REGULARITY RESULTS FOR EQUILIBRIA IN A VARIATIONAL MODEL FOR FRACTURE

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Abstract

In recent years models describing interactions between fracture and damage have been proposed in which the relaxed energy of the material is given by a functional involving bulk and interfacial terms, of the form

$$\mathcal{G}(K,u) := \int_{\Omega\setminus K} F(\nabla u)\,dx + \lambda \int_{\Omega\setminus K} |u-g|^q\,dx + \beta H^{N-1}(K\cap\Omega)\ ,$$

where Ω is an open, bounded subset of $\mathbb{R}^N$, $q \geq 1$, $g \in L^\infty(\Omega;\mathbb{R}^N)$, $\lambda, \beta > 0$, the bulk energy density F is quasiconvex, $K \subset \mathbb{R}^N$ is closed, and the admissible deformation $u : \Omega \to \mathbb{R}^N$ is C^1 in $\Omega \setminus K$. One of the main issues has to do with regularity properties of the "crack site" K for a minimizing pair (K,u). In the scalar case, i.e. when $u : \Omega \to \mathbb{R}$, similar models were adopted to image segmentation problems, and the regularity of the "edge" set K has been successfully resolved for a quite broad class of convex functions F with growth $p > 1$ at infinity. In turn, this regularity entails the existence of classical solutions. The methods thus used cannot be carried out to the vector-valued case, except for a very restrictive class of integrands. In this paper regularity on the plane is obtained for minimizers of $\mathcal{G}$ corresponding to bulk energy densities of the form

$$F(\xi) = \frac{1}{2}|\xi|^2 + h(\det \xi),$$

where the convex function h grows linearly at infinity.

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1. Introduction

In recent years models involving bulk and interfacial energies have been used in the contexts of fracture mechanics, phase transitions, and image segmentation in computer vision (see [BZ], [DGCL], [FFr], [MS]). The underlying quasistatic problems deal with minimization of an energy functional of the form

$$\mathcal{G}(K,u) := \int_{\Omega\setminus K} F(\nabla u)\,dx + \lambda \int_{\Omega\setminus K} |u-g|^q\,dx + \beta H^{N-1}(K\cap\Omega)$$

where Ω is an open, bounded subset of $\mathbb{R}^N$, $q>1$, $g\in L^\infty(\Omega;\mathbb{R}^k)$, $\lambda,\beta>0$, and H^{N-1} stands for the $(N-1)$-dimensional Hausdorff measure, among all pairs (K,u) with K closed in $\mathbb{R}^N$ and $u:\Omega\to\mathbb{R}^k$ is smooth; the first two terms in $\mathcal{G}$ represent the bulk energy and the last one accounts for interfacial energy. In order to attack this problem De Giorgi and Ambrosio (see [A1], [DGA]) introduced the space SBV of functions of *special bounded variation*, i.e. BV functions u whose distributional derivative Du, which is a finite, Radon measure, may be decomposed into an absolutely continuous part $\nabla u\,\mathcal{L}^N$ with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$, and a singular part whose support is an $(N-1)$-dimensional rectifiable set S_u. This is the "jump" set of u, in the sense that u has traces $u^+\neq u^-$ H^{N-1}-a.e. on the two sides of S_u (for example, the Cantor-Vitali function is BV but not SBV). Then to the functional $\mathcal{G}$ one may associate the functional $\mathcal{F}$ defined on SBV as

$$\mathcal{F}(u) := \int_\Omega F(\nabla u)\,dx + \lambda\int_\Omega |u-g|^q\,dx + \beta H^{N-1}(S_u\cap\Omega)\,.$$

Due to the relaxation result of Fonseca and Francfort (see [FFr]), and under suitable growth conditions for the bulk energy density F, it is not restrictive to take F quasiconvex. We recall that F is said to be *quasiconvex* if

$$F(\xi) \le \int_Q F(\xi+\nabla\varphi(x))\,dx$$

for all $k\times N$ matrix ξ, $\varphi\in C_0^\infty(Q;\mathbb{R}^k)$, and where $Q=(0,1)^N$. A particular class of quasiconvex functions which plays an important role in elasticity is the class of *polyconvex* functions, i.e. convex functions of all minors of the matrix ξ.

Under the quasiconvexity assumption, and if F is superlinear and g is bounded, the lower semicontinuity and compactness results of Ambrosio (see [A1], [A2], [A3]) yield the existence of a SBV minimizer of $\mathcal{F}$. Note that, in general, if $u\in SBV$ then the set S_u is far from being closed, i.e. $H^{N-1}\big((\overline{S_u}\setminus S_u)\cap\Omega\big)>0$.

In the scalar-valued case where $u:\Omega\to\mathbb{R}$, quasiconvexity reduces to convexity and De Giorgi, Carriero and Leaci [DGCL] proved that if

$$F(\xi)=|\xi|^2$$

and if g is bounded, then

$$H^{N-1}\big((\overline{S_u} \setminus S_u) \cap \Omega\big) = 0$$

for any minimizer $u \in SBV(\Omega;\mathbb{R})$ of $\mathcal{F}$. From this property one immediately concludes that the pair $(\overline{S_u}, u)$ is a minimizer of the original functional $\mathcal{G}$. The result of [DGCL] has been extended to a quite broad class of convex functions F with growth $p > 1$ at infinity, and further regularity on the set S_u has been obtained (see [AFP], [AP], [B], [DG], [DS], [FFu]).

In this paper we deal with a two-dimensional, vector-valued, polyconvex case, considering a class of functionals $\mathcal{G}$ which includes in particular the model

$$\int_{\Omega\setminus K} \Big[\frac{1}{2}\,|\nabla u|^2 + |\det \nabla u|\Big]\,dx + \lambda \int_{\Omega\setminus K} |u-g|^q\,dx + \beta H^1(K\cap\Omega)\ ,$$

where $u \in W^{1,2}(\Omega\setminus K;\mathbb{R}^2)$ and g is bounded. In Theorem 2.1 we prove that this functional admits a minimizing pair of the form $(\overline{S_u}, u)$, where $u \in SBV(\Omega;\mathbb{R}^2)$ and, moreover, $u \in C^{0,\alpha}(\Omega \setminus \overline{S_u};\mathbb{R}^2)$ for every $0 < \alpha < 1$.

2. Statements and Auxiliary Results

If $\Omega \subset \mathbb{R}^N$ is open, we say that a function of *bounded variation* $u \in BV(\Omega;\mathbb{R}^k)$ is a function of *special bounded variation*, $u \in SBV(\Omega;\mathbb{R}^k)$, if, denoting by S_u the complement of the set of Lebesgue points of u, the distributional derivative Du is represented by

$$Du = \nabla u\,\mathcal{L}^N + (u^+ - u^-)\otimes \nu\, H^{N-1}\lfloor S_u\ ,$$

where ∇u is the Radon-Nikodym derivative of the finite, Radon measure Du with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$, ν is the normal to the rectifiable set S_u, u^+ and u^- are the traces of u on S_u, and H^{N-1} denotes the $(N-1)$-dimensional Hausdorff measure.

We recall that the *recession function* h^∞ of a convex function $h : \mathbb{R} \to [0,+\infty)$ is defined by

$$h^\infty(t) := \lim_{s\to+\infty} \frac{h(st)}{s}\ ,$$

and it is convex and positively homogeneous of degree one.

In this paper we prove the following theorem.

Theorem 2.1. *Let Ω be a bounded, open subset of $\mathbb{R}^2$. Let $h : \mathbb{R} \to [0,\infty)$ be a convex function such that $h(0) \le h(t) \le C(1+|t|)$, and*

$$\left| h^\infty(t) - \frac{h(st)}{s}\right| \le \frac{C}{s^m}$$

for some $0 < m < 1$, and for all $t \in \mathbb{R}$, $s \ge s_0 > 0$. Let $\lambda, \beta > 0$ and define

$$\mathcal{G}(K,u) := \int_{\Omega\setminus K} \Big[\frac{1}{2}|\nabla u|^2 + h(\det \nabla u)\Big]\,dx + \lambda \int_{\Omega\setminus K} |u-g|^q\,dx + \beta H^1(\Omega\cap K)\ .$$

There exists a minimizer of $\mathcal{G}(\cdot,\cdot)$ of the form $(\overline{S_u}, u)$, with $u \in SBV(\Omega, \mathbb{R}^2)$, among all pairs (K, v) with $K \subset \Omega$ closed and $v \in W^{1,2}(\Omega \setminus K; \mathbb{R}^2)$. Moreover,

$$H^1((\overline{S_u} \setminus S_u) \cap \Omega) = 0 \, .$$

In order to prove Theorem 2.1, we invite the reader to follow the same steps taken in the proof of Theorem 4.1 of [FFu]. It is easy to verify that all the auxiliary lemmas hold in this case, with one exception: the Density Lower Bound ([FFu], Lemma 4.3) uses the Decay Lemma ([FFu], Lemma 4.4), which was proved by Carriero and Leaci [CL] only in the p-harmonic case, i.e. $F(\xi) = |\xi|^p$. Knowing that local minimizers are locally Lipschitz functions played a fundamental role in the analysis.

We introduce the notation

$$\mathcal{F}_0(u, c, A) := \int_A \Big[\frac{1}{2}|\nabla u|^2 + h^\infty(\det \nabla u)\Big]\, dx + cH^1(S_u \cap \overline{A}) \, ,$$

$$\Phi_0(u, c, A) := \inf\{\mathcal{F}_0(w, c, A) : w \in SBV(\Omega; \mathbb{R}^2), w = u \text{ in } \Omega \setminus \overline{A}\} \, ,$$

$$\Psi_0(u, c, A) := \mathcal{F}_0(u, c, A) - \Phi_0(u, c, A) \, ,$$

where A is an open subset of Ω, $c > 0$, $u \in SBV(\Omega; \mathbb{R}^2)$. In the case where $u \in W^{1,2}(A; \mathbb{R}^2)$ we simply write

$$\mathcal{F}_0(u, A) := \int_A \Big[\frac{1}{2}|\nabla u|^2 + h^\infty(\det \nabla u)\Big]\, dx \, .$$

In the sequel, we use the notation B_R, $R > 0$, to denote a generic open ball of radius R, centered at $x \in \Omega$, such that $B_R \subset \Omega$.

Definition. *We say that $u \in W^{1,2}(\Omega)$ is a $W^{1,2}$-*local minimizer *of*

$$I(v; \Omega) := \int_\Omega F(\nabla v)\, dx \, , \qquad v \in W^{1,2}(\Omega)$$

if

$$I(u; B_R(x_0)) = \min\Big\{ I(v; B_R(x_0)) \ : \ v \in u + W_0^{1,2}(B_R(x_0)) \Big\}$$

for all balls $B_R(x_0) \subset \Omega$.

We now state the version of the decay lemma which holds in our case. The density lower bound, and thus Theorem 2.1, follow from the decay lemma by the same argument used in [FFu].

Lemma 2.2. [Decay Lemma] *For all $\gamma \in (0, 1)$ there exists $\tau_\gamma \in (0, 1)$ such that for every $\tau \in (0, \tau_\gamma)$ and for every $c > 0$ there exist $\varepsilon = \varepsilon(c, \tau, \gamma)$, $\theta = \theta(c, \tau, \gamma)$, $R_0 = R_0(c, \tau, \gamma)$, such that if $0 < \rho < R_0$, and if $u \in SBV(\Omega; \mathbb{R}^2)$ is such that $\mathcal{F}_0(u, c, B_\rho) \le \varepsilon^2 \rho$ and $\Psi_0(u, c, B_\rho) \le \theta \mathcal{F}_0(u, c, B_\rho)$, then*

$$\mathcal{F}_0(u, c, B_{\tau\rho}) \le \tau^{2-\gamma} \mathcal{F}_0(u, c, B_\rho) \, .$$

In order to prove Lemma 2.2, we follow the proof of Lemma 3.12 in [FFu], and we suppose that the result is not true; then there exist $\gamma \in (0,1)$, for every $\tau_\gamma \in (0,1)$ there exists $\tau \in (0,\tau_\gamma)$, there exist $c > 0$, two sequences $\{\varepsilon_h\}$, $\{\theta_h\}$, with $\lim_h \varepsilon_h = \lim_h \theta_h = 0$, a sequence $\{u_h\} \subset SBV(\Omega;\mathbb{R}^2)$, and a sequence of balls $B_{R_h}(x_h) \subset\subset \Omega$ with $\lim_h R_h = 0$ such that

$$\mathcal{F}_0\big(u_h, c, B_{R_h}(x_h)\big) = \varepsilon_h R_h \ , \quad \Psi_0\big(u_h, c, B_{R_h}(x_h)\big) = \theta_h \mathcal{F}_0\big(u_h, c, B_{R_h}(x_h)\big) \ ,$$

and

$$\mathcal{F}_0\big(u_h, c, B_{\tau R_h}(x_h)\big) > \tau^{2-\gamma} \mathcal{F}_0\big(u_h, c, B_{R_h}(x_h)\big) \ .$$

After rescaling, it is easily seen that the rest of the proof can be carried out in a similar way to Lemma 3.12 in [FFu], provided an estimate of the type

$$\int_{B_\tau} \Big[\frac{1}{2}|\nabla u|^2 + h^\infty(\det \nabla u)\Big]\, dx \le C\tau^{2-\alpha} \int_{B_1} \Big[\frac{1}{2}|\nabla u|^2 + h^\infty(\det \nabla u)\Big]\, dx$$

holds for local minimizers of $\mathcal{F}_0$ in $W^{1,2}(B_1;\mathbb{R}^2)$, for any $0 < \alpha < 1$ or, equivalently,

Theorem 2.3. *If $u \in W^{1,2}(B_1, \mathbb{R}^2)$ is a $W^{1,2}$-local minimizer of $\mathcal{F}_0$ then*

$$\int_{B_\tau} |\nabla u|^2\, dx \le C\tau^{2-\alpha} \int_{B_1} |\nabla u|^2\, dx \tag{2.1}$$

for all $\alpha, \tau \in (0,1)$ and some constant $C > 0$, $C \equiv C(\alpha, h^\infty)$.

This is, therefore, the only result still needed to prove Theorem 2.1. As an immediate consequence of (2.1), it follows that $u \in C^{0,\alpha}$ for all $\alpha \in (0,1)$ (see [G], Theorem 1.1, Chapter 3).

The rest of the paper will be dedicated to proving Theorem 2.3. Although we will rely strongly on the arguments used by Dougherty [D] to obtain higher integrability of local minimizers in the case where $h \in C^1(\mathbb{R})$, we could not find an easier way to adapt directly his proof to more general Lipschitz functions h by means of a simple approximation and density approach.

The following lemma may be found in [G] (Chap. 3, Lemma 2.1) in the case where with $\gamma = \beta$; although stated in a slightly weaker form, its proof yields the result below.

Lemma 2.4. *Let $\phi : [0,+\infty) \to [0,+\infty)$ be a nonnegative, nondecreasing function, such that*

$$\phi(\rho) \le H\Big(\frac{\rho}{R}\Big)^\alpha \phi(R) + KR^\beta$$

for all $0 < \rho < R \le R_0$ and for some constants $H, K \ge 0$ and $0 < \beta < \alpha$. Then for every $\gamma \in [\beta, \alpha)$ there exists a constant $C = C(H, \alpha, \beta, \gamma)$ such that

$$\phi(\rho) \le C\Big[\Big(\frac{\rho}{R}\Big)^\gamma \phi(R) + K\rho^\beta\Big]$$

for all $0 < \rho < R \le R_0$.

Next, we prove a decay estimate for solutions of a well known elliptic equation.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain. If $f \in L^p(\Omega;\mathbb{R}^N)$, $p \geq 2$, and if $v \in W^{1,2}(\Omega)$ satisfies*

$$\Delta v = \operatorname{div} f \quad \text{in } \Omega$$

then for any $\delta \in (0, 2N/p)$ there exists a constant $C = C(p,N,\delta)$ such that if $B_R \subset \Omega$ and $0 < \rho < R$ then

$$\int_{B_\rho} |Dv|^2\,dx \leq C\Big[\Big(\frac{\rho}{R}\Big)^{N-\delta}\int_{B_R}|Dv|^2\,dx + \rho^{N-2N/p}\Big(\int_{B_R}|f|^p\,dx\Big)^{2/p}\Big]. \tag{2.2}$$

Proof. Fix a ball $B_R \subset \Omega$, and let w be the harmonic function which minimizes

$$z \mapsto \int_{B_R} |\nabla z|^2\,dx\,, \quad z \in v + W_0^{1,2}(B_R)\,.$$

Since $|Dw|^2$ is subharmonic we get

$$\int_{B_\rho} |Dw|^2\,dx \leq \Big(\frac{\rho}{R}\Big)^N \int_{B_R}|Dw|^2\,dx \tag{2.3}$$

for all $\rho < R$.

We claim that if $\rho < R$ then

$$\int_{B_\rho} |Dv|^2\,dx \leq 2\Big(\frac{\rho}{R}\Big)^N\int_{B_R}|Dv|^2\,dx + C\,R^{N-2N/p}\Big(\int_{B_R}|f|^p\,dx\Big)^{2/p}. \tag{2.4}$$

If (2.4) holds, then (2.2) follows from Lemma 2.4, setting

$$\phi(\rho) := \int_{B_\rho}|\nabla v|^2\,dx\,, \quad \alpha := N\,, \quad \beta := N - \frac{2N}{p}\,, \quad \gamma := N - \delta\,.$$

To prove the claim, we start by noting that

$$\begin{aligned}&\int_{B_R} Dv\cdot(Dv - Dw)\,dx = \int_{B_R} f\cdot(Dv-Dw)\,dx\,,\\ &\int_{B_R} Dw\cdot(Dv-Dw)\,dx = 0\,,\end{aligned}$$

so that

$$\int_{B_R}|Dv-Dw|^2\,dx = \int_{B_R} f\cdot(Dv-Dw)\,dx\,.$$

Using Cauchy-Schwartz, Young's, and Hölder's inequality, we obtain

$$\int_{B_R}|Dv-Dw|^2\,dx \leq \int_{B_R}|f|^2\,dx \leq C\,R^{N-2N/p}\Big(\int_{B_R}|f|^p\,dx\Big)^{2/p},$$

which, together with (2.3), yields

$$\int_{B_\rho} |Dv|^2\,dx \le 2\Big(\int_{B_\rho} |Dw|^2\,dx + \int_{B_R} |Dv-Dw|^2\,dx\Big)$$
$$\le 2\Big(\frac{\rho}{R}\Big)^N \int_{B_R} |Dw|^2\,dx + C\,R^{N-2N/p}\Big(\int_{B_R} |f|^p\,dx\Big)^{2/p}.$$

This inequality reduces to (2.4) since, by definition of w,

$$\int_{B_R} |Dw|^2\,dx \le \int_{B_R} |Dv|^2\,dx\,.$$

□

3. The Main Theorem

The main result of this section is the following.

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^2$, and let $u \in W^{1,2}(\Omega,\mathbb{R}^2)$ be a $W^{1,2}$-local minimizer of*

$$\mathbb{F}(v;\Omega) := \int_\Omega \Big[\frac{1}{2}|Dv|^2 + M|\det Dv|\Big]\,dx$$

with $M > 0$. If $\alpha \in (0,1)$, $B_R \subset \Omega$, and if $\rho < R$, then

$$\int_{B_\rho} |Du|^2\,dx \le C\Big(\frac{\rho}{R}\Big)^{2\alpha} \int_{B_R} |Du|^2\,dx$$

for some constant C. In particular, $u \in C^{0,\alpha}(\Omega;\mathbb{R}^2)$.

Before proving this result, we show how Theorem 2.3 can be derived as a simple corollary.

Proof of Theorem 2.3. Let u be a $W^{1,2}$-local minimizer of

$$\mathcal{F}_0(v,A) := \int_A \Big[\frac{1}{2}|\nabla v|^2 + h^\infty(\det\nabla v)\Big]\,dx\,, \quad v \in W^{1,2}(\Omega;\mathbb{R}^2)\,,$$

where the recession function is given by

$$h^\infty(t) = \begin{cases} at & \text{if } t > 0 \\ -bt & \text{if } t < 0 \end{cases}$$

for some $a, b \ge 0$. Since $v \mapsto \det\nabla v$ is a null-lagrangian, i.e.

$$\int_A \det\nabla v\,dx = \int_A \det\nabla w\,dx$$

whenever $v, w \in W^{1,2}(\Omega;\mathbb{R}^2)$, $A \subset \Omega$ has Lipschitz boundary and $v = w$ on ∂A, we conclude that u is a $W^{1,2}$-local minimizer of

$$\begin{aligned}\mathcal{F}_1(v;A) &= \int_A \Big[\frac{1}{2}|\nabla v|^2 + a(\det\nabla v)^+ + b(\det\nabla v)^- + \frac{b-a}{2}\det\nabla v\Big]\,dx \\ &= \int_A \Big[\frac{1}{2}|\nabla v|^2 + \frac{a+b}{2}|\det\nabla v|\Big]\,dx\,.\end{aligned}$$

The result now follows from Theorem 3.1. □

Next we recall some algebraic inequalities used by Dougherty [D], and which will enter in the proof of Theorem 3.1. For completeness, we include the proofs.

Consider a 2×2 matrix

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\det D = ad - bc\,,\ \operatorname{adj} D = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix},\ \operatorname{tr}(D^T D) = |D|^2 = a^2+b^2+c^2+d^2\,,$$

where $\operatorname{adj} D^T D = (\det D)^2\mathbf{I}$, and we define

$$A := \frac{1}{2}(a^2+c^2-b^2-d^2)\,,\ B := ab+cd\,,\ \sigma := \operatorname{sign}(\det D) \text{ whenever } \det D \neq 0.$$

Lemma 3.2. *The following inequalities hold:*

(i) $|A| + |B| \le |D|^2$;

(ii) $|D|^2 \le 2(|A|+|B|)$ *if* $\det D = 0$;

(iii) $|\sigma \operatorname{adj} D - D| \le 4\sqrt{|A|+|B|}$ *if* $\det D \neq 0$.

Proof. It is clear that

$$\begin{aligned}|A| + |B| &\le \frac{1}{2}(|a|+|b|)^2 + \frac{1}{2}(|c|+|d|)^2 \\ &\le a^2+b^2+c^2+d^2\,,\end{aligned}$$

proving (i). Also, consider the right stretching tensor $U := \sqrt{D^T D}$: then its eigenvalues $\nu_1 \ge \nu_2 \ge 0$ are the principal stretches of D, and it can be seen easily that

$$|\det D| = \nu_1\nu_2\,,\quad |D|^2 = \nu_1^2 + \nu_2^2\,.$$

By virtue of the polar decomposition, we may find rotations $R, Q \in SO(2)$ such that

$$D = RU\,,\quad U = Q^T\begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} Q\,,$$

and so

$$\begin{aligned}\begin{pmatrix} A & B \\ B & -A \end{pmatrix} &= D^T D - \frac{1}{2}|D|^2\mathbf{I} \\ &= \frac{1}{2} Q^T \begin{pmatrix} \nu_1^2 - \nu_2^2 & 0 \\ 0 & \nu_2^2 - \nu_1^2 \end{pmatrix} Q \,.\end{aligned}$$

Hence

$$\nu_1^2 - \nu_2^2 = 2\sqrt{|A|^2 + |B|^2}\,. \tag{3.1}$$

If $\det D = 0$ then $\nu_2 = 0$, and

$$|D|^2 = \nu_1^2 = 2\sqrt{|A|^2 + |B|^2} \leq 2(|A| + |B|)\,,$$

and we conclude (ii). Now consider the matrix

$$\tilde{D} = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} :$$

with obvious notation we have

$$|A| = |\tilde{A}|\,,\ |B| = |\tilde{B}|\,,\ |D| = |\tilde{D}|\,,\ |\sigma \operatorname{adj} D - D| = |\tilde{\sigma} \operatorname{adj} \tilde{D} - \tilde{D}|$$

but $\tilde{\sigma} = -\sigma$; thus to prove (iii) we may confine ourselves to the case $\sigma = +1$, eventually replacing D by $\tilde{D}$. If $\det D > 0$ then

$$\begin{aligned} |\operatorname{adj} D - D| &= \left| \begin{pmatrix} d-a & -c-b \\ -b-c & a-d \end{pmatrix} \right| \\ &= \sqrt{2|a-d|^2 + 2|c+b|^2} \\ &\leq 2(\nu_1 - \nu_2)\,, \end{aligned} \tag{3.2}$$

because (as a direct computation shows)

$$\begin{aligned} |a-d| &= \left| \operatorname{tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} D \right] \right| \\ &= \operatorname{tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} RQ^T \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} Q \right] \Bigg| \\ &\leq \nu_1 - \nu_2\,, \end{aligned}$$

$$\begin{aligned} |b+c| &= |D_{12} + D_{21}| \\ &= \left| \left[RQ^T \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} Q \right]_{12} + \left[RQ^T \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} Q \right]_{21} \right| \\ &\leq \nu_1 - \nu_2\,. \end{aligned}$$

Finally, (3.1), (3.2), and (i) yield

$$\begin{aligned} |\operatorname{adj} D - D| &\leq 2(\nu_1 - \nu_2) = 4\frac{\sqrt{|A|^2 + |B|^2}}{\nu_1 + \nu_2} \\ &\leq 4\frac{\sqrt{|A|^2 + |B|^2}}{|D|} \leq 4\frac{\sqrt{|A|^2 + |B|^2}}{\sqrt{|A| + |B|}} \leq 4\sqrt{|A| + |B|}\,. \end{aligned}$$

□

To avoid overburdening the reader with indices, we change the notation just for the length of the proof of Theorem 3.1, conforming to the one employed in [D]: we denote by $\mathcal{U}$ the local minimizer of $\mathbb{F}$, replacing the former u. Then we are free to denote by (u, v) the components of $\mathcal{U}$, whereas the variables are $X = (x, y)$. We also use $\Phi = (\phi, \psi)$ to designate any smooth function $\Phi \in C_0^1(\Omega; \mathbb{R}^2)$ with support in a ball $B_R \subset \Omega$.

Proof of Theorem 3.1. *Step 1.* Following the argument of Dougherty [D], we consider a variation of the domain for the local minimizer $\mathcal{U}$, i.e. we study the variation $\mathcal{U}(I + \varepsilon\Phi)$ near $\varepsilon = 0$, obtaining a first set of Euler-Lagrange equations.

Let $B_R \subset\subset \Omega$, fix $\Phi \in C_0^1(B_R, \mathbb{R}^2)$, and $\varepsilon > 0$ small. Setting $\mathcal{U}_\varepsilon(Y) := \mathcal{U}(Y + \varepsilon\Phi(Y))$, due to the minimality of $\mathcal{U}$ we have

$$\mathbb{F}(\mathcal{U}_\varepsilon; B_R) - \mathbb{F}(\mathcal{U}; B_R) \geq 0 ,$$

and since a simple change of variables yields

$$\int_{B_R} |\det D\mathcal{U}_\varepsilon| \, dY = \int_{B_R} |\det D\mathcal{U}| \, dX ,$$

we deduce that

$$\int_{B_R} \left(|D\mathcal{U}_\varepsilon|^2 - |D\mathcal{U}|^2\right) dY \geq 0 ,$$

and so

$$\int_{B_R} A(\phi_x - \psi_y) + B(\phi_y + \psi_x) \, dX = 0 , \tag{3.3}$$

where, using the notations introduced above,

$$D = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} ,$$

$$\begin{aligned} A &:= \frac{1}{2}(u_x^2 + v_x^2 - u_y^2 - v_y^2) \\ B &:= u_x u_y + v_x v_y \ . \end{aligned}$$

Hence $A, B \in L^1(\Omega)$, and (3.3) can be written, in the sense of distributions, as

$$\begin{cases} A_x + B_y = 0 \\ B_x - A_y = 0 , \end{cases}$$

and so A, B are harmonic in B_R. In particular,

$$\sup_{B_{R/2}} (|A| + |B|) \leq \frac{4}{\pi R^2} \int_{B_R} (|A| + |B|) \, dX . \tag{3.4}$$

Step 2 . For $t \geq 0$ define

$$\begin{aligned} \Omega_t^+ &:= \{X \in \Omega : \det D\mathcal{U} > t\} \, , \\ \Omega_t^- &:= \{X \in \Omega : \det D\mathcal{U} < -t\} \, , \\ \Omega_t &:= \{X \in \Omega : |\det D\mathcal{U}| \leq t\} \, . \end{aligned}$$

Let $\sigma(X)$ denote the sign of $\det D\mathcal{U}(X)$ whenever the determinant is not equal to zero. Cleary, if $t > 0$ and ε is sufficiently small then $X \in \Omega_t^+$ implies that $\det(D\mathcal{U}+\varepsilon\Phi)(X) > 0$, while $\det(D\mathcal{U}+\varepsilon\Phi)(X) < 0$ whenever $X \in \Omega_t^-$. Therefore

$$\int_\Omega \frac{1}{2}\frac{|D(\mathcal{U}+\varepsilon\Phi)|^2 - |D\mathcal{U}|^2}{\varepsilon}\,dx + M\int_{\Omega_t^+}\frac{\det D(\mathcal{U}+\varepsilon\Phi) - \det D\mathcal{U}}{\varepsilon}\,dx$$
$$-M\int_{\Omega_t^-}\frac{\det D(\mathcal{U}+\varepsilon\Phi) - \det D\mathcal{U}}{\varepsilon}dx + M\int_{\Omega_t}\frac{|\det D(\mathcal{U}+\varepsilon\Phi)| - |\det D\mathcal{U}|}{\varepsilon}dx \geq 0 \, .$$

Since

$$\det D(\mathcal{U}+\varepsilon\Phi) = \det D\mathcal{U} + \varepsilon \operatorname{adj} D\mathcal{U}\cdot D\Phi + \varepsilon^2 \det D\Phi \, ,$$

and

$$|\operatorname{adj} D\mathcal{U}\cdot D\Phi| \geq \frac{|\det D\mathcal{U} + \varepsilon \operatorname{adj} D\mathcal{U}\cdot D\Phi| - |\det D\mathcal{U}|}{\varepsilon} \, ,$$

we obtain, letting $\varepsilon \to 0$,

$$\int_\Omega D\mathcal{U}\cdot D\Phi\,dX + M\int_{\Omega\setminus\Omega_t}\sigma \operatorname{adj} D\mathcal{U}\cdot D\Phi\,dX + M\int_{\Omega_t}|\operatorname{adj} D\mathcal{U}\cdot D\Phi|\,dX \geq 0 \, .$$

Replacing Φ by $-\Phi$, this inequality reduces to

$$\int_\Omega D\mathcal{U}\cdot D\Phi\,dX + M\int_{\Omega\setminus\Omega_t}\sigma \operatorname{adj} D\mathcal{U}\cdot D\Phi\,dX - M\int_{\Omega_t}|\operatorname{adj} D\mathcal{U}\cdot D\Phi|\,dX \leq 0 \, ,$$

or, equivalently,

$$\begin{aligned} (M+1)\int_\Omega D\mathcal{U}\cdot D\Phi\,dX &+ M\int_{\Omega\setminus\Omega_t}(\sigma \operatorname{adj} D\mathcal{U}\cdot D\Phi - D\mathcal{U}\cdot D\Phi)\,dX \\ &\leq M\int_{\Omega_t}(|\operatorname{adj} D\mathcal{U}\cdot D\Phi| + D\mathcal{U}\cdot D\Phi)\,dX \, . \end{aligned}$$

Letting $t \to 0$, and setting $M' = 1 + 1/M$, we conclude that

$$\begin{aligned} M'\int_\Omega D\mathcal{U}\cdot D\Phi\,dX &+ \int_{\Omega\setminus\Omega_0}(\sigma \operatorname{adj} D\mathcal{U}\cdot D\Phi - D\mathcal{U}\cdot D\Phi)\,dX \\ &\leq \int_{\Omega_0}(|\operatorname{adj} D\mathcal{U}\cdot D\Phi| + D\mathcal{U}\cdot D\Phi)\,dX \, . \end{aligned} \tag{3.5}$$

Step 3 . We compare $\mathcal{U}$ with the solution of a more regular problem: fix $B_R \subset \Omega$, and let $\mathcal{V} \in W^{1,2}(B_{R/2};\mathbb{R}^2)$ be the solution of

$$\begin{cases} M'\Delta\mathcal{V} = -\mathrm{div}\left[\chi_{\Omega\setminus\Omega_0}(\sigma\ \mathrm{adj}\, D\mathcal{U} - D\mathcal{U})\right] & \text{in } B_{R/2} \\ \mathcal{V} = \mathcal{U} & \text{on } \partial B_{R/2}\ . \end{cases} \tag{3.6}$$

By Lemma 3.2 (iii) it follows that the right hand side of the equation is the divergence of an L^2 function, and so we have, in weak form,

$$\int_{B_{R/2}} M' D\mathcal{V}\cdot D\Phi\, dX + \int_{B_{R/2}\setminus\Omega_0} (\sigma\ \mathrm{adj}\, D\mathcal{U} - D\mathcal{U})\cdot D\Phi\, dX = 0$$

for all $\Phi \in W_0^{1,2}(B_{R/2};\mathbb{R}^2)$. Subtracting this equation from (3.5), and choosing $\Phi = \mathcal{U} - \mathcal{V}$, we obtain

$$\int_{B_{R/2}} M'|D\mathcal{U} - D\mathcal{V}|^2\, dX \le \int_{B_{R/2}\cap\Omega_0} (|\,\mathrm{adj}\, D\mathcal{U}| + |D\mathcal{U}|)|D\mathcal{U} - D\mathcal{V}|\, dX\ ,$$

and noticing that $|\,\mathrm{adj}\, D\mathcal{U}| = |D\mathcal{U}|$, by virtue of Cauchy-Schwartz inequality and Lemma 3.2 (ii), we have

$$\begin{aligned} \int_{B_{R/2}} |D\mathcal{U} - D\mathcal{V}|^2\, dX &\le 4\int_{B_{R/2}\cap\Omega_0} |D\mathcal{U}|^2\, dX \\ &\le 8\int_{B_{R/2}\cap\Omega_0} (|A| + |B|)\, dX\ . \end{aligned} \tag{3.7}$$

Step 4 . Fix $B_R \subset \Omega$. Since A and B are harmonic functions, by Lemma 3.2 (iii) we have $\chi_{\Omega\setminus\Omega_0}(\sigma\ \mathrm{adj}\, D\mathcal{U} - D\mathcal{U}) \in L^p_{\mathrm{loc}}(\Omega)$ for every $p \ge 2$, and so, applying Lemma 2.5 to (3.6) with $\delta = 2/p$, we obtain

$$\begin{aligned} &\int_{B_\rho} |D\mathcal{V}|^2\, dX \\ &\quad \le C\Big[\Big(\frac{\rho}{R}\Big)^{2-2/p}\int_{B_{R/2}} |D\mathcal{V}|^2\, dX + \rho^{2-4/p}\Big(\int_{B_{R/2}} (|A|^{p/2} + |B|^{p/2})\, dX\Big)^{2/p}\Big] \end{aligned}$$

for all $\rho < R/2$, where $C = C(p)$. This estimate, together with (3.7), yields

$$
\begin{aligned}
&\int_{B_\rho} |D\mathcal{U}|^2\,dX \\
&\quad \le 2\int_{B_\rho} |D\mathcal{V}|^2 + 2\int_{B_{R/2}} |D\mathcal{U} - D\mathcal{V}|^2\,dX \\
&\quad \le C\Big(\frac{\rho}{R}\Big)^{2-2/p}\int_{B_{R/2}} |D\mathcal{V}|^2\,dX + C\rho^{2-4/p}\Big(\int_{B_{R/2}} (|A|^{p/2} + |B|^{p/2})\,dX\Big)^{2/p} \\
&\qquad\qquad + 2\int_{B_{R/2}} |D\mathcal{U} - D\mathcal{V}|^2\,dX \\
&\quad \le C\Big[\Big(\frac{\rho}{R}\Big)^{2-2/p}\int_{B_{R/2}} |D\mathcal{U}|^2\,dX + \rho^{2-4/p}\Big(\int_{B_{R/2}} (|A|^{p/2} + |B|^{p/2})\,dX\Big)^{2/p} \\
&\qquad\qquad + \int_{B_{R/2}} |D\mathcal{U} - D\mathcal{V}|^2\,dX\Big] \\
&\quad \le C\Big[\Big(\frac{\rho}{R}\Big)^{2-2/p}\int_{B_{R/2}} |D\mathcal{U}|^2\,dX + \rho^{2-4/p}\Big(\int_{B_{R/2}} (|A|^{p/2} + |B|^{p/2})\,dX\Big)^{2/p} \\
&\qquad\qquad + \int_{B_{R/2}} (|A| + |B|)\,dX\Big] \\
&\quad \le C\Big[\Big(\frac{\rho}{R}\Big)^{2-2/p}\int_{B_{R/2}} |D\mathcal{U}|^2\,dX + R^{2-4/p}\Big(\int_{B_{R/2}} (|A|^{p/2} + |B|^{p/2})\,dX\Big)^{2/p}\Big]\,.
\end{aligned}
$$

Finally, by Lemma 2.4 with $\gamma = \beta := 2 - 4/p$, $\alpha = 2 - 2/p$, we conclude that

$$
\begin{aligned}
&\int_{B_\rho} |D\mathcal{U}|^2\,dX \\
&\quad \le C\Big[\Big(\frac{\rho}{R}\Big)^{2-4/p}\int_{B_{R/2}} |D\mathcal{U}|^2\,dX + \rho^{2-4/p}\Big(\int_{B_{R/2}} (|A|^{p/2} + |B|^{p/2})\,dX\Big)^{2/p}\Big]\,,
\end{aligned}
$$

for all $\rho < R/2$, where the constant C does not depend on R. From (3.4) and Lemma 3.2 (i), it follows that

$$
\begin{aligned}
&\int_{B_\rho} |D\mathcal{U}|^2\,dX \\
&\quad \le C\Big(\frac{\rho}{R}\Big)^{2-4/p}\int_{B_{R/2}} |D\mathcal{U}|^2\,dX + C\frac{\rho^{2-4/p}}{R^{-4/p}}\sup_{B_{R/2}}(|A| + |B|) \\
&\quad \le C\Big(\frac{\rho}{R}\Big)^{2-4/p}\Big[\int_{B_R} |D\mathcal{U}|^2\,dX + \int_{B_R} (|A| + |B|)\,dX\Big] \\
&\quad \le C\Big(\frac{\rho}{R}\Big)^{2-4/p}\int_{B_R} |D\mathcal{U}|^2\,dX
\end{aligned}
$$

for all $\rho < R/2$, and the arbitrariness of $p \ge 2$, together with standard $C^{0,\alpha}$ regularity results (see [G], Chapter 3, Theorem 1.1), concludes the proof. □

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