REGULARITY OF MINIMIZERS FOR A CLASS OF MEMBRANE ENERGIES

Emilio Acerbi [1] Irene Fonseca [2] Nicola Fusco [3]

Abstract Regularity properties for (local) minimizers of elastic energies have been challenging mathematical techniques for many years. Recently the interest has resurfaced due in part to the fact that existing partial regularity results do not suffice to ensure existence of (classical) solutions to problems involving free discontinuity sets. The analysis of such questions was started with the fundamental work of De Giorgi in the early 80's in connection with the Mumford-Shah model for image segmentation in computer vision, and later applied to some models for fracture mechanics, thin films, and membranes ([1], [18], [20]). In this paper it is shown that local minimizers in $W^{1,2}(\Omega;\mathbf{R}^d)$ of the functional

$$\mathcal{F}_0(u,\Omega) := \int_\Omega \left[\frac{1}{2}|Du|^2 + f(|\nu(u)|)\right] dx$$

are Hölder continuous of any exponent $\gamma \in (0,1)$, where $\Omega \subset \mathbf{R}^2$ is an open, bounded set, f is a (not necessarily convex) function growing linearly at infinity, and $\nu(u)$ stands for the vector of all 2×2 minors of Du. As a consequence, it is possible to obtain existence of "classical" minimizers in $SBV(\Omega;\mathbf{R}^2)$ of

$$\mathcal{F}(u,\Omega) := \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + f(|\nu(u)|)\right] dx + \beta\int_\Omega |u-g|^q\,dx + \gamma H^{N-1}(S_u\cap\Omega)$$

where $g \in L^\infty(\Omega;\mathbf{R}^d)$, $q > 1$, $\beta,\gamma > 0$. These minimizers are "classical" in the sense that $H^{N-1}\big((\overline{S_u}\setminus S_u)\cap\Omega\big) = 0$ and $u \in W^{1,2}(\Omega\setminus\overline{S_u})$.

1991 Mathematics subject classification (Amer. Math. Soc.): 35J20, 49Q20, 49J45, 49N60

Key Words : functions of special bounded variation, Morrey spaces, Hölder continuity

[1] Dipartimento di Matematica, Via Massimo D'Azeglio 85/A, 43100 Parma (Italy).
Research supported by MURST, Gruppo Nazionale 40%.
[2] Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213.
Research partially supported by the Army Research Office and the National Science Foundation through the Center for Nonlinear Analysis, and by the National Science Foundation under Grant No. DMS-9500531.
[3] Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/a, 50131 Firenze (Italy).
Research supported by MURST, Gruppo Nazionale 40%.

1. Introduction

De Giorgi's seminal work in the early 80's in the study of free discontinuity problems with relation to the model of Mumford Shah for image segmentation in computer vision ([9], [10], [11], [11], [13], [14], [25]) has opened many doors into the study of mathematical questions relevant to the understanding of the behavior of thin films, membranes, and fractured elastic media (see [7], [13], [15], [18]). Although very different in physical nature and motivation, models for these problems have as a common feature the fact that, when searching for quasistatic stable or metastable solutions, one is led to minimizing among all pairs (K,u) an energy involving bulk and interfacial terms,

$$\mathcal{G}(K,u) := \int_{\Omega\setminus K} F(Du)\,dx + \beta\int_{\Omega\setminus K} |u-g|^q\,dx + \gamma H^{N-1}(K\cap\Omega)$$

where Ω is an open, bounded subset of $\mathbf{R}^N$, $q>1$, $g\in L^\infty(\Omega;\mathbf{R}^d)$, $\beta,\gamma>0$, H^{N-1} stands for the $(N-1)$-dimensional Hausdorff measure, K is closed in $\mathbf{R}^N$ and $u:\Omega\to\mathbf{R}^d$ is smooth.

In order to find minima for this energy, as it is usual one relaxes the spaces of admissible fields where lower semicontinuity and compactness results may be found more easily ([2], [3], [4]), and through regularity arguments one concludes that, indeed, these solutions live in the more restricted set of "classical" fields. De Giorgi and Ambrosio (see [2], [4], [15]) introduced the space SBV of functions of *special bounded variation*, i.e. BV functions u whose distributional derivative Du, which is a finite, Radon measure, may be decomposed into an absolutely continuous part $\nabla u\,\mathcal{L}^N$ with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$, and a singular part whose support is an $(N-1)$-dimensional rectifiable set S_u. This is the "jump set" of u, in the sense that u has traces $u^+\neq u^-$ H^{N-1} a.e. on the two sides of S_u. They showed that

$$\mathcal{F}(u;\Omega) := \int_\Omega F(\nabla u)\,dx + \beta\int_\Omega |u-g|^q\,dx + \gamma H^{N-1}(S_u\cap\Omega) \tag{1.1}$$

admits a minimizer in SBV provided F is a convex function growing superlinearly at infinity and coercive. In the scalar-valued case, where $u:\Omega\to\mathbf{R}$, De Giorgi, Carriero and Leaci [16] proved that if $F(\xi)=|\xi|^2$ then

$$H^{N-1}\big((\overline{S_u}\setminus S_u)\cap\Omega\big)=0$$

for any minimizer $u\in SBV(\Omega;\mathbf{R})$ of $\mathcal{F}$, and so the pair $(\overline{S_u},u)$ is a "classical" minimizer of the original functional $\mathcal{G}$. Note that, in general, if $u\in SBV$ then the set S_u is far from being closed, i.e. $H^{N-1}\big((\overline{S_u}\setminus S_u)\cap\Omega\big)>0$, and it may be even dense in Ω. The result of [16] has been extended to a quite broad class of convex functions F with growth $p>1$ at infinity, and further regularity on the set S_u has been obtained (see [1], [5], [6], [11], [20]).

In the vectorial case $u:\Omega\to\mathbf{R}^d$, $d\geq 2$, Fonseca and Francfort (see [18]) showed that functionals of the type (1.1) appear naturally in the study of effective energies for fractured elastic materials.

In this paper we show that local minimizers $u \in W^{1,2}(\Omega;\mathbf{R}^d)$ of the functional

$$\mathcal{F}_0 : v \in W^{1,2}(\Omega;\mathbf{R}^d) \mapsto \int_\Omega \left[\frac{1}{2}|Du|^2 + f(|\nu(u)|)\right] dx$$

are in $C^{0,\gamma}_{\rm loc}(\Omega;\mathbf{R}^d)$ for all $\gamma \in (0,1)$, where $\Omega \subset \mathbf{R}^2$ is open and bounded, f grows linearly at infinity, and $\nu(u)$ stands for the vector of all 2×2 minors of Du. In turn, this regularity will entail that minimizers $u \in SBV(\Omega;\mathbf{R}^d)$ of

$$\mathcal{F}(u,\Omega) := \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + f(|\nu(u)|)\right] dx + \beta\int_\Omega |u-g|^q\, dx + \gamma H^{N-1}(S_u\cap\Omega)$$

are "classical" mimimizers in that $u \in W^{1,2}(\Omega\setminus\overline{S_u};\mathbf{R}^2)$ for every $0<\gamma<1$, and $H^{N-1}\big((\overline{S_u}\setminus S_u)\cap\Omega\big) = 0$. In the case where $d=3$ we may consider these energies as associated to thin films or membranes (see [7], [12], [19], [23]). We remark that f does not need to be convex. This regularity result was already obtained for $N=2$, $\nu(u) = \det\nabla u$, and when f is convex (see [1]). As in that paper, here we use an argument similar to the one introduced by Bauman, Phillips and Owen [8], and used by Dougherty [17]; precisely, we obtain regularity of local minimizers by means of the higher integrability of two auxilliary combinations of the derivatives of u, $A := (|D_1u|^2 - |D_2u|^2)/2, B := D_1u\wedge D_2u$, which turn out to be harmonic functions in the case where $d=2$ (see Proposition 3.2).

2. Statements and Preliminary Results

If $\Omega\subset\mathbf{R}^N$ is open, we say that a function of *bounded variation* $u\in BV(\Omega;\mathbf{R}^d)$ is a function of *special bounded variation* (see [2], [3], [4], [14], [15]]), $u \in SBV(\Omega;\mathbf{R}^d)$, if, denoting by S_u the complement of the set of Lebesgue points of u, the distributional derivative Du is represented by

$$Du = \nabla u\,\mathcal{L}^N + (u^+ - u^-)\otimes\nu\, H^{N-1}\lfloor S_u\ ,$$

where ∇u is the Radon-Nikodym derivative of the finite, Radon measure Du with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$, ν is the normal to the rectifiable set S_u, u^+ and u^- are the traces of u on S_u, and H^{N-1} denotes the $(N-1)$-dimensional Hausdorff measure.

In the sequel we consider Ω to be a bounded, open subset of $\mathbf{R}^2$, and we let $f:[0,+\infty)\to[0,\infty)$ to be a C^1 function such that

(H1) $f(t)\le C(1+t)$ for some $C>1$;

(H2) there exist $M\in[0,+\infty)$ such that

$$\lim_{t\to+\infty} f'(t) = M;$$

(H3) there exist $\alpha, C>0$ such that for all $t\ge 1$

$$\left|f'(t) - \frac{f(t)}{t}\right| \le \frac{C}{t^\alpha}.$$

It is not restrictive to assume that

$$0 < \alpha < 1$$

and in what follows we will work under this assumption. Also, in order to simplify the notation the value of the constant C may change from one line to the next, and B_R, $R > 0$, will denote a generic open ball of radius R, centered at $x \in \Omega$, and such that $B_R \subset \Omega$.

Given $u \in SBV(\Omega; \mathbf{R}^d)$ we define

$$\nu(u) := \frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2},$$

the 2-covector whose components are the 2×2 subdeterminants of ∇u.

Consider the energies

$$\mathcal{G}(K, u) := \int_{\Omega \setminus K} \Big[\frac{1}{2}|\nabla u|^2 + f(|\nu(u)|)\Big]\, dx + \beta \int_{\Omega \setminus K} |u - g|^q\, dx + \gamma H^1(\Omega \cap K),$$

$$\mathcal{F}(u; \Omega) := \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + f(|\nu(u)|)\right] dx + \beta \int_\Omega |u - g|^q\, dx + \gamma H^{N-1}(S_u \cap \Omega),$$

and

$$\mathcal{F}_0(u; \Omega) := \int_\Omega \left[\frac{1}{2}|Du|^2 + f(|\nu(u)|)\right] dx.$$

The following lemma may be found in [21] (Chap. 3, Lemma 2.1).

Lemma 2.1. *Let $\phi : [0, +\infty) \to [0, +\infty)$ be a nonnegative, nondecreasing function, such that*

$$\phi(\rho) \le H \left[\left(\frac{\rho}{R}\right)^\gamma + \varepsilon\right] \phi(R) + KR^\beta$$

for all $0 < \rho < R \le R_0$ and for some constants $H, K \ge 0$ and $0 < \beta < \gamma$. Then there exist constants $\varepsilon_0 = \varepsilon_0(H, \gamma, \beta), C = C(H, \gamma, \beta)$ such that

$$\phi(\rho) \le C\Big[\left(\frac{\rho}{R}\right)^\beta \phi(R) + K\rho^\beta\Big]$$

for all $0 < \rho < R \le R_0$.

Definition 2.2. *We say that $u \in W^{1,2}(\Omega; \mathbf{R}^d)$ is a $W^{1,2}$*-local minimizer *of*

$$I(v; \Omega) := \int_\Omega F(\nabla v)\, dx\ , \qquad v \in W^{1,2}(\Omega; \mathbf{R}^d)$$

if

$$I(u; B_R(x_0)) = \min\Big\{I(v; B_R(x_0))\ :\ v \in u + W_0^{1,2}(B_R(x_0); \mathbf{R}^d)\Big\}$$

for all balls $B_R(x_0) \subset \Omega$.

The main result of this paper is the following theorem.

Theorem 2.3. *If* $u \in W^{1,2}(\Omega, \mathbf{R}^d)$ *is a* $W^{1,2}$*-local minimizer of* $\mathcal{F}_0$ *then* $u \in C^{0,\gamma}_{\mathrm{loc}}$ *for all* $\gamma \in (0,1)$.

In the proof of Theorem 2.3 we will use classical arguments of regularity theory within the framework of the *Morrey spaces* $L^{p,\lambda}$; for a detailed study of these methods we refer the reader to [21], [24].

Definition 2.4. *Given* $\lambda \geq 0$ *we say that* $f \in L^{p,\lambda}(\Omega; \mathbf{R})$ *if there exists a constant* $C > 0$ *such that*

$$\int_{B_\rho} |f|^p \, dx \leq C\rho^\lambda$$

for alll $x \in \Omega$ *and* $0 < \rho < \operatorname{diam} \Omega$. *The function* f *is said to be in* $L^{p,\lambda}_{\mathrm{loc}}(\Omega; \mathbf{R})$ *if* $f \in L^{p,\lambda}(\Omega'; \mathbf{R})$ *for all* $\Omega' \subset\subset \Omega$.

It can be shown that, with $\Omega \subset \mathbf{R}^2$,

$$L^{p,0}(\Omega) = L^p(\Omega), \; L^{p,2}(\Omega) = L^\infty(\Omega), \; L^{p,\lambda}(\Omega) = \{0\} \quad \text{if } \lambda > 2,$$

and that $L^{p,\lambda}(\Omega)$ is a Banach space endowed with the norm

$$||f||_{L^{p,\lambda}(\Omega)} := \left\{ \sup_{x \in \Omega, 0 < \rho < \operatorname{diam} \Omega} \rho^{-\lambda} \int_{B(x,\rho)} |f|^p \, dx \right\}^{\frac{1}{p}}.$$

Morrey proved that (see Theorem 3.5.2, [24])

Lemma 2.5. *If* $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *and* $Du \in L^{2,\lambda}_{loc}(\Omega)$ *for some* $0 < \lambda < 2$ *then* $u \in C^{0,\lambda/2}_{\mathrm{loc}}(\Omega)$.

In light of Lemma 2.5, we will prove Theorem 2.3 by showing that if u is a $W^{1,2}$-local minimizer of $\mathcal{F}_0$ then for all $0 \leq \lambda < 2$

$$\int_{B_\rho} |Du|^2 \, dx \leq C \left(\frac{\rho}{R}\right)^\lambda \int_{B_R} |Du|^2 \, dx + C\rho^\lambda \tag{2.1}$$

for all $0 < \rho < R$ with $B_R \subset\subset \Omega$.

As a corollary we obtain,

Corollary 2.6. *Let* $u \in SBV(\Omega; \mathbf{R}^d)$ *be a minimizer for* $\mathcal{F}$. *Then* $(\overline{S_u}, u)$ *is a minimizer for* $\mathcal{G}$ *among all pairs* (K, v) *with* $K \subset \Omega$ *closed and* $v \in W^{1,2}(\Omega \setminus K; \mathbf{R}^d)$. *Moreover,*

$$H^1((\overline{S_u} \setminus S_u) \cap \Omega) = 0.$$

Following the argument introduced by De Giorgi, Carriero and Leaci [16], and outlined in [1], the corollary holds provided we can show that $W^{1,2}$ local minimizers of

$$v \in W^{1,2}(B_1;\mathbf{R}^d) \mapsto \int_{B_1} \left[\frac{1}{2}|Du|^2 + M|\nu(u)|\right] dx$$

satisfy an estimate of the type

$$\int_{B_\rho} \left[\frac{1}{2}|Du|^2 + M|\nu(u)||\right] dx \le C\rho^\lambda \int_{B_1} \left[\frac{1}{2}|Du|^2 + M|\nu(u)|\right] dx + C\rho^\lambda,$$

for some $0 < \lambda < 2$ and $0 < 0 \le 1$ or, equivalently,

$$\int_{B_\rho} |\nabla u|^2\, dx \le C\rho^\lambda \int_{B_1} |Du|^2\, dx + C\rho^\lambda.$$

We conclude that the assertion of the corollary holds true provided we prove (2.1).

The following lemma may be found in [21], Theorem 3.1, Chapter 3, page 87.

Lemma 2.7. *Let $\lambda < 2$, let $f \in L^{2,\lambda}(B_R;\mathbf{R}^2)$, and let $v \in W^{1,2}(B_R;\mathbf{R})$ satisfy*

$$\Delta v = \operatorname{div} f \quad \textit{in } B_R.$$

Then $Dv \in L^{2,\lambda}_{\mathrm{loc}}(B_R;\mathbf{R}^2)$, and for every $\rho \le R$

$$\int_{B_\rho} |Dv|^2\, dx \le C\left(\frac{\rho}{R}\right)^\lambda \int_{B_R} |Dv|^2\, dx + C\rho^\lambda\, ||f||^2_{L^{2,\lambda}(B_R)}.$$

Lemma 2.8. *Let $p > 1$ and $0 \le \lambda < 2$. If $f_{i,j} \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$ for $i,j \in \{1,2\}$ and $u \in L^1_{\mathrm{loc}}(\Omega)$ is a distributional solution of*

$$\Delta u = \sum D^2_{i,j} f_{i,j}$$

then $u \in L^{p,\lambda}_{\mathrm{loc}}(\Omega)$.

Proof. Let $B_R \subset\subset \Omega$ and for every i,j let $v_{i,j}$ be the solution of (see Theorem 9.15 and Lemma 9.17, [22])

$$\begin{cases} \Delta v_{i,j} = f_{i,j} \\ v_{i,j} \in W^{1,p}_0(B_R) \cap W^{2,p}(B_R), \end{cases}$$

and we set

$$w := \sum D_{i,j} v_{i,j}.$$

Then $w \in L^p(B_R)$ and $||w||_{L^p(B_R)} \le C \sum ||f_{i,j}||_{L^p(B_R)}$. In addition, $\Delta w = \sum D_{i,j} f_{i,j}$ in $\mathcal{D}'$, so that the function

$$v := u - w$$

is harmonic, i.e. $\Delta v = 0$. Hence

$$\sup_{B_{R/2}} |v| \le C(p) \left(\frac{1}{|B_R|} \int_{B_R} |v|^p \, dx \right)^{1/p},$$

from which we deduce that for every $\rho \le R/2$ (thus, for all $0 < \rho \le R$)

$$\int_{B_\rho} |v|^p \, dx \le C \left(\frac{\rho}{R} \right)^2 \int_{B_R} |v|^p \, dx.$$

We have

$$\begin{aligned} \int_{B_\rho} |u|^p \, dx &\le C \int_{B_\rho} (|v|^p + |w|^p) \, dx \\ &\le C \left(\frac{\rho}{R} \right)^2 \int_{B_R} |v|^p \, dx + C \int_{B_R} |w|^p \, dx \\ &\le C \left(\frac{\rho}{R} \right)^2 \int_{B_R} |u|^p \, dx + C \int_{B_R} |w|^p \, dx \\ &\le C \left(\frac{\rho}{R} \right)^2 \int_{B_R} |u|^p \, dx + C R^\lambda \end{aligned}.$$

By Lemma 2.1 we deduce that for all $0 < \rho \le R$

$$\begin{aligned} \int_{B_\rho} |u|^p \, dx &\le C \left(\frac{\rho}{R} \right)^\lambda \int_{B_R} |u|^p \, dx + C \rho^\lambda \\ &\le \rho^\lambda \left[\frac{C}{R^\lambda} \int_{B_R} |u|^p \, dx + C \right], \end{aligned}$$

and so $u \in L^{p,\lambda}_{\text{loc}}(\Omega)$. □

We end this section with a list of algebraic inequalities, following an argument introduced [8] (see also [17]).

Let $P, Q \in \mathbf{R}^d$ and set

$$A := \frac{|P|^2 - |Q|^2}{2}, \quad B := P \cdot Q, \quad \nu := P \wedge Q.$$

Lemma 2.9. *We have*

i) $2\sqrt{A^2 + B^2} \le |P|^2 + |Q|^2$;

ii) $0 \le |P|^2 + |Q|^2 - 2|\nu| \le 2\sqrt{A^2 + B^2}$;

iii) if $\nu = 0$ *then* $|P|^2 + |Q|^2 = 2\sqrt{A^2 + B^2}$;

iv) if $\alpha, \beta \in \mathbf{R}^N$ *and* $\nu \ne 0$ *then*

$$\left| \frac{1}{|\nu|} \nu \cdot (P \wedge \beta + \alpha \wedge Q) - (P \cdot \alpha + Q \cdot \beta) \right|^2 \le 4\sqrt{A^2 + B^2} \, (|\alpha|^2 + |\beta|^2).$$

Proof. Since

$$|\nu|^2 = \sum_{i<j} |P_i Q_j - P_j Q_i|^2 = \frac{1}{2} \sum_{i,j} |P_i Q_j - P_j Q_i|^2 = |P|^2 |Q|^2 - (P \cdot Q)^2,$$

we have

$$|P|^2|Q|^2 = B^2 + |\nu|^2,$$

and so

$$4A^2 = (|P|^2 + |Q|^2)^2 - 4|P|^2|Q|^2 = (|P|^2 + |Q|^2)^2 - 4(B^2 + |\nu|^2),$$

and

$$4(A^2 + B^2) = (|P|^2 + |Q|^2)^2 - 4|\nu|^2.$$

Clearly i) and iii) follow. In addition, we have that

$$(|P|^2 + |Q|^2)^2 - 4|\nu|^2 \geq 0$$

hence

$$0 \leq |P|^2 + |Q|^2 - 2|\nu| \leq \sqrt{(|P|^2 + |Q|^2)^2 - 4|\nu|^2} = 2\sqrt{A^2 + B^2},$$

which yields assertion ii).

Now remark that if $\nu \neq 0$ then $P \neq 0$ and, setting

$$Q' := Q - \frac{P \cdot Q}{|P|^2} P,$$

then also $Q' \neq 0$. Define the orthonormal vectors

$$P_1 := \frac{P}{|P|}, \quad Q_1 := \frac{Q'}{|Q'|}.$$

We write

$$P = p\, P_1, \quad Q = s\, P_1 + q\, Q_1$$

with

$$p := |P|, \quad q := |Q'|, \quad s := \frac{P.Q}{|P|}.$$

Note that

$$\nu = pq\, P_1 \wedge Q_1, \quad |\nu| = pq,$$

and that if $v \in \mathbf{R}^N$ then

$$(P_1 \wedge Q_1) \cdot (P_1 \wedge v) = v \cdot Q_1, \quad (P_1 \wedge Q_1) \cdot (v \wedge Q_1) = v \cdot P_1.$$

We have

$$\begin{aligned}
&\frac{1}{|\nu|}\nu \cdot (P \wedge \beta + \alpha \wedge Q) - (P \cdot \alpha + Q \cdot \beta) \\
&= (P_1 \wedge Q_1) \cdot (p\, P_1 \wedge \beta - s\, P_1 \wedge \alpha + q\, \alpha \wedge Q_1) - (p\, P_1 \cdot \alpha + s\, P_1 \cdot \beta + q\, Q_1 \cdot \beta) \\
&= [(q - p)\, P_1 - s\, Q_1] \cdot \alpha + [-s\, P_1 + (p - q)\, Q_1] \cdot \beta \\
&= v_1 \cdot \alpha + v_2 \cdot \beta,
\end{aligned}$$

with

$$v_1 := (q - p)\, P_1 - s\, Q_1 \qquad \text{and} \quad v_2 := -s\, P_1 + (p - q)\, Q_1.$$

We have

$$|v_1 \cdot \alpha + v_2 \cdot \beta|^2 \leq (|v_1|^2 + |v_2|^2)(|\alpha|^2 + |\beta|^2) = 2(|P|^2 + |Q|^2 - 2|\nu|)(|\alpha|^2 + |\beta|^2),$$

which, together with ii), concludes the proof of iv). □

3. Proof of the Regularity Theorem

In this section we assume that $u \in W^{1,2}(\Omega;\mathbf{R}^d)$ is a local minimizer of $\mathcal{F}_0$.

Proposition 3.1. *If $Du \in L^{2,\lambda}_{\rm loc}(\Omega;\mathbf{R}^d)$ for some $0 \le \lambda < 2$ then $Du \in L^{2,q_0(\lambda)}_{\rm loc}(\Omega;\mathbf{R}^d)$, where $q_0(\lambda) := \alpha + \lambda(1-\alpha/2)$.*

Before proceeding with the proof of this result, we remark that using an iterative scheme where

$$\lambda_0 := 0, \quad \lambda_{k+1} := q_0(\lambda_k)$$

then

$$\lim_{k\to+\infty} \lambda_k = \lim_{k\to+\infty} \alpha \sum_{i=0}^{k} \left(1 - \frac{\alpha}{2}\right)^i = 2,$$

hence (2.1) will follow for all $0 \le \lambda < 2$ and, as justified in Section 2, this suffices to assert Theorem 2.3.

The proof of Proposition 3.1 uses higher integrability properties of the functions

$$A := \frac{|D_1u|^2 - |D_2u|^2}{2}, \quad B := (D_1u)\cdot(D_2u),$$

where D_1u and D_2u stand for the column vectors in $\mathbf{R}^d$ of the derivatives of u with respect to x_1 and to x_2, respectively.

Proposition 3.2. *The functions A and B solve the system*

$$\begin{cases} \Delta A = D^2_{11}g - D^2_{22}g \\ \\ \Delta B = 2D^2_{12}g, \end{cases}$$

where

$$g := f(|\nu(u)|) - |\nu(u)|\, f'(|\nu(u)|).$$

In addition, if $Du \in L^{2,\lambda}_{\rm loc}(\Omega;\mathbf{R}^{2d})$ for some $0 \le \lambda < 2$ then $\sqrt{|A|+|B|} \in L^{2,2\alpha+\lambda(1-\alpha)}_{\rm loc}(\Omega;\mathbf{R})$.

Proof. Consider $\Phi := (\varphi,\psi) \in C^1_0(\Omega;\mathbf{R}^2)$, and let $\varepsilon > 0$ be small enough so that with $\Phi_\varepsilon(x) := x + \varepsilon\Phi(x)$, then $\Phi_\varepsilon : \Omega \to \Omega$ is a smooth diffeomorphism satisfying

$$\det D\Phi_\varepsilon(x) = 1 + \varepsilon \operatorname{div} \Phi(x) + \omega_1(x,\varepsilon),$$

$$\det D\Phi_\varepsilon^{-1}(y) = 1 - \varepsilon \operatorname{div} \Phi(\Phi_\varepsilon^{-1}(y)) + \omega_2(y,\varepsilon),$$

where $\omega_i(\cdot,\varepsilon)/\varepsilon \to 0$, as $\varepsilon \to 0$, uniformly in Ω. Set

$$u_\varepsilon(y) := u\left(\Phi_\varepsilon^{-1}(y)\right), \quad y \in \Omega.$$

We have

$$\int_\Omega |Du_\varepsilon(y)|^2 dy = \int_\Omega |Du(\mathbf{I} - \varepsilon D\Phi)|^2 (\Phi_\varepsilon^{-1}(y))\, dy + o(\varepsilon)$$

$$= \int_\Omega |Du(\mathbf{I} - \varepsilon D\Phi)|^2 (1 + \varepsilon \operatorname{div} \Phi)\, dx + o(\varepsilon)$$

$$= \int_\Omega |Du|^2\, dx + \varepsilon \int_\Omega \left[|Du|^2 \operatorname{div} \Phi - 2 Du D\Phi \cdot Du \right]\, dx + o(\varepsilon),$$

where the inner product of two $d \times 2$ matrices ξ and η is defines as $\xi \cdot \eta := \operatorname{trace}(\xi^T \eta)$.

On the other hand, since

$$\nu(u_\varepsilon(y)) = [\det D\Phi_\varepsilon^{-1}(y)]\, \nu(u)(\Phi_\varepsilon^{-1}(y)),$$

we also have that, setting $\Omega_\varepsilon := \{x \in \Omega : \, |\varepsilon \operatorname{div} \Phi - \omega_2||\nu(u)| \neq 0\}$,

$$\int_\Omega f(|\nu(u_\varepsilon(y))|)\, dy = \int_\Omega f\left((1 - \varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|\right) \det D\Phi_\varepsilon\, dx$$

$$= \int_{\Omega_\varepsilon} [f(|\nu(u)|) + (-\varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|\, f'(|\nu(u)|)] \det D\Phi_\varepsilon\, dx$$

$$+ \int_{\Omega_\varepsilon} \left[\frac{f((1 - \varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|) - f(|\nu(u)|)}{(-\varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|} - f'(|\nu(u)|) \right] (-\varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|) \det D\Phi_\varepsilon\, dx$$

$$+ \int_{\Omega \setminus \Omega_\varepsilon} f(|\nu(u)|) \det D\Phi_\varepsilon\, dx$$

$$= \int_\Omega f(|\nu(u)|) \det D\Phi_\varepsilon\, dx + \int_\Omega (-\varepsilon \operatorname{div} \Phi + \omega_2)|\nu(u)|\, f'(|\nu(u)|) \det D\Phi_\varepsilon\, dx + o(\varepsilon),$$

$$= \int_\Omega f(|\nu(u)|)\, dx + \varepsilon \int_\Omega [f(|\nu(u)|) - |\nu(u)|\, f'(|\nu(u)|)] \operatorname{div} \Phi\, dx + o(\varepsilon),$$

because by Lebesgue's dominated convergence, by (H1), and due to the boundedness of f',

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\left|\frac{f((1-\varepsilon\operatorname{div}\Phi+\omega_2)|\nu(u)|)-f(|\nu(u)|)}{(-\varepsilon\operatorname{div}\Phi+\omega_2)|\nu(u)|}-f'(|\nu(u)|)\right|$$
$$|\nu(u)|\left|\operatorname{div}\Phi-\frac{\omega_2}{\varepsilon}\right|\,|1+\varepsilon\operatorname{div}\Phi+\omega_1|\,dx=0.$$

By the local minimality of u we have $\mathcal{F}_0(u_\varepsilon)-\mathcal{F}_0(u)\geq 0$, from which the Euler-Lagrange equation can be easily obtained,

$$\int_\Omega\left[\frac{1}{2}|Du|^2\operatorname{div}\Phi-DuD\Phi.Du\right]\,dx=\int_\Omega\left[\,|\nu(u)|f'(|\nu(u)|)-f(|\nu(u)|)\,\right]\operatorname{div}\Phi\,dx$$

for every $\Phi=(\varphi,\psi)\in C^1_0(\Omega;\mathbf{R}^2)$. This equation may be rewritten as

$$\int_\Omega[A(D_2\psi-D_1\varphi)-B(D_1\psi+D_2\varphi)]\,dx=\int_\Omega -g(D_1\varphi+D_2\psi)\,dx,$$

that is,

$$\begin{cases} D_1A+D_2B=D_1g \\ \\ D_2A-D_1B=-D_2g,\end{cases}$$

and the first assertion follows. By (H3)

$$|g|\leq C(1+|\nu(u)|^{1-\alpha})$$

and so, assuming that $Du\in L^{2,\lambda}_{\rm loc}(\Omega;\mathbf{R}^{2d})$ we have that $|\nu(u)|\in L^{1,\lambda}_{\rm loc}(\Omega;\mathbf{R})$ and

$$g\in L^{\frac{1}{1-\alpha},\lambda}_{\rm loc}(\Omega).$$

We may now use Lemma 2.8 to obtain that

$$A,B\in L^{\frac{1}{1-\alpha},\lambda}_{\rm loc}(\Omega),$$

and by Hölder inequality we conclude that

$$\sqrt{|A|+|B|}\in L^{2,2\alpha+\lambda(1-\alpha)}_{\rm loc}(\Omega).$$

□

Finally, in order to prove Proposition 3.1 we introduce the following notation:

$$q(\lambda):=2\alpha+\lambda(1-\alpha),$$

$$\Omega_0:=\{x\in\Omega:\ |\nu(u)|=0\},$$

$$\Omega_0':=\{x\in\Omega:\ |\nu(u)|>0\},$$

$$\Omega_K:=\{x\in\Omega:\ 0<|\nu(u)|\leq K\},$$

$$\Omega_K':=\{x\in\Omega:\ |\nu(u)|>K\}.$$

Proof of Proposition 3.1. Fix $\phi \in W_0^{1,2}(\Omega;\mathbf{R}^d)$ and assume that $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega;\mathbf{R}^{2d})$ for some $0 \le \lambda < 2$. For $\varepsilon \in \mathbf{R}$ set $u_\varepsilon(x) := u(x) + \varepsilon\phi(x)$. Define

$$P := D_1 u, \quad Q := D_2 u, \quad \alpha := D_1\phi, \quad \beta = D_2\phi, \quad \nu := \nu(u).$$

Since

$$\nu(u_\varepsilon) = \nu(u) + \varepsilon P \wedge \beta + \varepsilon\alpha \wedge Q + \varepsilon^2 \alpha \wedge \beta,$$

we have

$$\int_\Omega f(|\nu(u_\varepsilon)|)\,dx - \int_\Omega f(|\nu|)\,dx = \varepsilon \int_{\Omega_0'} f'(|\nu|)\frac{\nu}{|\nu|}\cdot(P\wedge\beta + \alpha\wedge Q)\,dx + |\varepsilon| \int_{\Omega_0} f'(0)|P\wedge\beta+\alpha\wedge Q|\,dx + o(\varepsilon).$$

Local minimality of u entails

$$\limsup_{\varepsilon\to 0^-} \frac{\mathcal{F}_0(u_\varepsilon;\Omega) - \mathcal{F}_0(u;\Omega)}{\varepsilon} \le 0,$$

and so

$$\int_\Omega Du\cdot D\phi\,dx + \int_{\Omega_0'} f'(|\nu|)\frac{\nu}{|\nu|}\cdot(P\wedge\beta+\alpha\wedge Q)\,dx \le \int_{\Omega_0} f'(0)|P\wedge\beta+\alpha\wedge Q|\,dx.$$

We have

$$\begin{aligned}(M+1)\int_\Omega Du\cdot D\phi\,dx &+ M\int_{\Omega_0'}\left[\frac{\nu}{|\nu|}\cdot(P\wedge\beta+\alpha\wedge Q) - (P\cdot\alpha + Q\cdot\beta)\right]dx \\ &+ \int_{\Omega_0'\cap\Omega_K}(f'(|\nu|)-M)\frac{\nu}{|\nu|}\cdot(P\wedge\beta+\alpha\wedge Q)\,dx \\ &\le C\int_{\Omega_0}|Du||D\phi|\,dx + \omega_K\int_{\Omega_K'}|Du||D\phi|\,dx,\end{aligned}$$

where

$$\omega_K := \sup_{t\ge K}|M - f'(t)|.$$

We recall that by (H2)

$$\omega_K \to 0 \qquad \text{as } k \to +\infty.$$

By Lemma 2.9 iii), iv) we deduce that

$$(M+1)\int_\Omega Du\cdot D\phi\,dx + \int_\Omega G\cdot D\phi\,dx \le C\int_\Omega \sqrt{|A|+|B|}|D\phi|\,dx + \omega_K\int_\Omega |Du||D\phi|\,dx \tag{3.1}$$

with

$$G_1 := \chi_{\Omega_0'\cap\Omega_K}(M - f'(|\nu|))\frac{\nu}{|\nu|} \wedge Q$$
$$G_2 := \chi_{\Omega_0'\cap\Omega_K}(f'(|\nu|) - M)\frac{\nu}{|\nu|} \wedge P$$

and where χ_A stands for the characteristic function of the set A. By Lemma 2.9 ii), iii), and recalling that on Ω_K we have $|\nu| \le K$, we have

$$|G| \le C(K)(1 + \sqrt{|A| + |B|}), \qquad \text{a.e. in } \Omega, \tag{3.2}$$

and by Proposition 3.2 we deduce that $G \in L^{2,q(\lambda)}(\Omega;\mathbf{R}^d)$. Next, for a fixed ball $B_R \subset\subset \Omega$ we compare u with the solution of the Dirichlet problem

$$\begin{cases} (M+1)\Delta v = \operatorname{div} G \quad \text{in } B_R \\ \\ v - u \in W_0^{1,2}(B_R;\mathbf{R}). \end{cases} \tag{3.3}$$

By Lemma 2.7 $Dv \in L^{2,q(\lambda)}_{\mathrm{loc}}(B_R;\mathbf{R}^2)$ and for all $0 < \rho \le R$

$$\int_{B_\rho} |Dv|^2\,dx \le C\left(\frac{\rho}{R}\right)^{q(\lambda)} \int_{B_R} |Dv|^2\,dx + C(K)\rho^{q(\lambda)}. \tag{3.4}$$

¿From (3.1) and (3.3) we have for all $\phi \in W_0^{1,2}(B_R;\mathbf{R}^d)$

$$(M+1)\int_{B_R}(Du - Dv)\cdot D\phi dx \le C\int_{\Omega\cap B_R}\sqrt{|A| + |B|}|D\phi|dx + \omega_K\int_{B_R}|Du||D\phi|dx.$$

Therefore, taking $\phi := u - v$, and using the fact that Lemma 2.9 i) and (3.2) yield

$$|G| \le C|Du|, \qquad \int_{B_R}|Dv|^2 \le C\int_{B_R}|Du|^2,$$

we have

$$\int_{B_R}|Du - Dv|^2\,dx \le C\int_{B_R}(|A| + |B|)\,dx + C\omega_K\int_{B_R}|Du|^2\,dx.$$

Using (3.4) we now obtain

$$\int_{B_\rho}|Du|^2\,dx \le C\left[\left(\frac{\rho}{R}\right)^{q(\lambda)} + \omega_K\right]\int_{B_R}|Du|^2\,dx + C(K)R^{q(\lambda)},$$

and if K is large enough, so that ω_K is small, from Lemma 2.1 we conclude that for all $0 < \lambda' < q(\lambda)$

$$\int_{B_\rho}|Du|^2\,dx \le C\left(\frac{\rho}{R}\right)^{\lambda'}\int_{B_R}|Du|^2\,dx + C\rho^{\lambda'}, \tag{3.4}$$

and thus (3.4) holds true for $\lambda' = q_0(\lambda)$. □

References

1 Acerbi, E., I. Fonseca, N. Fusco, N. Fusco, Regularity results for equilibria in a variational model for fracture. To appear in *Proc. R. Soc. Edin.*
2 Ambrosio, L., A compactness theorem for a new class of functions of bounded variation, *Boll. Un. Mat. Ital.* **3-B** (1989), 857–881.
3 Ambrosio, L., A new proof of the *SBV* compactness theorem, *Calc. Var.* **3** (1995), 127–137.
4 Ambrosio, L., On the lower semicontinuity of quasiconvex integrals in $SBV(\Omega,\mathbf{R}^k)$, *Nonlinear Anal.* To appear.
5 Ambrosio, L., N. Fusco and D. Pallara, Partial regularity of free discontinuity sets II, *Preprint Dip. Mat. e Appl. Napoli*, (1995).
6 Ambrosio, L. and D. Pallara, Partial regularity of free discontinuity sets I, *Preprint Sc. Norm Sup. Pisa*, (1994).
7 Bhattacharya, K., R. James, in preparation.
8 Bauman, P., N. C. Owen and D. Phillips, Maximum principles and apriori estimates for a class of problems from nonlinear elasticity, *Ann. Inst. H. Poincaré* **8** (1991), 119–157.
9 Blake, A., and A. Zisserman, Visual Reconstruction, *The MIT Press, Cambridge, Massachussets, 1985.*
10 Bonnet, A., On the regularity of edges in the Mumford-Shah model for image segmentation. To appear.
11 Carriero, M. and A. Leaci, S^k-valued maps minimizing the L^p norm of the gradient with free discontinuities, *Ann. Sc. Norm. Sup. Pisa* **18** (1991), 321–352.
12 Ciarlet, P. G., P. Destuynder, A justification of a nonlinear model in plate theory, *Comput. Methods Appl. Mech. Engrg.* **17/18** (1979), 227-258.
13 David, G. and S. Semmes, On the singular set of minimizers of the Mumford-Shah functional, *J. Math. Pures et Appl.* To appear.
14 De Giorgi, E., *Free Discontinuity Problems in the Calculus of Variations*, a collection of papers dedicated to J. L. Lions on the occasion of his 60^{th} birthday, North Holland (R. Dautray ed.), 1991.
15 De Giorgi, E. and L. Ambrosio, Un nuovo tipo di funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199–210.
16 De Giorgi, E., M. Carriero and A. Leaci, Existence theorem for a minimum problem with free discontinuity set, *Arch. Rat. Mech. Anal.* **108** (1989), 195–218.
17 Dougherty, M., Higher integrability of the gradient for minimizers of certain polyconvex functionals in the calculus of variations. Preprint.
18 Fonseca, I. and G. Francfort, Relaxation in *BV* versus quasiconvexification in $W^{1,p}$; a model for the interaction between fracture and damage, *Calc. Var.* **3** (1995), 407-446.
19 Fonseca, I. and G. Francfort, Optimal design problems in elastic membranes. To appear.
20 Fonseca, I. and N. Fusco, Regularity results for anisotropic image segmentation models, *Ann. Sc. Norm. Sup. di Pisa.* To appear.
21 Giaquinta, M., *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Mathematics Studies, Princeton University Press, 1983.
22 Gilbarg, D., N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.
23 Le Dret, H., A. Raoult, The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity, *J. Math. Pures et Appl.* **74** (1995), 549-578.
24 Morrey, C. B., *Multiple integrals in the Calculus of Variations*, Springer, Berlin 1966.
25 Mumford, D. and J. Shah, Boundary detection by minimizing functionals, *Proc. IEEE Conf. on Computer Vision and Pattern Recognition (San Francisco, 1985).*