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Existence and regularity for mixtures of micromagnetic materials

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A new model for the energy of a mixture of micromagnetic materials is introduced within the context of functions with special bounded variation. Existence and regularity for the solution of an optimal design problem in micromagnetics are obtained.

Keywords: micromagnetics; constraints; quasi-minimizer; decay lemma; deviation

1. Introduction

The commonly adopted Weiss–Landau–Lifschitz model of micromagnetics applies to a single crystal of a magnetic material, and according to this theory the total energy associated with the magnetized crystal is given as a sum of several energy contributions, as described briefly in §2 (see Brown 1963; Landau & Lifschitz 1984; Visintin 1985; Anzellotti *et al.* 1991; Hubert & Schäfer 1998; Dacorogna & Fonseca 2000). When considering a body composed of several distinct magnetic materials, surface energy terms must be taken into account due to the interaction between grains with different magnetic properties, and this leads to the introduction in §3 of a new model for mixtures of magnetic materials, framed within the context of the space SBV of functions with special bounded variation (SBV).

In this model, all material information is encapsulated in a function $\boldsymbol{u}$, *the composite magnetization*, and the total magnetic energy associated with a body $\Omega \subset \mathbb{R}^3$ composed of a finite number K of different magnetic materials has the form

$$\mathcal{E}(\boldsymbol{u}) = \int_\Omega [a(|\boldsymbol{u}|)|\nabla \boldsymbol{u}|^2 + \phi(\boldsymbol{u}) - \boldsymbol{f}\cdot \boldsymbol{m}(\boldsymbol{u}) - \tilde{\boldsymbol{h}}[\boldsymbol{u}]\cdot \boldsymbol{m}(\boldsymbol{u})]\mathrm{d}x + \int_{J_{\boldsymbol{u}}} \gamma(\boldsymbol{u}^+, \boldsymbol{u}^-, \nu)\mathrm{d}\mathcal{H}^2,$$

where $\boldsymbol{f}$ is the external magnetic field, $J_{\boldsymbol{u}}$ is the set of discontinuity points of $\boldsymbol{u}$, and the composite magnetization must satisfy the pointwise constraint,

$$|\boldsymbol{u}| \in \{1, 2, \ldots, K\} \text{ a.e. in } \Omega.$$

In §4, we apply this model to an optimal design problem, that of minimizing the total energy of the body Ω for a fixed external magnetic field $\boldsymbol{f}$, given the K materials which Ω may be made of, and possibly under fixed volume fractions of

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each component. We prove existence of a solution, and by modifying appropriately the arguments in Carriero & Leaci (1991) and Ambrosio *et al.* (2000), we establish a regularity property for the optimal configuration.

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2. The Weiss–Landau–Lifschitz model for one crystal briefly visited

In this section, we summarize the Weiss–Landau–Lifschitz model of micromagnetics, which has proved to be suitable for the study of a magnetic material with moderate conductivity. Consider a single crystal of some magnetic material, and assume that it occupies a set $\Omega \subset \mathbb{R}^3$. Each of its material points is magnetized, i.e. it generates a magnetic field described by a vector field, the *magnetization*

$$\boldsymbol{m} : \Omega \to \mathbb{R}^3.$$

Below the Curie temperature, the magnetization has constant intensity,

$$|\boldsymbol{m}| \equiv m^{\mathrm{s}}, \quad \text{in } \Omega,$$

where the *magnetic saturation* intensity m^{s} and the Curie temperature are characteristic of the material. We assume throughout that the temperature is well below the Curie temperature of each material employed.

The *magnetic exchange energy* favours the alignment with $\boldsymbol{m}$ of the magnetization at neighbouring points. It depends on the gradient matrix $\nabla \boldsymbol{m}$ through a four-indices tensor A, also characteristic of the material, and is given by $\int_\Omega \langle \mathsf{A}\nabla \boldsymbol{m}, \nabla \boldsymbol{m}\rangle \mathrm{d}x$. A good approximation, which is commonly adopted, is that A is close to being a multiple of the identity, thus we set

$$\mathrm{Exch} = \int_\Omega a|\nabla \boldsymbol{m}|^2 \mathrm{d}x,$$

where the constant a is another characteristic of the material.

Owing to the structure of the crystal, there are some alignments of the magnetization $\boldsymbol{m}$ (the *easy axes*), which are preferred with respect to others. There is just one direction (and its opposite) for uniaxial crystals, namely the main axis of the crystal, whereas for different symmetry groups there are several easy directions. This preference is expressed through the *anisotropy energy*, which is usually described as the integral of a non-negative polynomial in $\boldsymbol{m}$, here generalized to read

$$\mathrm{Anis} = \int_\Omega \phi(\boldsymbol{m}) \mathrm{d}x,$$

where the continuous function

$$\phi : \partial B_{m^{\mathrm{s}}} \to [0, +\infty[$$

depends on the grain, not only through the material it is made of, but also through its orientation.

It may happen that the exchange and anisotropy energies compete. For example, consider a rod of uniaxial crystal, and assume that at the two ends of the rod the magnetization is forced by external conditions to be along the easy axis, but pointing in the two opposite directions. The anisotropy energy favours $\boldsymbol{m}$ to align with these two directions across a thin transition layer, whereas the exchange energy term favours a slow transition. Most theories agree that the total energy contribution of the transition layer is proportional to the area of the cross-section. Ultimately, the material reaches equilibrium partitioned into islands of constant $\boldsymbol{m}$, the *magnetic domains*, separated by thin layers where all the transitions take place, the *Bloch walls* (some of our considerations apply to other kinds of walls, such as *Néel walls*, only on a large-scale level). The diameter of magnetic domains is in the range of one-tenth of a micrometre to millimetres, whereas the thickness of the walls is about 10–100 atomic layers, and so, as is customary also in the physicists' practice, we will later take into account very thin transition layers by introducing a surface energy penalization.

The external magnetic field $\boldsymbol{f}$ interacts with the magnetization $\boldsymbol{m}$, thus producing another energy term (to simplify the notation we dropped a few constants, such as a couple of 1/2 in front of some integrals)

$$\mathrm{Ext} = -\int_{\Omega} \boldsymbol{f}\cdot\boldsymbol{m}\,\mathrm{d}x,$$

and in order to minimize it, $\boldsymbol{m}$ will seek to align with $\boldsymbol{f}$.

The last term is the *demagnetizing energy*. The magnetization $\boldsymbol{m}$ induces a field $\boldsymbol{h}[\boldsymbol{m}]$ in the whole space that is determined by Maxwell's equations,

$$\begin{cases} \operatorname{curl} \boldsymbol{h} = \boldsymbol{0}, & \text{in } \mathbb{R}^3, \\ \operatorname{div}(\boldsymbol{h} + \boldsymbol{m}\mathbb{1}_{\Omega}) = 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{2.1}$$

These are to be interpreted in the sense of distributions as

$$\begin{cases} \boldsymbol{h}\in L^2(\mathbb{R}^3;\mathbb{R}^3), & \operatorname{curl} \boldsymbol{h} = \boldsymbol{0}, \\ \int_{\mathbb{R}^3}\boldsymbol{h}\cdot\boldsymbol{v}\,\mathrm{d}x = -\int_{\Omega}\boldsymbol{m}\cdot\boldsymbol{v}\,\mathrm{d}x, & \text{for all } \boldsymbol{v}\in L^2(\mathbb{R}^3;\mathbb{R}^3), \text{ such that } \operatorname{curl} \boldsymbol{v} = \boldsymbol{0}. \end{cases}$$

The demagnetizing energy (which is a non-local term) is given by

$$\mathrm{Demag} = \int_{\mathbb{R}^3} |\boldsymbol{h}[\boldsymbol{m}]|^2 \mathrm{d}x.$$

Since (2.1) holds in $\mathbb{R}^3$, the generated magnetic field is zero if $\boldsymbol{m}$ is divergence-free and is tangent to the boundary of Ω, whereas it is large if $\boldsymbol{m}$ has constant direction. Thus, the demagnetizing energy has large effects and strongly interferes with the exchange and anisotropy energies, which have opposite preferences regarding the alignment of $\boldsymbol{m}$. For interesting microstructure problems arising from this situation, we refer, for example, to James & Kinderlehrer (1990) and DeSimone *et al.* (2000).

The total energy associated with a magnetization $\boldsymbol{m}$ of a single crystal is given by the sum of the four terms we discussed, i.e.

$$\begin{aligned} V(\boldsymbol{m}) &= \mathrm{Exch} + \mathrm{Anis} + \mathrm{Ext} + \mathrm{Demag} \\ &= \int_{\Omega} [a|\nabla \boldsymbol{m}|^2 + \phi(\boldsymbol{m}) - \boldsymbol{f}\cdot\boldsymbol{m}]\mathrm{d}x + \int_{\mathbb{R}^3} |\boldsymbol{h}[\boldsymbol{m}]|^2 \mathrm{d}x. \end{aligned}$$

We conclude this section by remarking that the mapping $\boldsymbol{m} \mapsto \boldsymbol{h}[\boldsymbol{m}]$ has some interesting properties (see DeSimone 1993). The theory of singular integrals (see Stein 1970) ensures that it is linear, continuous from $L^p(\Omega)$ to $L^p(\mathbb{R}^3)$ for any $p>1$, and

$$\int_{\mathbb{R}^3} |\boldsymbol{h}[\boldsymbol{m}]|^2 \mathrm{d}x = -\int_{\Omega} \mathrm{m} \cdot \boldsymbol{h}[\boldsymbol{m}] \mathrm{d}x.$$

From this formula we deduce, in particular, that

$$\boldsymbol{m} \mapsto \int_{\mathbb{R}^3} |\boldsymbol{h}[\boldsymbol{m}]|^2 \mathrm{d}x \quad \text{is continuous from } L^2(\Omega;\mathbb{R}^3) \text{ into } \mathbb{R}.$$

Also, although the term Demag is non-local, some local estimates may be recovered. If two magnetizations agree outside a ball, i.e. if $\boldsymbol{m}=\boldsymbol{m}'$ outside $B_\varrho \subset \Omega$, and if $|\boldsymbol{m}|, |\boldsymbol{m}'| \le L$, then we have for every $q>1$

$$\begin{aligned} &\left| \int_{\Omega} \boldsymbol{h}[\boldsymbol{m}] \cdot \boldsymbol{m} - \boldsymbol{h}[\boldsymbol{m}'] \cdot \boldsymbol{m}' \, \mathrm{d}x \right| \\ &\quad = \left| \int_{\Omega} (\boldsymbol{m} - \boldsymbol{m}') \cdot \boldsymbol{h}[(\boldsymbol{m} + \boldsymbol{m}')/2] + (\boldsymbol{m} + \boldsymbol{m}') \cdot \boldsymbol{h}[(\boldsymbol{m} - \boldsymbol{m}')/2] \mathrm{d}x \right| \\ &\quad \le c(L,q) \|\boldsymbol{m} - \boldsymbol{m}'\|_{q'} \le c\varrho^{3/q'}. \end{aligned} \tag{2.2}$$

Also, by the continuity of ϕ on the compact set $\partial B_{m^{\mathrm{s}}}$

$$\int_{\Omega} |\phi(\boldsymbol{m}) - \phi(\boldsymbol{m}')| \mathrm{d}x \le c\varrho^3. \tag{2.3}$$

Both estimates will be useful when proving regularity in §5.

3. Mixtures, their energy and the new model

We now turn to *mixtures* of magnetic materials. Suppose that Ω is composed of two crystals Ω_1 and Ω_2 of different materials, separated by a smooth surface Σ. We stress the fact that by 'different' we mean that the two grains may also be made of the same substance but with differently oriented crystallographic axes. Each magnetic material is identified by the exchange constant a, the magnetic saturation m^{s} and the anisotropy function ϕ, which contains all the necessary crystallographic information, thus we must consider two triples $(a_1, m_1^{\mathrm{s}}, \phi_1)$ and $(a_2, m_2^{\mathrm{s}}, \phi_2)$, and the energy contribution of the magnetizations $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ of the two grains is then

$$\begin{aligned} V(\boldsymbol{m}_1, \boldsymbol{m}_2) &= \int_{\Omega_1} [a_1 |\nabla \boldsymbol{m}_1|^2 + \phi_1(\boldsymbol{m}_1) - \boldsymbol{f} \cdot \boldsymbol{m}_1] \mathrm{d}x \\ &\quad + \int_{\Omega_2} [a_2 |\nabla \boldsymbol{m}_2|^2 + \phi_2(\boldsymbol{m}_2) - \boldsymbol{f} \cdot \boldsymbol{m}_2] \mathrm{d}x + \int_{\mathbb{R}^3} |\boldsymbol{h}|^2 \mathrm{d}x, \end{aligned}$$

where $\boldsymbol{h} = \boldsymbol{h}[\boldsymbol{m}_1 \mathbb{1}_{\Omega_1} + \boldsymbol{m}_2 \mathbb{1}_{\Omega_2}]$.

The presence of the dividing surface in Ω creates chemical and electric disturbances in the lattice atoms, possibly related also to the different magnetizations on the two sides, and maybe also on the direction of the normal vector to Σ (we are not aware of a physical interpretation for this, but mathematically it comes for free). Therefore, we must include in the energy a surface term, whose density depends on the two materials. Precisely, this density is the sum of a positive constant with a non-negative function of the traces $\mathrm{Tr}\,\boldsymbol{m}_1$ and $\mathrm{Tr}\,\boldsymbol{m}_2$ of the magnetizations on the two sides of Σ, and of the normal vector $\boldsymbol{\nu}$ to Σ, i.e.

$$S(\boldsymbol{m}_1, \boldsymbol{m}_2) = \int_\Sigma [\alpha_{1,2} + \beta_{1,2}(\mathrm{Tr}\,\boldsymbol{m}_1, \mathrm{Tr}\,\boldsymbol{m}_2, \boldsymbol{\nu})]\mathrm{d}\mathcal{H}^2.$$

The total energy becomes

$$E(\boldsymbol{m}_1, \boldsymbol{m}_2) = V(\boldsymbol{m}_1, \boldsymbol{m}_2) + S(\boldsymbol{m}_1, \boldsymbol{m}_2),$$

under the constraint that the modulus of each magnetization equals the respective magnetic saturation intensity.

In the case of K magnetic materials, each is characterized by a triple $(a_i, m_i^{\mathrm{s}}, \phi_i)$ and occupies an open subset Ω_i of Ω, where the sets Ω_i are pairwise disjoint and their union is all of Ω up to a two-dimensional set Σ. If we denote by $\boldsymbol{m}_i$ the magnetization in Ω_i, then the energy is given by

$$\sum_{i=1}^{K} \int_{\Omega_i} [a_i |\nabla \boldsymbol{m}_i|^2 + \phi_i(\boldsymbol{m}_i) - \boldsymbol{f}\cdot\boldsymbol{m}_i]\mathrm{d}x + \int_{\mathbb{R}^3} \left| \boldsymbol{h}\left[\sum_{i=1}^{K} \boldsymbol{m}_i \mathbb{1}_{\Omega_i}\right]\right|^2 \mathrm{d}x + \sum_{i\neq j} S(\boldsymbol{m}_i, \boldsymbol{m}_j), \tag{3.1}$$

where the terms $S(\boldsymbol{m}_i, \boldsymbol{m}_j)$ are surface integrals on subsets of Σ, and where we recall that

$$|\boldsymbol{m}_i| = m_i^{\mathrm{s}}, \quad \text{in } \Omega_i, \quad \boldsymbol{m}_i \in W^{1,2}(\Omega_i; \mathbb{R}^3). \tag{3.2}$$

Since the landscape of the subdomains Ω_i, $i = 1, \dots, K$, is an unknown of the problem in the optimal problem considered in §4, it is easy to see that this energy does not entail compactness of energy bounded sequences of magnetizations. Indeed, a magnetization $\boldsymbol{m}_i$ which is discontinuous along a surface $\sigma \subset \Omega_i$ is not admissible due to the Sobolev condition in (3.2), but it may be approached by a sequence of admissible magnetizations with equibounded energy, simply by fattening σ into an open set σ' and adding this to Ω_j for some $j \neq i$ (extend $\boldsymbol{m}_j$ to σ' as a constant). This then leads to a finite relaxed energy for the discontinuous function we selected. We remark that this is somewhat analogous to Gibbs' phenomenon in fluids (see Modica 1987).

The structure of the relaxed energy, which allows inner discontinuities but penalizes them, may be physically interpreted as keeping into account the possible discontinuities of the magnetic field inside a crystal, or as a simplification of the energy of a Bloch wall.

We are thus led to considering an energy which no longer forces the magnetizations to belong to $W^{1,2}$ inside each grain, but instead allows jumps, so it is natural to take as ambient space that of special functions of bounded variation, SBV. We recall that the distributional derivative of a function u with bounded variation in $\Omega \subset \mathbb{R}^n$ may be decomposed as the sum of an absolutely continuous term, $\nabla \boldsymbol{u} \llcorner \mathrm{d}x$, and a singular part $D^{\mathrm{s}}u$. Moreover (the precise representative of) u is discontinuous on a 'jump' set J_u, which is countably $\mathcal{H}^{n-1}$ rectifiable. The singular part may be further decomposed into a jump part $D^{\mathrm{j}}u = D^{\mathrm{s}}u \llcorner J_u$, which is supported on J_u, and a Cantor part $D^{\mathrm{c}}u = D^{\mathrm{s}}u - D^{\mathrm{j}}u$. The functions with SBV are defined as those BV functions whose distributional derivative has no Cantor part.

If $\boldsymbol{u} \in \mathrm{SBV}$, then the jump set has a normal $\boldsymbol{\nu}$ at $\mathcal{H}^{n-1}$-a.e. point, and the traces of u from the two sides are denoted u^+ and u^-. We will later use, without further description, the precise definition and several properties of the space SBV, and we refer the reader to Ambrosio *et al.* (2000) for a comprehensive treatise on the subject.

We may now relax the requirements in (3.2) to read $\boldsymbol{m}_i \in \mathrm{SBV}(\Omega_i; \mathbb{R}^3)$; in this setting, the energy may be written in a form which is different, but not much simpler than before, as we cannot charge all surface terms on the jump set of the overall magnetization, because some parts of the surface Σ may then be missing. Indeed, in two adjacent grains Ω_i and Ω_j one may well have $m_i^{\mathrm{s}} = m_j^{\mathrm{s}}$, thus the saturation magnetization might have no jump across the interface, although some energy has to be taken into account (due to the electric disturbances we mentioned). Then, at this stage it is impossible to replace the extra term on Σ by an integral on the jump set J, since, in general, $\Sigma \not\subset J$.

In order to overcome this problem, we will rescale the magnetizations $\boldsymbol{m}_i$ in order to obtain an auxiliary magnetization field $\boldsymbol{u}$ which will contain all the information, and which will allow us to write the energy in an easy, implicit form. We change $\boldsymbol{m}_i$ so that the magnetic saturation intensity in Ω_i becomes equal to i, thus the norm of the new magnetization will jump on Σ. Moreover, the same norm at any point of $\Omega \backslash \Sigma$ will tell us in which of the subsets Ω_i the point lies.

We set for $i = 1, \ldots, K$

$$\boldsymbol{u}_i := i \frac{\boldsymbol{m}_i}{m_i^{\mathrm{s}}}, \quad \boldsymbol{u} := \sum_1^K \boldsymbol{u}_i \mathbb{1}_{\Omega_i}, \quad \boldsymbol{m} := \sum_1^K \boldsymbol{m}_i \mathbb{1}_{\Omega_i},$$

so that $\boldsymbol{u} \in \mathrm{SBV}(\Omega; \mathbb{R}^3)$ and

$$\Omega_i = \{x : |\boldsymbol{u}(x)| = i\} \quad \text{and} \quad \mathbb{1}_{\Omega_i}(x) = (1 - ||\boldsymbol{u}(x)| - i|)^+ =: \mu_i(|\boldsymbol{u}(x)|), \tag{3.3}$$

$$\boldsymbol{m} = \boldsymbol{m}(\boldsymbol{u}) = \left(\sum_1^K \frac{m_i^{\mathrm{s}}}{i} \mathbb{1}_{\Omega_i} \right) \boldsymbol{u} = \left(\sum_1^K \frac{m_i^{\mathrm{s}}}{i} \mu_i(|\boldsymbol{u}|) \right) \boldsymbol{u} =: \lambda(|\boldsymbol{u}|) \boldsymbol{u},$$

and now the jump set $J_{\boldsymbol{u}}$ of $\boldsymbol{u}$ consists exactly of the union of both the interfaces between grains and the inner discontinuities of the magnetic field. We remark that given $\boldsymbol{u}$ one easily deduces $\boldsymbol{m}$ and may also decide whether a jump of $\boldsymbol{u}$ represents an interface or an inner discontinuity of the magnetic field. The former is also a jump of $|\boldsymbol{u}|$, the second is not.

We may now rescale the other factors. Fix any bounded, positive, continuous function a satisfying

$$a : [0, +\infty[\to]0, +\infty[, \quad a(i) = \left(\frac{m_i^{\mathrm{s}}}{i} \right)^2 a_i, \quad \text{for } i = 1, \ldots, K,$$

and we have

$$\mathrm{Exch} = \sum_{i=1}^{K} \int_{\Omega_i} a_i |\nabla \boldsymbol{m}_i|^2 \mathrm{d}x = \int_{\Omega} a(|\boldsymbol{u}|)|\nabla \boldsymbol{u}|^2 \mathrm{d}x.$$

Note that the definition of the function a, outside the numbers $1, \ldots, K$, allows us to extend naturally this energy to all SBV. Analogously, take any bounded, non-negative, continuous function satisfying

$$\phi : \mathbb{R}^3 \to [0, +\infty[, \quad \phi\big|_{\partial B_i}(\boldsymbol{z}) = \phi_i\left(\frac{m_i^{\mathrm{s}} \boldsymbol{z}}{i}\right), \quad \text{for } i = 1, \ldots, K,$$

and

$$\mathrm{Anis} = \sum_{1}^{K} \int_{\Omega_i} \phi_i(\boldsymbol{m}_i) \mathrm{d}x = \int_{\Omega} \phi(\boldsymbol{u}) \mathrm{d}x.$$

Now, since the mapping $\boldsymbol{u} \mapsto \boldsymbol{m}(\boldsymbol{u}) = \lambda(|\boldsymbol{u}|)\boldsymbol{u}$ is continuous in every L^p, so is the mapping

$$\boldsymbol{u} \mapsto \boldsymbol{h}[\boldsymbol{m}(\boldsymbol{u})] =: \tilde{\boldsymbol{h}}[\boldsymbol{u}],$$

although it is no longer linear because λ is not; only additivity with disjoint supports is preserved. We may write the total energy as

$$\mathcal{E}(\boldsymbol{u}) = \int_{\Omega} [a(|\boldsymbol{u}|)|\nabla \boldsymbol{u}|^2 + \phi(\boldsymbol{u}) - \boldsymbol{f} \cdot \boldsymbol{m}(\boldsymbol{u}) - \tilde{\boldsymbol{h}}[\boldsymbol{u}]\boldsymbol{m}(\boldsymbol{u})] \mathrm{d}x + \int_{J_u} \gamma(\boldsymbol{u}^+, \boldsymbol{u}^-, \nu) \mathrm{d}\mathcal{H}^2,$$

where the function γ encompasses all the surface terms we described earlier.

In order to have semi-continuity of the energy on the set

$$\{\boldsymbol{v} \in \mathrm{SBV}(\Omega) : |\boldsymbol{v}| = i \text{ a.e. in } \Omega_i\},$$

or simply in $\mathrm{SBV}(\Omega)$, one has to impose on γ a restriction of elementary geometric nature, equivalent to saying that if (as we did earlier) one interposes between two adjacent grains an infinitesimal layer of another material, the energy will not decrease. Mathematically, this leads to the introduction of *jointly convex* functions (see Ambrosio *et al.* 2000, §5.3).

Although the results in §4 hold for the general case of a jointly convex γ satisfying $\gamma \geq c > 0$ and the standard assumptions in Ambrosio *et al.* (2000, ch. 5), we opt to consider in the sequel

$$\gamma \equiv 1.$$

This will considerably reduce the amount of writing while leaving intact the main points in the proof. To keep the balance even, we generalize to the n-dimensional case in the obvious way (e.g. replace 2 and 3 by $(n-1)$ and n, respectively), only $\tilde{\boldsymbol{h}}$ needs some care, as $\boldsymbol{h}$ was defined in terms of Maxwell's equations and in the general case we take it to be a continuous function mapping from every $L^p(\Omega)$ into $L^p(\mathbb{R}^n)$.

4. Existence and regularity for an optimal design problem

We test our model by applying it to an optimal design problem. Assume that

H1 Ω is a bounded, open domain of $\mathbb{R}^n$;
H2 $\boldsymbol{f}$ is a given vector field in $L^1(\Omega; \mathbb{R}^n)$;

H3 a is a positive, continuous real function defined on $[0,+\infty[$;
H4 ϕ is a non-negative, continuous real function defined on $\mathbb{R}^n$;
H5 $\boldsymbol{h}$ is a mapping from $L^2(\Omega;\mathbb{R}^n)$ to $L^2(\mathbb{R}^n;\mathbb{R}^n)$ which is continuous in every strong L^p topology (on both domain and target), and such that if $\boldsymbol{u}$ and $\boldsymbol{u}'$ have disjoint supports, then $\tilde{\boldsymbol{h}}[\boldsymbol{u}+\boldsymbol{u}']=\tilde{\boldsymbol{h}}[\boldsymbol{u}]+\tilde{\boldsymbol{h}}[\boldsymbol{u}']$;
H6 $m_1^{\mathrm{s}},\dots,m_K^{\mathrm{s}}$ are positive real numbers;
H7 $\boldsymbol{f}\in L^q(\Omega;\mathbb{R}^n)$ for some $q>n$.

Set

$$\mu_i:[0,+\infty[\to\mathbb{R},\quad \mu_i(t):=(1-|t-i|)^+,\quad \text{for } i=1,\dots,K, \tag{4.1}$$

$$\boldsymbol{m}:\mathbb{R}^n\to\mathbb{R}^n,\quad \boldsymbol{m}(\boldsymbol{z}):=\left(\sum_1^K\frac{\boldsymbol{m}_i^{\mathrm{s}}}{i}\mu_i(|\boldsymbol{z}|)\right)z,\quad \text{for } \boldsymbol{z}\in\mathbb{R}^n.$$

Define for every $\boldsymbol{u}\in\mathrm{SBV}(\Omega;\mathbb{R}^n)$

$$\mathcal{E}(\boldsymbol{u})=\int_\Omega[a(|\boldsymbol{u}|)|\nabla\boldsymbol{u}|^2+\phi(\boldsymbol{u})-\boldsymbol{f}\cdot\boldsymbol{m}(\boldsymbol{u})-\tilde{\boldsymbol{h}}[\boldsymbol{u}]\cdot\boldsymbol{m}(\boldsymbol{u})]\mathrm{d}x+\mathcal{H}^{n-1}(J_{\boldsymbol{u}}),$$

and consider the set of admissible functions

$$\mathcal{A}:=\{\boldsymbol{u}\in\mathrm{SBV}(\Omega;\mathbb{R}^n):|\boldsymbol{u}|=1,\dots,K \text{ a.e. in } \Omega\}.$$

The optimal design problem consists in finding a partition of Ω into K open sets Ω_i, a $(n-1)$-dimensional (relatively) closed set C, and a function $\boldsymbol{u}:\Omega\to\mathbb{R}^n$, such that $\mathcal{H}^{n-1}(C)<+\infty$, $|\boldsymbol{u}|\equiv i$ in each Ω_i, $\boldsymbol{u}\in W^{1,2}(\Omega\backslash C;\mathbb{R}^n)$ and $\boldsymbol{u}$ minimizes

$$\mathcal{OPT}(\boldsymbol{v},\{\Omega_i\},C)=\int_{\Omega\backslash C}[a(|\boldsymbol{v}|)|\nabla\boldsymbol{v}|^2+\phi(\boldsymbol{v})-\boldsymbol{f}\cdot\boldsymbol{m}(\boldsymbol{v})-\tilde{\boldsymbol{h}}[\boldsymbol{v}]\cdot\boldsymbol{m}(\boldsymbol{v})]\mathrm{d}x+\mathcal{H}^{n-1}(C)$$

among all possible choices of $(\boldsymbol{v},\{\Omega_i\},C)$ satisfying the constraints above.

We first relax the problem to that of finding a minimizer $\boldsymbol{u}\in\mathcal{A}$ of the functional $\mathcal{E}$; we prove an existence result, and then we show using a regularity argument that a solution to the original optimal design problem actually exists. Precisely,

Theorem 4.1. *Assume that* H1, …, H6 *hold, and let* $\boldsymbol{m},\mathcal{E},\mathcal{A}$ *be defined as above. There exists* $\boldsymbol{u}\in\mathcal{A}$*, such that*

$$\mathcal{E}(\boldsymbol{u})\le\mathcal{E}(\boldsymbol{v}),\quad \textit{for all } \boldsymbol{v}\in\mathcal{A}.$$

Theorem 4.2. *Assume that* H1, …, H7 *hold, let* $\boldsymbol{u}\in\mathcal{A}$ *be the minimizer given by theorem 4.1 and set* $C=\Omega\cap\overline{J_{\boldsymbol{u}}}$. *Then* $\mathcal{H}^{n-1}(C\backslash J_{\boldsymbol{u}})=0$*, the function* $\boldsymbol{u}$ *is in* $W^{1,2}(\Omega\backslash C;\mathbb{R}^n)$*, and its modulus takes only the values* $1,\dots,K$ *in* $\Omega\backslash C$*. Setting* $\Omega_i:=\{x\in\Omega\backslash C:|\boldsymbol{u}(x)|=i\}$*, the triple* $(\boldsymbol{u},\{\Omega_i\},C)$ *minimizes* $\mathcal{OPT}$*. Finally, the function* $\boldsymbol{u}$ *is locally Hölder continuous in each connected component of* $\Omega\backslash C$*, except for a locally finite set of points.*

Now that the results have been stated, we make a typographic simplification, dropping the boldface vectorial notation and reverting to the standard one, thus $u,f,m,\tilde{h},z$ will appear in place of their boldface equivalents. We also remark that by our requirements on the admissible functions, we may assume without loss of generality that for some $L>0$

$$\frac{1}{L}\leq a(t)\leq L,\quad 0\leq \phi(z)\leq L,\quad \frac{1}{L}\leq m_i^{\mathrm{s}}\leq L. \tag{4.2}$$

We now prove theorem 4.1.

Proof. Clearly, the set $\mathcal{A}$ is not empty. Let $\{u_h\}$ be a sequence in $\mathcal{A}$, such that

$$\mathcal{E}(u_h)\to \inf_{\mathcal{A}} \mathcal{E}.$$

Denoting by c any constant depending only on n, L, K, f, Ω, and whose value may vary from line to line and expression to expression within a line, and remarking that $\|u_h\|_\infty \leq K$, we have

$$\int_\Omega |\nabla u_h|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_{u_h}) \leq c\mathcal{E}(u_h) + c\|f\|_1 + c \leq c. \tag{4.3}$$

We apply the compactness theorem 4.8 of Ambrosio *et al.* (2000) to obtain that, up to a subsequence, $u_h \to u \in \mathrm{SBV}(\Omega;\mathbb{R}^n)$ in the following sense:

$$u_h \to u \text{ a.e. and in every } L^p, p<+\infty, \tag{4.4}$$

$$\nabla u_h \rightharpoonup \nabla u \text{ weakly in } L^1,$$

$$\mathcal{H}^{n-1}(\omega\cap J_u)\leq \liminf \mathcal{H}^{n-1}(\omega\cap J_{u_h}), \tag{4.5}$$

$$\int_\omega |\nabla u|^2 \mathrm{d}x \leq \liminf \int_\omega |\nabla u_h|^2 \mathrm{d}x, \tag{4.6}$$

where (4.5) and (4.6) hold for every $\omega \subseteq \Omega$. Since $|u_h(x)|\in\{1,\dots,K\}$ a.e., (4.4) implies that $u\in\mathcal{A}$. Moreover, if we set

$$\Omega_i^h := \{x\in\Omega : |u_h(x)| = i\},\quad \Omega_i := \{x\in\Omega : |u(x)| = i\}$$

(where it is understood that we are using the precise representatives here and elsewhere), we have, recalling (3.3) and (4.1),

$$\mathbb{1}_{\Omega_i^h}(x) = \mu_i(|u_h(x)|) \to \mathbb{1}_{\Omega_i}(x) \tag{4.7}$$

strongly in every L^p. Since clearly,

$$v\mapsto \int_\Omega [\phi(v) - f\cdot m(v) - \tilde{h}[v]\cdot m(v)]\mathrm{d}x$$

is continuous in the L^2 topology on L^∞, we deduce that

$$\int_\Omega [\phi(u_h) - f\cdot m(u_h) - \tilde{h}[u_h]\cdot m(u_h)]\mathrm{d}x \to \int_\Omega [\phi(u) - f\cdot m(u) - \tilde{h}[u]\cdot m(u)]\mathrm{d}x. \tag{4.8}$$

The last term in $\mathcal{E}$ is dealt with using (4.5), which yields

$$\mathcal{H}^{n-1}(J_u)\leq \liminf \mathcal{H}^{n-1}(J_{u_h}). \tag{4.9}$$

By (4.2), (4.3) and (4.6), we have $a(|u|)\nabla u \in L^2(\Omega)$, and so given $\varepsilon > 0$ we may find $\delta > 0$ so small that

$$\int_{\Omega \setminus \omega} a(|u|)|\nabla u|^2 \mathrm{d}x < \varepsilon, \tag{4.10}$$

whenever $|\Omega \setminus \omega| < \delta$. By (4.7), and again up to subsequences, the characteristic functions converge quasi-uniformly, i.e. if $\delta > 0$ is chosen as above, then there exists $\omega_\delta \subset \Omega$, such that $|\Omega \setminus \omega_\delta| < \delta$ and

$$\mathbb{1}_{\Omega_i^h} \to \mathbb{1}_{\Omega_i}, \quad \text{uniformly in } \omega_\delta \text{ for } i = 1, \ldots, K.$$

However, for characteristic functions, uniform convergence reduces to equality (for large h), thus

$$\Omega_i^h \cap \omega_\delta = \Omega_i \cap \omega_\delta.$$

Then, $|u_h(x)| = |u(x)|$ in ω_δ and we have

$$\int_\Omega a(|u_h|)|\nabla u_h|^2 \mathrm{d}x \geq \int_{\omega_\delta} a(|u|)|\nabla u_h|^2 \mathrm{d}x = \sum_{i=1}^K a(i) \int_{\Omega_i \cap \omega_\delta} |\nabla u_h|^2 \mathrm{d}x,$$

and (4.6) implies

$$\liminf \int_\Omega a(|u_h|)|\nabla u_h|^2 \mathrm{d}x \geq \sum_{i=1}^K a(i) \int_{\Omega_i \cap \omega_\delta} |\nabla u|^2 \mathrm{d}x$$

$$= \int_{\omega_\delta} a(|u|)|\nabla u|^2 \mathrm{d}x \geq \int_\Omega a(|u|)|\nabla u|^2 \mathrm{d}x - \varepsilon$$

by (4.10). This, together with (4.8) and (4.9), yields

$$\mathcal{E}(u) \leq \liminf \mathcal{E}(u_h) = \inf_{\mathcal{A}} \mathcal{E},$$

which concludes the proof. ■

Remark 4.3. The existence theorem we just proved holds under more general conditions. The exponent 2 plays no special role and may be replaced by any $p > 1$, and the surface term may include an appropriate jointly convex function, in which case we would apply the lower semi-continuity theorem 5.22 in Ambrosio *et al.* (2000). We also remark that the latter part of the proof could have been supplied by Ioffe's theorem, but we preferred to show here how we take advantage of the special partition-like structure.

Remark 4.4. Since the characteristic functions of the sets Ω_i^h converge in L^1, in particular $|\Omega_i^h| \to |\Omega_i|$ for all i. Thus, if we modify the optimal design problem to account for fixed volume fractions, i.e. if we take K non-negative numbers α_i, such that $\alpha_1 + \cdots + \alpha_K = |\Omega|$, and if we consider as admissible only the functions in the set

$$\mathcal{A}_{\alpha_1, \ldots, \alpha_K} = \{u \in \mathrm{SBV}(\Omega; \mathbb{R}^n) : |\{x : |u(x)| = i\}| = \alpha_i\},$$

then the limit u of a minimizing sequence is still in the same class, and so also the fixed volume fractions optimal design problem has a solution.

5. A road map to theorem 4.2

In theorem 4.2, the only assertion to be proved is that

$$\mathcal{H}^{n-1}(\Omega \cap \overline{J_u} \backslash J_u) = 0. \tag{5.1}$$

Indeed, this will immediately entail that $\boldsymbol{u} \in W^{1,2}(\Omega \backslash C; \mathbb{R}^n)$, $|u(x)| \in \{1, \ldots, K\}$ in $\Omega \backslash C$, and $(\boldsymbol{u}, \{\Omega_i\}, C)$ minimizes $\mathcal{OPT}$, where $\Omega_i := \{x \in \Omega \backslash C : |\boldsymbol{u}(x)| = i\}$. It now suffices to apply corollary 5.2 of Hardt & Kinderlehrer (2000) to any ball contained in a single connected component of the open set $\Omega \backslash C$: indeed, in such a ball u also minimizes the standard micromagnetic energy considered in Hardt & Kinderlehrer (2000), and we may conclude that $\boldsymbol{u}$ is locally Hölder continuous in each connected component of $\Omega \backslash C$ except for a locally finite set of points.

Estimate (5.1) has been treated in full detail in the unconstrained case (see ch. 7 of Ambrosio *et al.* 2000) and in the case of a single constraint (i.e. $K=1$, see §§3 and 4 of Carriero & Leaci 1991). Here, we will not reproduce those parts in the proof which reduce to obvious adaptations of those in Carriero & Leaci (1991) and Ambrosio *et al.* (2000). Instead, we will sketch the proof, highlighting the points where departing from the existing results needs an explanation, and finally we will prove a decay lemma.

As we will frequently refer to results in ch. 7 of Ambrosio *et al.* (2000) (all quoted as 7.x) and in §§3 and 4 of Carriero & Leaci (1991), we will abbreviate the quotations to read 7.x and 3.x, 4.x, respectively. The key to (5.1) is the density lower bound (theorem 7.21 or lemma 4.9), stating that if u is a minimizer (and in a more general context, see definition 5.1), then in every sufficiently small ball $B_\varrho(x) \subset \Omega$ whose centre is in $\overline{J_u}$, the amount of jump set is not too small, i.e.

$$\mathcal{H}^{n-1}(B_\varrho(x) \cap J_u) \geq \theta_0 \varrho^{n-1} \tag{5.2}$$

for some $\theta_0 > 0$. This implies, in particular, that the $\mathcal{H}^{n-1}$-density of J_u at all points in $\overline{J_u}$ is not zero. However, a standard measure theoretic result ensures that this density is zero $\mathcal{H}^{n-1}$-a.e. outside J_u, and thus (5.2) implies (5.1). We must therefore concentrate on establishing the density lower bound (5.2).

In the standard non-constrained case, this is proved via a blow-up and comparison methods, considering a sequence of balls with vanishing radii, rescaling the balls to the same radius, say 1, and comparing the rescaled minimizer in each ball with suitable modifications of the same function. This requires some ingredients which we introduce now. Since bulk and surface terms rescale with different powers of the radius, one is forced to considering not only the functional

$$(\text{bulk part}) + \mathcal{H}^{n-1}(J_u),$$

i.e. with 'one' times the area of the jump, but also more generally

$$(\text{bulk part}) + c\mathcal{H}^{n-1}(J_u),$$

with any $c>0$. When rescaling a function in a ball to have radius 1, we also have the choice of either leaving the integral of $|\nabla u|^2$ intact or leaving the values taken by u intact, but not both. Thus, in the constrained case, either we change the main contribution in the bulk energy, or we take into account that the rescaled functions will satisfy not the original constraints, but a rescaled version of the constraints, say

$$|u(x)| \in \{t, 2t, \ldots, Kt\} \text{ a.e.,}$$

instead of $|u(x)| \in \{1, 2, \ldots, K\}$ a.e.

Now, we define for every $c, t > 0$, every Borel set $E \subset \Omega$, and every $u \in \mathrm{SBV}(\Omega; \mathbb{R}^n)$, such that $|\nabla u| \in L^2(\Omega)$,

$$F(u, c, E, t) := \int_E a\left(\frac{|u|}{t}\right)|\nabla u|^2 \mathrm{d}x + c\mathcal{H}^{n-1}(J_u \cap E), \quad \text{if } |u(x)| \in \{t, \ldots, Kt\} \text{ a.e. in } E.$$

In situations where the exact values of c and t are irrelevant, we omit the dependence of F on these parameters and we use the simplified notation

$$F(u, E) := \int_E a(|u|)|\nabla u|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_u \cap E), \quad \text{if } |u(x)| \in \{1, \ldots, K\} \text{ a.e. in } E. \tag{5.3}$$

Another useful tool is the deviation from minimality, which measures how far a function u is from being a minimizer,

$$\mathrm{Dev}(u, c, E, t) := F(u, c, E, t) - \inf\{F(v, c, E, t) : \{v \neq u\} \subset\subset E\}.$$

Clearly, the deviation from minimality is zero if and only if u is a minimizer on E. We apply to Dev the same convention regarding the meaning of the abbreviated $\mathrm{Dev}(u, E)$. The next definition, see Ambrosio *et al.* (2000), is crucial to link the energy F to our full energy $\mathcal{E}$.

Definition 5.1. Let $0 \leq \nu < 1$. A function $u \in \mathrm{SBV}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ is a ν-quasi-minimizer of F if there exist a constant $\kappa \geq 0$ and a radius $\varrho_0 > 0$, such that for every ball $B_\varrho \subset \Omega$ with $\varrho \leq \varrho_0$,

$$\mathrm{Dev}(u, B_\varrho) \leq \kappa \varrho^{n-\nu}.$$

If $\nu = 0$, we simply say u is a quasi-minimizer.

Proposition 5.2. *A minimizer of $\mathcal{E}$ is a ν-quasi-minimizer of $F(\cdot, \Omega)$ with $\nu = n/q$.*

Proof. It is enough to remark that if u is a minimizer of $\mathcal{E}$ and $\{v \neq u\} \subset\subset B_\varrho \subset \Omega$, then we may write the inequality $\mathcal{E}(u) \leq \mathcal{E}(v)$ as

$$F(u, B_\varrho) - F(v, B_\varrho) \leq \int_{B_\varrho} \phi(v) - \phi(u) + f \cdot (m(u) - m(v)) \mathrm{d}x + \int_\Omega \tilde{h}[u] \cdot m(u) - \tilde{h}[v] \cdot m(v) \mathrm{d}x.$$

Now H4 and (4.2) yield, as in (2.3),

$$\int_{B_\varrho} |\phi(v) - \phi(u)| \mathrm{d}x \leq c(L)\varrho^n,$$

from H2, H7 and (4.2) we deduce that

$$\int_{B_\varrho} |f \cdot (m(u) - m(v))| \mathrm{d}x \leq c(L, q)\|f\|_q \varrho^{n/q'},$$

and finally H5 with the same q as in H7 gives by (2.2)

$$\left|\int_{\Omega}\tilde{h}[u]\cdot m(u)-\tilde{h}[v]\cdot m(v)\mathrm{d}x\right|\leq c(L,q)\varrho^{n/q'}.$$

Collecting these estimates we have (for $\varrho\leq 1$)

$$F(u,B_{\varrho})-F(v,B_{\varrho})\leq c\varrho^{n/q'},$$

thus

$$\mathrm{Dev}(u,B_{\varrho})\leq c\varrho^{n-n/q}.$$

■

Remark 5.3. We note that the integrability condition $q>n$ required in H7 is needed to ensure that $\nu:=n/q\in(0,1)$ will conform with definition 5.1.

We now focus on proving the density lower bound (5.2) for ν-quasi-minimizers of F. We resume our road map. The proof of (5.2) for quasi-minimizers is largely independent on the particular features of the functional (see theorem 7.21 and lemma 4.9), and it rests solely on an energy upper bound (lemma 7.19, which is obviously true in our situation) and on a decay lemma. In fact, Ambrosio *et al.* (2000)only deals with quasi-minimizers, but the proof for ν-quasi-minimizers is entirely identical, and exhibiting the same technical difficulties (see Ambrosio *et al.* 1997; Ambrosio & Pallara 1997).

This decay lemma (lemma 7.14 and lemma 3.9) states that if in a certain ball the amount of jump set (in the $\mathcal{H}^{n-1}$ sense) is small and the deviation from minimality is also small (compared with the size of the functional), then on smaller concentric balls the value of F decays as a power of the radius. Its proof in turn is based on an auxiliary lemma (theorem 7.7 and theorem 3.6 with corollary 3.7) which is the true keypoint. Its statement is exactly what comes out when trying to prove the decay lemma by contradiction.

The one big difference from the unconstrained to the constrained case is the limitation in the comparison methods. In the unconstrained case, one may compare the minimizer u with suitable modifications of u in two ways: either by cutting away a part of u and replacing it with any function (this is permitted in BV, and clearly makes no harm if image constraints are added), or by patching the two via a smooth cut-off function. In the single-constraint case $|u|=1$, the latter method produces a function whose image no longer lies on the boundary of the unit ball, but with care (and using the fact that this has to be done only when the amount of jump set is small); the key lemma 3.5 provides a projection on ∂B_1 of the comparison function (which must be kept not too far from the correct value) which satisfies a Poincaré–Wirtinger inequality.

This road is not allowed in our case, because the image is not on ∂B_1 but on the union of many such spheres, and there is no way to be sure that the comparison function is always not too far from a single allowed value.

The remaining of this paper is dedicated to the proof of the decay lemma which in our situation reads

Lemma 5.4. *Let* $1/L<a(i)<L$ for $i=1,\dots,K$, *and let* F *be defined as in (5.3). There is a constant* C_0, *depending on* (n,L), *such that for every* $0<\tau<1$ *there* *exist* $\varepsilon, \theta > 0$, *both depending on* τ, *such that if* $B_\varrho(x) \subset \Omega$, $F(v, \overline{B_\varrho}) < +\infty$ *and if*

$$\mathcal{H}^{n-1}(J_v, \overline{B_\varrho}) \leq \varepsilon \varrho^{n-1}, \quad \mathrm{Dev}(v, \overline{B_\varrho}) \leq \theta F(v, \overline{B_\varrho}),$$

then

$$F(v, \overline{B_{\tau\varrho}}) \leq C_0 \tau^n F(v, \overline{B_\varrho}). \tag{5.4}$$

Before proceeding with the proof, we revisit lemma 3.9 in Carriero & Leaci (1991) for the decay estimate in the case of a single constraint. We set

$$F_1(w, E) = \int_E a(1) |\nabla w|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_w \cap E), \quad \text{if } |w| = 1 \text{ a.e.},$$

and we call Dev_1 the deviation from minimality relative to F_1. We claim that an argument entirely similar to that of the proof of lemma 3.9 in Carriero & Leaci (1991) entails the validity of the statement of lemma 5.4 with F_1 and D_1 in place of F and D, respectively. Indeed, combining (3.9) in Carriero & Leaci (1991) with the inequality that is found two lines before the end of that proof, we deduce that

$$F_1(v, \overline{B_{\tau\varrho}}) \leq C\tau^n F_1(v, \overline{B_\varrho}), \tag{5.5}$$

provided $C > 2^{p-1} c_0(\omega_n + 1)$, where ω_n is the $\mathcal{H}^{n-1}$ measure of the unit sphere in $\mathbb{R}^n$, and $c_0 = c_0(n, p)$ satisfies (see the theorem on p. 244 in Tolksdorff (1973)

$$\|\nabla u\|^p_{L^\infty(B(x, 2r/3); \mathbb{R}^n)} \leq \frac{c_0}{\omega_n r^n} \int_{B(x,r)} (1 + |\nabla u(y)|^p) \mathrm{d}y,$$

for all $B(x, r) \subset \Omega$ and all p-harmonic function $u \in W^{1,p}(\Omega; \mathbb{R}^n)$.

Proof. Let $C_0 := C + 1$, where C is the constant in (5.5). We argue by contradiction, assuming that for a certain τ there exist a sequence of balls B_{ϱ_h} and a sequence of functions $v_h \in \mathrm{SBV}(\Omega)$, such that

$$\frac{\mathcal{H}^{n-1}(J_{v_h} \cap \overline{B_{\varrho_h}})}{\varrho_h^{n-1}} \to 0, \quad \frac{\mathrm{Dev}(v_h, \overline{B_{\varrho_h}})}{F(v_h, \overline{B_{\varrho_h}})} \to 0,$$

but

$$F(v_h, \overline{B_{\tau\varrho_h}}) > C_0 \tau^n F(v_h, \overline{B_{\varrho_h}}).$$

Note that the latter justifies the division by $F(v_h, \overline{B_{\varrho_h}})$ in the line before. In order to set the relevant quantities for rescaling, we define

$$\varepsilon_h^2 := \varrho_h^{1-n} \mathcal{H}^{n-1}(J_{v_h} \cap \overline{B_{\varrho_h}}), \quad \theta_h := \frac{\mathrm{Dev}(v_h, \overline{B_{\varrho_h}})}{F(v_h, \overline{B_{\varrho_h}})},$$

and also

$$\sigma_h := \varrho_h^{n-1} / F(v_h, \overline{B_{\varrho_h}}), \quad t_h := \sqrt{\sigma_h / \varrho_h}.$$

Remark that in view of (5.3), $\varepsilon_h^2 \sigma_h \leq 1$ and thus

$$\theta_h' := \sigma_h \varepsilon_h^{2+1/(n-1)} \to 0.$$

Note also that

$$\varepsilon_h \to 0, \quad \theta_h \to 0, \quad t_h \to +\infty.$$

The latter holds because $F(v_h, \overline{B_{\varrho_h}})$ is bounded from above by a constant times ϱ_h^{n-1}. Indeed, v_h is 'almost' a minimizer in the sense that

$$(1-\theta_h)F(v_h, \overline{B_{\varrho_h}}) = F(v_h, \overline{B_{\varrho_h}}) - \mathrm{Dev}(v_h, \overline{B_{\varrho_h}})$$

$$= \inf\{F(v, \overline{B_{\varrho_h}}) : \{v \neq v_h\} \subset\subset B_{\varrho_h}\} \leq c\varrho_h^{n-1},$$

where in the last inequality, we used as test functions for $k \in \mathbb{N}$,

$$\bar{v}_k(x) := \begin{cases} \boldsymbol{e}_1, & \text{if } x \in B_{\varrho_h - 1/k} \\ v_h, & \text{if } x \in B_{\varrho_h} \setminus B_{\varrho_h - 1/k}, \end{cases}$$

where $\boldsymbol{e}_1$ is the first vector in the canonical orthonormal basis of $\mathbb{R}^n$, and we let $k \to +\infty$.

With x_h the centre of B_{ϱ_h}, set

$$u_h(x) := t_h v_h(x_h + \varrho_h x).$$

Clearly, u_h is defined in the unit ball B_1 and

$$\mathcal{H}^{n-1}(J_{u_h} \cap \overline{B_1}) = \varepsilon_h^2, \tag{5.6}$$

$$F(u_h, \boldsymbol{\sigma}_h, \overline{B_1}, t_h) = 1, \tag{5.7}$$

$$\mathrm{Dev}(u_h, \boldsymbol{\sigma}_h, \overline{B_1}, t_h) = \theta_h, \tag{5.8}$$

$$F(u_h, \boldsymbol{\sigma}_h, \overline{B_\tau}, t_h) > C_0 \tau^n. \tag{5.9}$$

In particular, as $J_{|u|} \subset J_u$, and since $J_{|u|}$ contains the sets, where $|u|$ jumps from one to another of the permitted values, we have

$$\text{for all } 1 \leq i \leq K, \quad \mathcal{H}^{n-1}(\partial\{|u_h| = it_h\} \cap B_1) \leq \varepsilon_h^2 \to 0,$$

so by the isoperimetric inequality (3.43) of Ambrosio *et al.* (2000), denoting by γ the isoperimetric constant for balls, we have that for all i one of the two sets

$$P_h^i = \{x \in B_1 : |u_h(x)| = it_h\}, \quad Q_h^i = \{x \in B_1 : |u_h(x)| \neq it_h\}$$

has n-dimensional measure not exceeding $\gamma \varepsilon_h^{2n/(n-1)}$. We claim that for h large there exist $i \in \{1, \ldots, K\}$, such that the measure of Q_h^i does not exceed $\gamma \varepsilon_h^{2n/(n-1)}$. Indeed, if this was not the case, then for all $i \in \{1, \ldots, K\}$ and for a subsequence (not relabelled)

$$|P_h^i| \leq \gamma \varepsilon_h^{2n/(n-1)},$$

and thus

$$|B_1| = |\cup_{i=1}^K P_h^i| \leq K\gamma \varepsilon_h^{2n/(n-1)},$$

what is clearly impossible since ε_h converge to zero. This asserts the claim. As the i's range over a finite set, we may assume (up to the extraction of a subsequence, not relabelled) that this happens always with the same i, say $i=1$, thus for all h

$$|\{x \in B_1 : |u_h(x)| \neq t_h\}| \leq \gamma \varepsilon_h^{2n/(n-1)}.$$

In particular, for any $0<R_h<1$,

$$\gamma\varepsilon_h^{2n/(n-1)} \geq \int_{R_h}^1 \mathcal{H}^{n-1}(\{|u_h| \neq t_h\} \cap \partial B_r)\mathrm{d}r,$$

thus for a suitable $r_h \in (R_h, 1)$,

$$\mathcal{H}^{n-1}(\{|u_h| \neq t_h\} \cap \partial B_{r_h}) \leq \frac{\gamma\varepsilon_h^{2n/(n-1)}}{1-R_h}, \tag{5.10}$$

and also (this happens for a.e. r_h)

$$\mathcal{H}^{n-1}(J_{u_h} \cap \partial B_{r_h}) = 0.$$

Choose R_h, such that

$$1-R_h = \gamma\varepsilon_h^{1/(n-1)},$$

and remark that $R_h \to 1$ and that the inequality (5.10) above reduces to

$$\mathcal{H}^{n-1}(\{|u_h| \neq t_h\} \cap \partial B_{r_h}) \leq \varepsilon_h^{2+1/(n-1)}. \tag{5.11}$$

We change the function u_h in B_{r_h} to get rid of all values of $|u_h|$ different from t_h by setting

$$w_h(x) := \begin{cases} u_h(x), & \text{if } |x| > r_h \\ u_h(x), & \text{if } |x| \leq r_h \text{ and } |u_h(x)| = t_h \\ t_h\boldsymbol{e}, & \text{otherwise,} \end{cases}$$

where $\boldsymbol{e}$ is any unit vector.

We claim that

$$F(w_h, \sigma_h, \overline{B_\tau}, t_h) \geq C_0\tau^n - 2\theta_h - 2\theta_h'. \tag{5.12}$$

First note that in the interior of the ball B_{r_h} we added no jump set because all possible jump points of w_h were already jump points of $|u_h|$, thus we may have added jump points only on the boundary of B_{r_h}. Hence for any $E \subseteq \overline{B_1}$,

$$J_{w_h} \cap E \subseteq (J_{u_h} \cap E) \cup (\{|u_h| \neq t_h\} \cap \partial B_{r_h}),$$

so by (5.6) and (5.11),

$$\mathcal{H}^{n-1}(J_{w_h} \cap \overline{B_{r_h}}) \leq \varepsilon_h^2(1+\varepsilon_h^{1/(n-1)}), \tag{5.13}$$

$$\sigma_h\mathcal{H}^{n-1}(J_{w_h} \cap E) \leq \sigma_h\mathcal{H}^{n-1}(J_{u_h} \cap E) + \theta_h'. \tag{5.14}$$

Below we use the bulk part of F, so we define

$$f(u, E, t) := \int_E a\left(\frac{|u|}{t}\right)|\nabla u|^2\mathrm{d}x, \quad \text{if } |u| \in \{t, 2t, \dots, Kt\} \text{ a.e.}$$

Clearly,

$$f(w_h, E, t_h) \leq f(u_h, E, t_h), \quad \text{for any } E \subset B_1. \tag{5.15}$$

Since $\{u_h \neq w_h\} \subset\subset B_1$, we have by (5.8)

$$F(u_h, \sigma_h, \overline{B_1}, t_h) \leq F(w_h, \sigma_h, \overline{B_1}, t_h) + \theta_h,$$

which implies by (5.14) and (5.15)

$$\begin{aligned} 0 &\leq f(u_h, B_1, t_h) - f(w_h, B_1, t_h) \\ &\leq \sigma_h \mathcal{H}^{n-1}(J_{w_h} \cap \overline{B_1}) - \sigma_h \mathcal{H}^{n-1}(J_{u_h} \cap \overline{B_1}) + \theta_h \\ &\leq \theta_h + \theta'_h. \end{aligned}$$

Therefore,

$$\sigma_h \mathcal{H}^{n-1}(J_{w_h} \cap \overline{B_1}) + \theta_h \geq \sigma_h \mathcal{H}^{n-1}(J_{u_h} \cap \overline{B_1}),$$

and using this inequality together with (5.14), we obtain

$$\begin{aligned} \sigma_h \mathcal{H}^{n-1}(J_{w_h} \cap \overline{B_\tau}) &= \sigma_h \mathcal{H}^{n-1}(J_{w_h} \cap \overline{B_1}) \\ &\quad - \sigma_h \mathcal{H}^{n-1}(J_{w_h} \cap (\overline{B_1} \setminus \overline{B_\tau})) \\ &\geq \sigma_h \mathcal{H}^{n-1}(J_{u_h} \cap \overline{B_1}) \\ &\quad - \theta_h - \sigma_h \mathcal{H}^{n-1}(J_{w_h} \cap (\overline{B_1} \setminus \overline{B_\tau})) \\ &\geq \sigma_h \mathcal{H}^{n-1}(J_{u_h} \cap \overline{B_1}) \\ &\quad - \theta_h - \sigma_h \mathcal{H}^{n-1}(J_{u_h} \cap (\overline{B_1} \setminus \overline{B_\tau})) - \theta'_h \\ &= \sigma_h \mathcal{H}^{n-1}(J_{u_h} \cap \overline{B_\tau}) - \theta_h - \theta'_h. \end{aligned} \tag{5.16}$$

Moreover, by (5.15) we may split the left-hand side of the inequality

$$f(u_h, B_1, t_h) - f(w_h, B_1, t_h) \leq \theta_h + \theta'_h$$

into two non-negative terms (the integrals inside and outside B_τ), and we deduce that

$$f(w_h, B_\tau, t_h) \geq f(u_h, B_\tau, t_h) - \theta_h - \theta'_h.$$

This inequality, (5.9) and (5.16), imply that

$$F(w_h, \sigma_h, \overline{B_\tau}, t_h) \geq F(u_h, \sigma_h, \overline{B_\tau}, t_h) - 2\theta_h - 2\theta'_h \geq C_0 \tau^n - 2\theta_h - 2\theta'_h,$$

and the claim is asserted.

Remark that $r_h \to 1$, so that with no loss of generality we may suppose $r_h \geq \tau$. Using (5.12), we get

$$F(w_h, \sigma_h, \overline{B_{r_h}}, t_h) \geq F(w_h, \sigma_h, \overline{B_\tau}, t_h) \geq C_0 \tau^n - 2\theta_h - 2\theta'_h. \tag{5.17}$$

Since $u_h = w_h$ in an outer annulus, if $\{v \neq w_h\} \subset\subset B_1$, then also $\{v \neq u_h\} \subset\subset B_1$, and so using (5.8), (5.14) and (5.15), we have

$$F(w_h, \sigma_h, \overline{B_1}, t_h) \leq F(u_h, \sigma_h, \overline{B_1}, t_h) + \theta'_h \leq F(v, \sigma_h, \overline{B_1}, t_h) + \theta_h + \theta'_h, \tag{5.18}$$

yielding

$$\mathrm{Dev}(w_h, \sigma_h, \overline{B_1}, t_h) \leq \theta_h + \theta'_h.$$

On the other hand, omitting in the following equation σ_h and t_h,

$$F(w_h,\overline{B_{r_h}})-\inf\{F(v,\overline{B_{r_h}}):\{v\neq w_h\}\subset\subset B_{r_h}\}$$

$$=F(w_h,\overline{B_1})-\inf\{F(v,\overline{B_1}):\{v\neq w_h\}\subset\subset B_{r_h}\}$$

$$\leq F(w_h,\overline{B_1})-\inf\{F(v,\overline{B_1}):\{v\neq w_h\}\subset\subset B_1\},$$

and so

$$\mathrm{Dev}(w_h,\sigma_h,\overline{B_{r_h}},t_h)\leq \mathrm{Dev}(w_h,\sigma_h,\overline{B_1},t_h)\leq\theta_h+\theta_h'. \tag{5.19}$$

Also, using (5.7), (5.14) and (5.18), we may write

$$F(w_h,\sigma_h,\overline{B_{r_h}},t_h)\leq F(w_h,\sigma_h,\overline{B_1},t_h)\leq F(u_h,\sigma_h,\overline{B_1},t_h)+\theta_h'=1+\theta_h'. \tag{5.20}$$

Collecting (5.9), (5.12) (5.13), (5.17), (5.19) and (5.20), we have for h large enough

$$\mathcal{H}^{n-1}(J_{w_h}\cap\overline{B_{r_h}})\leq\varepsilon_h^2(1+\varepsilon_h^{1/(n-1)}),\quad \frac{F(w_h,\sigma_h,\overline{B_{r_h}},t_h)}{1+\theta_h'}\leq 1\leq\frac{F(w_h,\sigma_h,\overline{B_{r_h}},t_h)}{C_0\tau^n-2\theta_h-2\theta_h'},$$

$$\mathrm{Dev}(w_h,\sigma_h,\overline{B_{r_h}},t_h)\leq\theta_h+\theta_h'\leq\frac{\theta_h+\theta_h'}{C_0\tau^n-2\theta_h-2\theta_h'}F(w_h,\sigma_h,\overline{B_{r_h}},t_h),$$

$$F(w_h,\sigma_h,\overline{B_\tau},t_h)>C_0\tau^n-2\theta_h-2\theta_h'\geq\frac{C_0\tau^n-2\theta_h-2\theta_h'}{1+\theta_h'}F(w_h,\sigma_h,\overline{B_{r_h}},t_h).$$

Setting

$$\tau_h:=\tau/r_h,$$

the last line above may be rewritten as

$$F(w_h,\sigma_h,\overline{B_{\tau_h r_h}},t_h)>\frac{C_0r_h^n\tau_h^n-2\theta_h-2\theta_h'}{1+\theta_h'}F(w_h,\sigma_h,\overline{B_{r_h}},t_h).$$

Recalling that

$$\varepsilon_h\to 0,\quad \theta_h\to 0,\quad \theta_h'\to 0,\quad r_h\to 1,\quad \tau_h\to\tau,$$

and performing backwards the change of variables we made to obtain u_h from v_h, we get a new sequence of functions v_h', each defined in the ball of radius $\varrho_h':=r_h\varrho_h$, and such that (recall that τ is fixed, so it does no harm in the inequality relative to the deviation from minimality)

$$\mathcal{H}^{n-1}(J_{v_h'}\cap\overline{B_{\varrho_h'}})\leq\omega_h(\varrho_h')^{n-1},\quad \mathrm{Dev}(v_h',\overline{B_{\varrho_h'}})\leq\omega_h F(v_h',\overline{B_{\varrho_h'}})$$

and

$$F(v_h',\overline{B_{\tau_h\varrho_h'}})>(C_0-\omega_h)\tau_h^n F(v_h',\overline{B_{\varrho_h'}}), \tag{5.21}$$

with $\omega_h\to 0$. But now

$$|v_h'|=1\text{ a.e.},$$

therefore if we set

$$F_1(w,E)=\int_E a(1)|\nabla w|^2\mathrm{d}x+\mathcal{H}^{n-1}(J_w\cap E),\quad \text{if } |w|=1\text{ a.e.},$$

and we call Dev_1 the deviation from minimality relative to F_1, clearly

$$\mathrm{Dev}_1(v'_h, \overline{B_{\varrho'_h}}) \le \mathrm{Dev}(v'_h, \overline{B_{\varrho'_h}}) \le \omega_h F_1(v'_h, \overline{B_{\varrho'_h}}),$$

and by (5.21)

$$F_1(v'_h, \overline{B_{\tau_h \varrho'_h}}) > (C_0 - \omega_h)\tau_h^n F_1(v'_h, \overline{B_{\varrho'_h}}).$$

But for h large enough this violates the decay estimate (5.5), and we reached a contradiction. ■

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