta} \left(\fint_{B_{R/2}} |Du - \xi|^2\,dx\right)^{1+\delta},\end{aligned} \tag{2.24}$$

where we used (2.22). But by Lemma 2.2

$$\begin{aligned}\fint_{B_{R/2}} |Du - \xi|^2\,dx &\le c \fint_{B_{R/2}} \left(\mu^2 + |\xi|^2 + |Du|^2\right)^{(2-p)/2} |V(Du) - V(\xi)|^2\,dx \\ &\le c\left(\mu^2 + |\xi|^2\right)^{(2-p)/2} \Phi(x_0, R),\end{aligned} \tag{2.25}$$

using again (2.19). Then from (2.23),(2.24) we deduce

$$\Phi(x_0,\tau R)\le c\left(\frac{\mu^2+|\xi|^2}{\mu^2+|(Du)_{\tau R}|^2}\right)^{(2-p)/2}\left[\tau^2\,\Phi(x_0,R)+\tau^{-n}\left(\mu^2+|\xi|^2\right)^{-\delta p/2}\left(\Phi(x_0,R)\right)^{1+\delta}\right],$$

and (2.18) implies

$$\Phi(x_0,\tau R)\le c\left(\frac{\mu^2+|\xi|^2}{\mu^2+|(Du)_{\tau R}|^2}\right)^{(2-p)/2}(\tau^2+\tau^{-n}\,\varepsilon^\delta)\,\Phi(x_0,R). \tag{2.26}$$

We prove that the ratio appearing at the right-hand side is bounded: using (2.25) and (2.18),

$$\begin{aligned}|\xi|^2&\le 2\big(|\xi-(Du)_{\tau R}|^2+|(Du_{\tau R}|^2\big)\\&\le 2\left(\fint_{B_{\tau R}}|Du-\xi|^2\,dx+|(Du)_{\tau R}|^2\right)\\&\le c\left(\tau^{-n}\fint_{B_{R/2}}|Du-\xi|^2\,dx+|(Du)_{\tau R}|^2\right)\\&\le c\left[\tau^{-n}\,\varepsilon\left(\mu^2+|\xi|^2\right)+|(Du)_{\tau R}|^2\right].\end{aligned}$$

If $\tau^{-n}\,\varepsilon<1/2c$ we obtain

$$|\xi|^2\le c\left(\mu^2+|(Du)_{\tau R}|^2\right),$$

therefore in (2.26) it is enough to choose $\varepsilon<\tau^{(n+2)/\delta}$ to conclude the proof. ∎

Proposition 2.8 and Lemma 2.10 are the only two estimates needed to prove

Proposition 2.11 . *There are two constants $c>0$ and $\sigma<1$, both independent of μ, such that*

$$\sup_{B_{R/2}}|Du|^p\le c\fint_{B_R}\left(\mu^p+|Du|^p\right)dx$$

$$\Phi(x_0,\varrho)\le c\left(\frac{\varrho}{R}\right)^\sigma\Phi(x_0,R)$$

for every $B_R\subset\subset\Omega$ and $\varrho<R$.

The proof is the same as Lemma 3.1 and Theorem 3.1 of [4]. To extend this result to the case $\mu=0$ we will approximate the function f.

Lemma 2.12 . *Let f satisfy* (H1), …,(H5) *with $\mu=0$, and for $0<\varepsilon<1$ set $g^\varepsilon(t^2)=g(\varepsilon^2+t^2)$. Then the function $f^\varepsilon(\xi)=g^\varepsilon\left(|\xi|^2\right)$ satisfies* (H1), …,(H5) *with $\mu=\varepsilon$, the same α and c_1 as f, and with c independent of ε.*

Proof . It is easy to derive from (H1), …,(H5) the properties of g:

$$c_1\,|t|^p\le g(t^2)\le c\,|t|^p; \tag{G1}$$

$$\begin{cases}\frac12|t|^{p-2}\le g'(t^2)\le c\,|t|^{p-2}\\ |g''(t^2)|\le c\,|t|^{p-4}\end{cases}\qquad\text{for all } t\ne 0; \tag{G2}$$

$$g'(t^2)+2g''(t^2)t^2\ge |t|^{p-2}/2\qquad\text{for all } t\ne 0; \tag{G3}$$

$$|g'(t^2)-g'(s^2)|+|g''(t^2)t^2-g''(s^2)s^2|\le c\,|t|^{p-2}\,|s|^{p-2}\,|t^2+s^2|^{(2-p-\alpha)/2}\,|t-s|^\alpha\qquad\text{for } t,s\ne 0. \tag{G4}$$

Then the properties (H1), …,(H4) of f^ε are immediately verified, and (H5) requires little effort. ∎

Proposition 2.13 . *The result of Proposition* 2.11 *holds also in the case* $\mu = 0$.

PROOF . Fix a ball $B \subset\subset \Omega$, and for every $\varepsilon \in (0,1)$ let u_ε be the (only) minimum point of

$$\int_B f^\varepsilon(Dv)\,dx$$

in the space $u + W_0^{1,p}(B)$. Then

$$\int_B |Du_\varepsilon|^p\,dx \le c\int_B f^\varepsilon(Du_\varepsilon)\,dx \le c\int_B f^\varepsilon(Du)\,dx \le c\int_B \big(1+|Du|^2\big)^{p/2}\,dx;$$

moreover by (2.8), if B_R is any ball contained in B,

$$\int_{B_{R/2}} |Du_\varepsilon|^p\,dx \le \frac{c}{R^p}\int_{B_R} \big(\varepsilon^2+|Du_\varepsilon|^2\big)^{p/2}\,dx \le \frac{c}{R^p}\int_B \big(1+|Du|^2\big)^{p/2}\,dx;$$

therefore, at least for a subsequence,

$$u_\varepsilon \to u_0 \text{ weakly in } W^{2,p}_{\rm loc}(B) \text{ and weakly in } u + W_0^{1,p}(B).$$

Since $Du_\varepsilon \to Du_0$ a.e., it is easy to check that u_0 is a minimum point of $\int_B f(Dv)\,dx$ in $u+W_0^{1,p}(B)$, so that $u_0 \equiv u$ because f is strictly convex due to (H4). By (2.6) we then have

$$\big(\varepsilon^2+|Du_\varepsilon|^2\big)^{(p-2)/4}\,Du_\varepsilon \to |Du|^{(p-2)/2}\,Du \qquad \text{weakly in } W^{1,2}_{\rm loc}(B),$$

so the result follows by letting $\varepsilon \to 0$ in Proposition 2.11. ∎

Remark 2.14 . *In the case* $\mu > 0$, *from* (2.7), (2.14) *we deduce that for every* $B_R \subset\subset \Omega$

$$\fint_{B_{R/2}} |D^2u|^2\,dx \le \frac{c}{R^2}\left(\fint_{B_R} H(Du)\,dx\right)^{2/p},$$

and the discussion above shows that this inequality holds also in the case $\mu = 0$, *thus implying* $u \in W^{2,2}_{\rm loc}$.

PROOF OF THEOREM 1.1 . Fix $B_R(x_0) \subset\subset \Omega$ and $y_0 \in B_{R/2}(x_0)$, then take $B_\varrho(y_0) \subset\subset B_{R/2}(x_0)$: from Propositions 2.11 and 2.13 we deduce

$$\Phi(y_0,\varrho) \le c\left(\frac{\varrho}{R}\right)^\sigma \Phi\Big(y_0,\frac{R}{2}\Big) \le c(R)\,\varrho^\sigma,$$

and also

$$\sup_{B_\varrho(y_0)} |Du|^p \le \sup_{B_{R/2}(x_0)} |Du|^p \le c(R).$$

Then

$$\big|\big(V(Du)\big)_{y_0,\varrho}\big| \le c(R),$$

so if ξ is such that $V(\xi) = \big(V(Du)\big)_{y_0,\varrho}$ we have

$$|\xi| \le c(R),$$

and by Lemma 2.2

$$\begin{aligned}\fint_{B_\varrho} |Du-(Du)_\varrho|^2\,dx &\le \fint_{B_\varrho} |Du-\xi|^2\,dx \le c\fint_{B_\varrho} |V(Du)-V(\xi)|^2\,\big(\mu^2+|Du|^2+|\xi|^2\big)^{(2-p)/2}\,dx \\ &\le c(R)\,\Phi(y_0,\varrho) \le c(R)\,\varrho^\sigma.\end{aligned}$$

This inequality allows us to apply the regularity theorem of Campanato (Theorem 1.3, section III of [3]), which concludes the proof. ∎

Proof of Theorem 1.2

Deriving Theorem 1.2 from the decay estimate for Φ given in Propositions 2.11 and 2.13 is almost routine, and we shall often refer to [3],[4], giving only the statements and some proofs which are different from the case $p \geq 2$. In this section we always assume that f satisfies (H6),(H7), and we adopt the definitions of H, V and Φ given in section 2; it is not restrictive to assume $\mu \leq 1$.

As its proof depends only on (H1), again we have a higher integrability result for H:

Lemma 3.1 . *Let $\mu \geq 0$. Then for every $B_R \subset\subset \Omega$*

$$\left(\fint_{B_{R/2}} H^q(Du)\, dx \right)^{1/q} \leq c \fint_{B_R} H(Du)\, dx,$$

with $q > 1$ and $c > 0$ both independent of μ, R.

If a function happens to be a global minimizer whose boundary value has some extra regularity, then the local result of Lemma 3.1 becomes global:

Remark 3.2 . *Assume f satisfies* (H1) *and B is a ball; if v is a minimizer of $\int f(Dw)\,dx$ in the class $u + W_0^{1,p}(B)$, with $u \in W^{1,p+\varepsilon}(B)$ for some $\varepsilon > 0$, then $H(Dv) \in L^q(B)$ for some $q > 1$, and*

$$\left(\fint_B H^q(Dv)\, dx \right)^{1/q} \leq c \left(\fint_B H^{(p+\varepsilon)/p}(Du)\, dx \right)^{p/(p+\varepsilon)}.$$

For the proof, see [3], page 152.

In order to use the estimates of section 2 we compare u with the solution of a problem independent of (x, u):

Lemma 3.3 . *There are two positive constants c, β, both independent of $\mu \geq 0$, such that if $B_R(x_0) \subset\subset \Omega$ and v is the minimum point of*

$$\int_{B_{R/2}} f\big(x_0, (u)_{x_0,R}, Dw\big)\, dx$$

in the space $u + W_0^{1,p}(B_{R/2})$, then

$$\fint_{B_{R/2}} |V(Du) - V(Dv)|^2\, dx \leq c\, K(|u_{x_0,R}|) \fint_{B_R} H(Du)\, dx \left(R^p \fint_{B_R} \big(1 + |Du|^p\big)\, dx \right)^{\beta}.$$

Proof . We may assume that the exponents q in Lemma 3.1 and Remark 3.2 are the same, and that $q\gamma > p(q-1)$, where γ appears in (H7). To deal simultaneously with the cases $\mu = 0$ and $\mu > 0$, set

$$g^0(t) = g(x_0, u_{x_0,R}, t)$$

and define for all $\varepsilon \geq 0$

$$f^\varepsilon(\xi) = g^0\big(\varepsilon^2 + |\xi|^2\big)$$

(compare Lemma 2.12). We may write

$$\begin{aligned} &\int_{B_{R/2}} [f^\varepsilon(Du) - f^\varepsilon(Dv)]\, dx \\ &\quad = \int_{B_{R/2}} f^\varepsilon_{\xi^i_\alpha}(Dv)(D_\alpha u^i - D_\alpha v^i)\, dx \\ &\qquad + \int_{B_{R/2}} \int_0^1 (1-s)\, f^\varepsilon_{\xi^i_\alpha \xi^j_\beta}\big(Dv + s(Du - Dv)\big)\, ds\, (D_\alpha u^i - D_\alpha v^i)(D_\beta u^j - D_\beta v^j)\, dx \\ &\quad = I_1^\varepsilon + I_2^\varepsilon; \end{aligned} \tag{3.1}$$

since (2.8) holds for f^ε, we have easily

$$\lim_\varepsilon I_1^\varepsilon = \int_{B_{R/2}} f^0_{\xi^i_\alpha}(Dv)(D_\alpha u^i - D_\alpha v^i)\,dx = 0$$

by the minimality of v, whereas (H4) and Lemmas 2.1 and 2.2 imply

$$I_2^\varepsilon \geq c \int_{B_{R/2}} \left|\left(\varepsilon^2+\mu^2+|Du|^2\right)^{(p-2)/4} Du - \left(\varepsilon^2+\mu^2+|Dv|^2\right)^{(p-2)/4} Dv\right|^2 dx,$$

and by Fatou's lemma

$$\liminf_\varepsilon I_2^\varepsilon \geq c \int_{B_{R/2}} |V(Du)-V(Dv)|^2\,dx;$$

letting $\varepsilon \to 0$ in (3.1) we have by (H1)

$$\fint_{B_{R/2}} [f^0(Du)-f^0(Dv)]\,dx \geq c \fint_{B_{R/2}} |V(Du)-V(Dv)|^2\,dx. \tag{3.2}$$

On the other hand, the left-hand side of (3.2) may be written

$$\begin{aligned} S_1+S_2+S_3 = &\fint_{B_{R/2}} [f(x_0,u_{x_0,R},Du)-f(x,u,Du)]\,dx \\ &+\fint_{B_{R/2}} [f(x,u,Du)-f(x,v,Dv)]\,dx \\ &+\fint_{B_{R/2}} [f(x,v,Dv)-f(x_0,u_{x_0,R},Dv)]\,dx. \end{aligned}$$

Here,

$$S_2 \leq 0 \tag{3.3}$$

by the minimality of u; by (H7) and Lemma 3.1

$$\begin{aligned} S_1 &\leq c\,K(|u_{x_0,R}|0 \fint_{B_{R/2}} H(Du)\left(\min\{L, R+|u-u_{x_0,R}|\}\right)^\gamma dx \\ &\leq c(L)\,K(|u_R|) \fint_{B_R} H(Du)\,dx \left(\fint_{B_R} \left(R^p+|u-u_R|^p\right)dx\right)^{(q-1)/q} \\ &\leq c\,K(|u_R|) \fint_{B_R} H(Du)\,dx \left(R^p \fint_{B_R} \left(1+|Du|^p\right)dx\right)^{(q-1)/q}. \end{aligned}$$

Analogously by (H7) and Remark 3.2

$$\begin{aligned} S_3 &\leq c\,K(|u_R|) \fint_{B_R} H(Du)\,dx \left(\fint_{B_{R/2}} \left(R^p+|v-u|^p+|u-u_R|^p\right)dx\right)^{(q-1)/q} \\ &\leq c\,K(|u_R|) \fint_{B_R} H(Du)\,dx \left(R^p \fint_{B_R} \left(1+|Du|^p\right)dx\right)^{(q-1)/q}, \end{aligned}$$

and the result follows by (3.2),(3.3),(3.4). ■

Proposition 3.4 . *There exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{0,\lambda}(\Omega_0)$ for every $\lambda < 1$, and Hausdorff measure $\mathcal{H}_{n-p-\varepsilon}(\Omega \setminus \Omega_0) = 0$ for some $\varepsilon > 0$.*

Proof . For every $B_\varrho(x_0) \subset\subset \Omega$ we set

$$\varphi(x_0, \varrho) = \varrho^p \fint_{B_\varrho(x_0)} H(Du)\, dx;$$

fix a particular $B_R(x_0)$, and let v be the function defined in the statement of Lemma 3.3. If $0 < \tau < 1/4$ we have

$$\varphi(x_0, \tau R) \le c\,(\tau R)^p \fint_{B_{\tau R}} \big(H(Dv) + |Du - Dv|^p\big)\, dx; \tag{3.5}$$

by Propositions 2.11 and 2.13

$$\fint_{B_{\tau R}} H(Dv)\, dx \le \sup_{B_{R/4}} H(Dv) \le c \fint_{B_{R/2}} H(Dv)\, dx \le c\, R^{-p}\, \varphi(x_0, R). \tag{3.6}$$

As for the second term in the integral in (3.5), by Lemmas 2.2 and 3.3

$$\begin{aligned}
\fint_{B_{\tau R}} |Du - Dv|^p\, dx &\le \tau^{-n} \fint_{B_{R/2}} |Du - Dv|^p\, dx \\
&\le c\tau^{-n} \fint_{B_{R/2}} \Big(|V(Du) - V(Dv)|\, \big(\mu^2 + |Du|^2 + |Dv|^2\big)^{(2-p)/2}\Big)^p\, dx \\
&\le c\,\tau^{-n} \bigg[K(|u_R|) \fint_{B_R} H(Du)\, dx \bigg(R^p \fint_{B_R} \big(1 + |Du|^p\big)\, dx\bigg)^{\beta}\bigg]^{p/2} \cdot \\
&\qquad \cdot \bigg(\fint_{B_{R/2}} \big(\mu^2 + |Du|^2 + |Dv|^2\big)^{p/2}\, dx\bigg)^{(2-p)/2} \\
&\le c\,\tau^{-n}\, \big(K(|u_R|)\big)^{p/2} \fint_{B_R} H(Du)\, dx \bigg(R^p \fint_{B_R} \big(1 + |Du|^p\big)\, dx\bigg)^{p\beta/2}.
\end{aligned}$$

By (3.5),(3.6) it then follows

$$\varphi(x_0, \tau R) \le c\,\tau^p\, \varphi(x_0, R)\, \Big(1 + \tau^{-n}\, \big(K(|u_R|)\big)^{p/2} \big[R^p + \varphi(x_0, R)\big]^{p\beta/2}\Big).$$

The result follows from this inequality as in [3], pp.170–174. ∎

Remark 3.5 . *As in the case $p \ge 2$, one may prove that*

$$\Omega \setminus \Omega_0 \subset \{x\colon \sup_R |u_{x,R}| = +\infty\} \cup \{x\colon \liminf_{R \to 0} R^p \fint_{B_R(x)} |Du|^p\, dy > 0\};$$

in addition, for every M there are ε_0, R_0 such that

$$\Omega_0 \supset \{x\colon \sup_{R < R_0} |u_{x,R}| \le M\} \cap \{x\colon \inf_{R < R_0} R^p \fint_{B_R(x)} |Du|^p\, dy \le \varepsilon_0\}.$$

Proof of Theorem 1.2 . See the proof of Theorem 4.3 in [4]. ∎

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