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Regularity Results for a Class of Functionals with Non-Standard Growth

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Abstract

We consider the integral functional

$$\int f(x, Du)\, dx$$

under non-standard growth assumptions that we call $p(x)$ type: namely, we assume that

$$|z|^{p(x)} \leqq f(x, z) \leqq L(1 + |z|^{p(x)}),$$

a relevant model case being the functional

$$\int |Du|^{p(x)}\, dx.$$

Under sharp assumptions on the continuous function $p(x) > 1$ we prove regularity of minimizers. Energies exhibiting this growth appear in several models from mathematical physics.

1. Introduction

The aim of this paper is the study of the regularity properties of (local) minimizers of integral functionals of the type

$$\mathcal{F}(u, \Omega) := \int_\Omega f(x, Du)\, dx,$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $f : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Carathéodory integrand and $u \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^N)$. Under the assumption of p-growth

$$|z|^p \leqq f(x, z) \leqq L(1 + |z|^p), \quad p > 1, \tag{1.1}$$

the regularity theory for minimizers was succesfully carried out under fairly natural assumptions of convexity (or quasiconvexity) of f (see [17,11,15,2]). Quite recently, integrands satisfying more general growth conditions have been considered. Ten years ago Marcellini replaced (1.1) with the more flexible (p,q)-growth,

$$|z|^p \leqq f(x,z) \leqq L(1+|z|^q), \qquad q > p > 1, \tag{1.2}$$

and proved several regularity results for minimizers. Subsequently the theory of integrals with these non-standard growth conditions received contributions from various authors (see the references in [21–25]).

A borderline case lying between (1.1) and (1.2) is the one of $p(x)$-growth,

$$|z|^{p(x)} \leqq f(x,z) \leqq L(1+|z|^{p(x)}), \qquad p(x) > 1, \tag{1.3}$$

a prominent model example being, with $\mu \geqq 0$,

$$\mathcal{F}(u,\Omega) := \int_\Omega (\mu^2 + |Du|^2)^{p(x)/2}\,dx.$$

This kind of integral was first considered by ZHIKOV in the context of homogenization (see [34]), and in recent years the subject gained more and more importance by providing variational models for many problems from mathematical physics. For instance, very recently Rajagopal and Růžička elaborated a model for the electrorheological fluids: these are special non-Newtonian fluids which are characterized by their ability to change their mechanical properties in the presence of an electromagnetic field $\mathbf{E}(x)$; in this case the model for the steady case is

$$-\operatorname{div} S(x,\mathcal{E}(v)) = g(x,v,Dv), \qquad \operatorname{div} v = 0,$$

where v is the velocity of the fluid, $\mathcal{E}(v)$ is the symmetric part of the gradient Dv and the "extra stress" tensor S satisfies standard monotonicity conditions in the Leray-Lions fashion but with $p(x)$-growth. In particular

$$D^2S(x,z) \geqq \nu(1+|z|^2)^{(p(x)-2)/2}\mathrm{Id},$$

where $p(x) \equiv p(|\mathbf{E}|^2)$ and $\mathbf{E}$ is given (see [26,28]). Moreover other models of this type arise for fluids whose viscosity is influenced in a similar way by the temperature (see [33]). The differential system modelling the so called "thermistor problem" (see [31–33]) includes equations like

$$-\operatorname{div}(p(x)|Du|^{p(x)-2}Du) = 0,$$

whose solutions correspond to minimizers of $\mathcal{F}$ when $\mu = 0$. In this last case $p(x)$ also appears as an unknown of the system itself, and this eventually leads us to look for minimal regularity assumptions on it. Leaving to a forthcoming paper [5] the analysis of the vector-valued case $N > 1$, we restrict ourselves here to the scalar case $N = 1$.

In this paper we want to offer essentially optimal regularity results for minimizers of functionals with $p(x)$-growth that, together with the preexisting ones, allow us to give a complete picture of the regularity theory for such integrals in the scalar case. In the more general framework of functionals with (p,q)-growth, the first regularity result was given by MARCELLINI [22], who in the scalar case obtained Lipschitz regularity of minimizers provided $p(x) \geqq 2$ is of class C^1. As far as lower regularity is concerned, ZHIKOV proved that, if $\omega(R)$ denotes a modulus of continuity for $p(x)$, then the condition

$$\limsup_{R\to 0} \omega(R)\log\left(\frac{1}{R}\right) < +\infty \tag{1.4}$$

ensures (see [31]) higher integrability of the gradient of minimizers of $\mathcal{F}$ under the $p(x)$-growth hypothesis (1.3) alone. Later on, building on Zhikov's work, FAN XIANGLING & ZHAO DUN AND ALKHUTOV ([12,6], see also [4] for a special case) proved local $C^{0,\alpha}$ continuity of minimizers, for some $\alpha > 0$ (see also [7]), again under assumption (1.4). These results, all valid only in the scalar case, draw a parallel to the theory of functionals with p-growth (that is: $p(x) \equiv$ constant) in view of the theorems of GIAQUINTA & GIUSTI [16].

Our purpose here is to push this parallel further by giving quite sharp assumptions, especially on $p(x)$, ensuring higher regularity of minimizers.

We recall that in the scalar, p-growth case (see [20]), when $f(x,z)$ satisfies suitable smoothness and convexity assumptions (that is, ellipticity of $D^2 f$) then local $C^{0,\alpha}$ regularity of minimizers for every $0 < \alpha < 1$ is known provided $f(x,z)$ is continuous with respect to the variable x; moreover this result is sharp (see [9] and the references included). Here we prove (Theorem 2.1) that the same result holds true in the case of functionals with $p(x)$-growth, provided condition (1.4) is reinforced into

$$\limsup_{R\to 0} \omega(R)\log\left(\frac{1}{R}\right) = 0, \tag{1.5}$$

in clear accordance with the theory of functionals with p-growth where, as just described, an additional continuity assumption (with respect to x) is required to reach any exponent $\alpha < 1$. Moreover, we also observe that in order to prove Hölder continuity up to a certain exponent $\alpha < 1$, (1.5) can be substituted by a suitable smallness condition (see Remark 3.3).

An interesting fact is that the technical reason for condition (1.5) to arise is quite different from the origin of (1.4). We stress that condition (1.4) is sharp since (see [32]), in general, dropping it causes the loss of any type of regularity of minimizers, like Hölder continuity and even higher integrability. Moreover condition (1.4) seems to play a central role in the theory of functionals with $p(x)$-growth since ZHIKOV proved (see [32]) that such functionals exhibit the so called Lavrentiev phenomenon if and only if (1.4) is violated, while in [1] it is proved that the singular part of the measure representation of relaxed integrals with this growth disappears if and only if (1.4) holds true. More significantly, in order to highlight the importance of condition (1.4) it is useful to note that all the counterexamples cited above are valid already in the case of the model functional

$$\int_\Omega |Du|^{p(x)}\,dx. \tag{1.6}$$

We actually prove something more, extending some recent results in [9] valid in the p-growth case, where Hölder continuity for any exponent $\alpha < 1$ is proved for non-smooth integrals: indeed our Hölder continuity theorem holds without requiring any differentiability assumption on $f(x, z)$ with respect to z. This is of interest since this type of regularity is usually obtained by differentiating the Euler equation of the functional, though here the Euler equation itself cannot even be written.

Again as in the theory of functionals with p-growth, p constant (see Theorem 2.2), in order to get Hölder continuity of Du the Hölder continuity of $p(x)$ will be assumed too (see also [8]). Also this result is sharp as shown by counterexamples valid even in the case of functionals with quadratic growth. In this way we also extend to the case of the "$p(x)$ Laplacian" (1.6) a classical result due to URAL'TSEVA [30].

Finally we say a few words about the techniques. In order to get our estimates we employ a careful perturbation-comparison argument, based on the freezing method; this procedure turns out to be delicate since the perturbation is performed in the growth exponent. Then we combine this technique with some recent estimates, due to IWANIEC (see [19]), for the theory of $L \log L(\Omega)$ spaces. In the case of Theorem 2.1 this method cannot be used directly and it must be incorporated in a fine approximation argument to overcome the lack of differentiability of the functional.

2. Notation and statements

In what follows, Ω will denote an open bounded domain in $\mathbb{R}^n$, and $B(x, R)$ the open ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. If u is an integrable function defined on $B(x, R)$, we will set

$$(u)_{x,R} = \fint_{B(x,R)} u(x)\,dx = \frac{1}{\omega_n R^n} \int_{B(x,R)} u(x)\,dx,$$

where ω_n is the Lebesgue measure of $B(0, 1)$. We shall also adopt the convention of writing B_R and $(u)_R$ instead of $B(x, R)$ and $(u)_{x,R}$ respectively, when the center will not be relevant, or it is clear from the context; moreover, unless otherwise stated, all balls considered will have the same center. Finally, the letter c will freely denote a constant, not necessarily the same in any two occurences, while only the dependences on relevant quantities will be highlighted.

We are going to deal with the integral functional

$$\mathcal{F}(u, \Omega) = \int_{\Omega} f(x, Du)\,dx, \tag{2.1}$$

defined on $W^{1,1}_{\mathrm{loc}}(\Omega)$. The Carathéodory function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ will be supposed to satisfy a growth condition of the following type:

$$L^{-1}|z|^{p(x)} \leqq f(x, z) \leqq L(1 + |z|^{p(x)}) \tag{2.2}$$

for any $z \in \mathbb{R}^n$, $x \in \Omega$, where $p : \Omega \to (1, +\infty)$ is a continuous function and $L \geqq 1$. With this type of non-standard growth condition we adopt the following notion of a (local) minimizer:

Definition 1. We say that a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a *local minimizer* of $\mathcal{F}$ if $|Du|^{p(x)} \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_{\mathrm{spt}\varphi} f(x, Du)\,dx \leqq \int_{\mathrm{spt}\varphi} f(x, Du + D\varphi)\,dx$$

for any $\varphi \in W^{1,1}_0(\Omega;\mathbb{R}^N)$ with compact support in Ω.

We shall consider the following growth, ellipticity and continuity conditions:

$$L^{-1}(\mu^2 + |z|^2)^{p(x)/2} \leqq f(x,z) \leqq L(\mu^2 + |z|^2)^{p(x)/2}, \tag{2.3}$$

$$\begin{aligned}\int_{Q_1} &[f(x_0, z_0 + D\varphi) - f(x_0, z_0)]\,dx \\ &\geqq L^{-1}\int_{Q_1} (\mu^2 + |z_0|^2 + |D\varphi|^2)^{(p(x_0)-2)/2}|D\varphi|^2\,dx\end{aligned} \tag{2.4}$$

for each $z_0 \in \mathbb{R}^n,\ x_0 \in \Omega$ and each $\varphi \in C^\infty_0(Q_1)$ where $0 \leqq \mu \leqq 1$, $Q_1 = (0,1)^n$, and

$$\begin{aligned}&|f(x,z) - f(x_0,z)| \\ &\leqq L\omega(|x - x_0|)\big((\mu^2 + |z|^2)^{p(x)/2} + (\mu^2 + |z|^2)^{p(x_0)/2}\big)\big[1 + |\log(\mu^2 + |z|^2)|\big]\end{aligned} \tag{2.5}$$

for any $z_0 \in \mathbb{R}^n,\ x, x_0 \in \Omega$ and where $L \geqq 1$; here $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing continuous function vanishing at zero which represents the modulus of continuity of $p(x)$:

$$|p(x) - p(y)| \leqq \omega(|x - y|).$$

We will always assume ω to satisfy (1.4), thus in particular without loss of generality we may assume

$$\omega(R) \leqq L|\log R|^{-1} \tag{2.6}$$

for all $R < 1$. No differentiability will ever be assumed with respect to x (and not even with respect to z, in the case of Theorem 2.1), thus the symbol Df will always denote differentiation with repect to z.

Remark 2.1. We observe that our regularity results need no other growth assumptions, in particular on the second derivatives of the function f. We recall that if (2.2) holds then (2.4) implies (see, e.g., [2]) the following growth property for Df (when it exists):

$$|Df(x_0, z)| \leqq c(1 + |z|)^{p(x_0)-1}$$

with $c \equiv c(L, \gamma_1, \gamma_2)$, for any $z \in \mathbb{R}^n$ and $x_0 \in \Omega$. Moreover, although condition (2.4) may appear a little involved in its formulation, it is very general (see [14]): indeed in the scalar case $N = 1$ it provides a qualified form of convexity which, for example, covers all integrands of the form

$$f(x,z) = (\mu^2 + |z|^2)^{p(x)/2} + h(x,z)$$

where h is a convex function of z satisfying (2.5) and such that $0 \leqq h(x,z) \leqq L(\mu^2 + |z|^2)^{p(x)/2}$. Condition (2.4) is also similar to the uniform strict quasiconvexity introduced by Evans [11], useful in the vector-valued case $N > 1$.

Theorem 2.1. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}(\cdot,\Omega)$ where f is a continuous function satisfying* (2.3)–(2.5). *Suppose moreover that*

$$\limsup_{R\to 0} \omega(R)\log\left(\frac{1}{R}\right) = 0. \tag{2.7}$$

Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for any $0 < \alpha < 1$.

When we impose higher regularity on the functions f and p, we recover the classical $C^{1,\alpha}$ regularity of local minimizers (see [30, 8]):

Theorem 2.2. *Under the hypotheses of Theorem* 2.1*, suppose that*

$$\omega(R) \leqq LR^{\alpha}$$

for some α in the range $0 < \alpha \leqq 1$ and all $R \leqq 1$, and that f is of class C^2 with respect to the variable z in $\Omega \times (\mathbb{R}^n \setminus \{0\})$ with $D^2 f$ satisfying

$$L^{-1}(\mu^2 + |z|^2)^{(p(x)-2)/2}|\lambda|^2 \leqq D^2 f(x,z)\lambda \otimes \lambda \leqq L(\mu^2 + |z|^2)^{(p(x)-2)/2}|\lambda|^2$$

for all $\lambda \in \mathbb{R}^n$. Then Du is locally Hölder continuous in Ω.

Remark 2.2. In a matter already burdened with technicalities we preferred to avoid the full generality in order to highlight only the main ideas. However, our results can be carried out for more general functionals of the type

$$\int_\Omega f(x, u, Du)\, dx$$

with f satisfying (2.3)–(2.5), or the assumption in Theorem 2.2 for higher regularity, and a continuity assumption with respect to u such as

$$|f(x,u,z) - f(x,u_0,z)| \leqq L\omega(|u-u_0|)(\mu^2 + |z|^2)^{p(x)/2},$$

and where $\omega(R) \leqq LR^{\alpha}$, or else ω satisfies (2.7) in order to get the result of Theorem 2.1 but with a more accurate argument.

3. Proof of the results

We prove Theorem 2.1. In this section, since all our results are local in nature, without loss of generality we shall suppose that

$$1 < \gamma_1 \leqq p(x) \leqq \gamma_2 \quad \forall x \in \Omega, \qquad \int_\Omega |Du|^{p(x)}\, dx < +\infty.$$

Although we stated our theorems in the scalar case, some of the following results will be valid also when u is vector valued. This is the case of the next higher integrability result, due to ZHIKOV, that in a slightly less general statement appears in [32] (see also [8]):

Theorem 3.1. *Let $\bar{u} \in W^{1,1}_{\mathrm{loc}}(\mathcal{O}, \mathbb{R}^N)$ be a local minimizer of the functional $w \mapsto \int_{\mathcal{O}} \bar{f}(x, Dw)\, dx$ with $\bar{f} : \mathcal{O} \times \mathbb{R}^{nN} \to \mathbb{R}$ satisfying* (2.2), (2.6) *and $\mathcal{O}$ an open subset of Ω. Suppose also that*

$$\int_{\mathcal{O}} |D\bar{u}|^{p(x)}\, dx \leqq M_1,$$

for some fixed constant $M_1 < +\infty$. Then there exist two positive constants $c_0, \delta \equiv c_0, \delta(\gamma_1, \gamma_2, L, M_1)$ such that if $B_R \subset\subset \mathcal{O}$, then

$$\left(\fint_{B_{R/2}} |D\bar{u}|^{p(x)(1+\delta)}\, dx \right)^{1/(1+\delta)} \leqq c_0 \fint_{B_R} |D\bar{u}|^{p(x)}\, dx + c_0. \tag{3.1}$$

Remark 3.1. The way Theorem 3.1 can be obtained involves a standard combination of a suitable Caccioppoli type inequality and the Gehring lemma in the version of Giaquinta-Modica (see [32]); since the reverse Hölder type inequalities involved are verified only on balls $B_R \subset\subset \mathcal{O}$ with $R \leqq R_0 \equiv R_0(n)$, it is also useful to refer to general statements as in [29]. For future convenience we stress that the higher integrability constants c_0, δ are independent of the function $\bar{f}$ and also of the minimizer $\bar{u}$; they only depend on the growth constants and on the quantity M_1 above. So, once the quantities $(\gamma_1, \gamma_2, L, M_1)$ are fixed, the constants δ and c_0 are determined independently of the function $\bar{f}$ and the minimizer $\bar{u}$ considered. Of course, in (3.1), δ may be replaced at will by smaller constants.

The following result is taken from [13], see also [10]:

Theorem 3.2. *Let $g(z) : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying* (2.3),(2.4) *with constant $p(x) \equiv p$, $\gamma_1 \leqq p \leqq \gamma_2$, and let $\bar{u} \in W^{1,p}(\Omega)$ be a local minimizer of the functional $w \mapsto \int_{B_R} g(Dw)\, dx$ with $B_R \subset\subset \Omega$. Then $D\bar{u}$ is locally bounded and, moreover, if $0 < \rho < R/2$, then*

$$\int_{B_\rho} (\mu^2 + |D\bar{u}|^2)^{p/2}\, dx \leqq c \left(\frac{\rho}{R} \right)^n \int_{B_R} (\mu^2 + |D\bar{u}|^2)^{p/2}\, dx$$

with $c \equiv c(L, \gamma_1, \gamma_2)$.

The next lemma is an up-to-the-boundary higher integrability result, which we restate from [9], Lemma 2.7, in a slightly different form.

Lemma 3.1. *Let $g(z) : \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying*

$$\tilde{L}^{-1}|z|^p \leqq g(z) \leqq \tilde{L}(|z|^p + 1),$$

where $\tilde{L} \geqq 1$, $\gamma_1 \leqq p \leqq \gamma_2$. Let $\bar{u} \in W^{1,q}(B_{2R})$, $p < q$, $B_{2R} \subset\subset \Omega$ and $v \in \bar{u} + W^{1,p}_0(B_R)$ be a minimizer of the functional $w \mapsto \int_{B_R} g(Dw)\, dx$ in the Dirichlet class $\bar{u} + W^{1,p}_0(B_R)$. Then there exist $c, \varepsilon \equiv c, \varepsilon(\gamma_1, \gamma_2, \tilde{L})$ with $0 < \varepsilon < (q-p)/p$, both independent of R, v and $\bar{u}$, such that

$$\left(\fint_{B_R} |Dv|^{p(1+\varepsilon)}\, dx \right)^{1/p(1+\varepsilon)} \leqq c \left(\fint_{B_R} |Dv|^p\, dx \right)^{1/p} + c \left(\fint_{B_{2R}} |D\bar{u}|^q\, dx \right)^{1/q}.$$

We remark that although from [9] it seems that the exponent ε depends on p, the proof of [9] itself and an accurate inspection of the statements of the various versions of the Gehring lemma appearing in the literature (again, see for instance [29], or [18], Proposition 6.1), reveal that ε can be chosen to be bounded uniformly away from zero as p varies in a compact subset of $]1,+\infty[$ such as our $[\gamma_1,\gamma_2]$.

The following is a slight modification of a technical iteration lemma (see for instance [17], Lemma 7.3), in which the assumption that Φ is increasing is weakened.

Lemma 3.2. *Let $\Phi : [0,a[\to \mathbb{R}, \quad 0 < a < 1$, be a positive bounded function such that $\Phi(s) \leqq 2\,\Phi(t)$ whenever $s \leqq t$, and such that:*

$$\Phi(\rho) \leqq C\left[\left(\frac{\rho}{R}\right)^n + \varepsilon\right]\Phi(R) + CR^n$$

with $0 < C < +\infty$, whenever $0 < \rho \leqq R/8$. Then, for any τ such that $0 < \tau < n$ there exist $c, \varepsilon_0 \equiv c, \varepsilon_0(C,\tau,n) > 0$ such that if $\varepsilon \leqq \varepsilon_0$, then:

$$\Phi(\rho) \leqq c\left(\frac{\rho}{R}\right)^{n-\tau}\left[\Phi(R) + R^{n-\tau}\right]$$

whenever $0 < \rho \leqq R/16$.

Basic facts from the theory of Orlicz spaces. Let $A : \mathbb{R}^+ \to \mathbb{R}^+$ be an Orlicz function, that is, A is convex, strictly increasing and such that $A(0) = 0$ and $(A(t)/t) \to +\infty$ as $t \to +\infty$. We consider the Orlicz space generated by A, that is the Banach space $L^A(\Omega)$ (we refer the reader to the classical monograph [27] for a complete account of the theory), equipped with the following Luxemburg norm:

$$\|h\|_A := \inf\left\{\lambda > 0 : \ \fint_\Omega A\left(\frac{|h|}{\lambda}\right)dx \leqq 1\right\}.$$

In the case $A(t) = t^p/p, \ p > 1$, this quantity is equivalent to the averaged L^p norm which we define for all $p \geqq 1$ as

$$\|h\|_p := \left(\fint_\Omega |h|^p\,dx\right)^{1/p}. \tag{3.2}$$

We will mainly be interested in the particular case $A(t) = t\log(e+t)$. With this choice the space $L^A(\Omega)$ is denoted by $L\log L(\Omega)$ and plays a fundamental role in various branches of analysis. When $p > 1$, $L^p(\Omega)$ is continuously embedded in this space, that is, there exists a constant $c \equiv c(p)$ such that

$$\|h\|_{L\log L(\Omega)} \equiv \|h\|_A \leqq c(p)\|h\|_p.$$

We note that the constant $c(p)$ in the previous inequality blows up as $p \to 1$. Anyway, the prominent fact for us will be an integral characterization of the quantity $\|h\|_{L\log L(\Omega)}$ recently discovered by IWANIEC (see [19]): the norm $\|h\|_{L\log L(\Omega)}$ is equivalent, via a constant that does not depend on the open set Ω, to the integral functional

$$\fint_{\Omega} |h| \log\Big(e + \frac{|h|}{\|h\|_1}\Big)\, dx$$

which thus turns out to define an equivalent, order preserving norm in $L \log L(\Omega)$. So, connecting the previous facts, we have the following inequality valid for any $h \in L^p(B_R)$ and $p > 1$:

$$\fint_{B_R} |h| \log\Big(e + \frac{|h|}{\|h\|_1}\Big)\, dx \leqq c(p) \left(\fint_{B_R} |h|^p \, dx \right)^{1/p}. \tag{3.3}$$

Choice of some relevant quantities. We start by applying Theorem 3.1 and Remark 3.1, thus getting the higher integrability exponent $\delta < \gamma_1 - 1$ determined by $(\gamma_1, \gamma_2, 8^{\gamma_2} L, M_1)$, where

$$M_1 := 8^{\gamma_2} L \int_{\Omega} \big(1 + |Du|^2\big)^{p(x)/2}\, dx; \tag{3.4}$$

the reasons for this peculiar choice are technical and will be clear later. We select $R_0 \equiv R_0(\gamma_1, \gamma_2, 8^{\gamma_2} L, M_1) > 0$ with the property that $\omega(8R_0) < \delta/4$; finally we fix a ball $B_{R_0} \subset\subset \Omega$ and we set

$$p_m := \max_{\overline{B_{R_0}}} p(x).$$

Then we consider balls $B(x_c, 4R) \equiv B_{4R} \subset\subset B_{R_0/4}$ and we define

$$p_2 := \max_{\overline{B_{4R}}} p(x), \qquad p_1 := \min_{\overline{B_{4R}}} p(x)$$

(note that p_1, p_2 depend on the ball). We remark that for a suitable $x_0 \in \overline{B_{4R}}$, not necessarily the center, we have $p_2 = p(x_0)$; also, we have $p_2 - p_1 \leqq \omega(8R) \leqq 8\omega(R)$. The preceding choices imply that

$$\begin{aligned} &p_2(1 + \delta/4) \leqq p(x)(1 + \delta/4 + \omega(8R)) \leqq p(x)(1 + \delta) && \text{in } B_{4R}, \\ &p_m(1 + \delta/4) \leqq p(x)(1 + \delta) && \text{in } B_{R_0}. \end{aligned} \tag{3.5}$$

Finally, without loss of generality we shall always choose $16R \leqq R_0 \leqq 1$: this is again a technicality that will be explained later. We are now ready to state the main proposition of the section; this will provide the necessary *a priori* estimates.

Proposition 3.1. *Let $\bar{u} \in W^{1,1}(B_{R_0})$ be a local minimizer of the functional $w \mapsto \int_{B_{R_0}} \bar{f}(x, Dw)\, dx$ with $\bar{f}(x, z) : B_{R_0} \times \mathbb{R}^n \to \mathbb{R}$ of class C^2 with respect to the variable z and satisfying* (2.3)–(2.7) *with L replaced by* $8^{\gamma_2} L$ *and $\mu > 0$. Moreover suppose*

$$\int_{B_{R_0}} |D\bar{u}|^{p(x)}\, dx \leqq M_1, \qquad \int_{B_{R_0/4}} |D\bar{u}|^{p_m}\, dx \leqq M_2,$$

where M_1 is defined in (3.4) *and $M_2 < +\infty$ is a constant. Then if $B(x_c, 4R) \subset\subset B_{R_0/4}$, not necessarily concentric with B_{R_0}, for every $0 < \tau < n$ there exist*

$0 < R_1 < R/8$ *and* $c > 0$, *both depending on* $(\gamma_1, \gamma_2, L, M_1, M_2, \omega, \tau)$ *but not on* $\bar{u}$, μ *and* $\bar{f}$, *such that*

$$\int_{B(x_c,\rho)} |D\bar{u}|^{\gamma_1}\, dx \leqq c\rho^{n-\tau} \tag{3.6}$$

whenever $0 < \rho < R_1$.

Proof. Let $B(x_c, 4R) \equiv B_{4R} \subset\subset B_{R_0/4}$ be a ball as described above, and hence not necessarily concentric with B_{R_0}; from now on, when not otherwise specified, all the balls considered (except B_{R_0}) will have x_c as their center. With the notation introduced above, we observe that the exponent p_2 is actually a function of R, thus

$$p_2 := \max_{\overline{B_{4R}}} p(x) \equiv p_2(R).$$

We first remark that by Theorem 3.1 and by (3.5) we have $\bar{u} \in W^{1,p_2(1+\delta/4)}(B_{4R})$. We define $v \in \bar{u} + W_0^{1,p_2}(B_R)$ as the unique solution to the following Dirichlet problem:

$$\min\left\{\int_{B_R} \bar{f}(x_0, Dw)\, dx : w \in \bar{u} + W_0^{1,p_2}(B_R)\right\},$$

where as above $p_2 = p(x_0)$; note that it may well happen that x_0 does not belong to B_R. We observe that the function $g(z) := \bar{f}(x_0, z)$ satisfies the hypotheses of Theorem 3.2 and Lemma 3.1 with $p \equiv p_2$, $\gamma_1 \leqq p_2 \leqq \gamma_2$. Hence, by the minimality of v it follows that there exist $c \equiv c(\gamma_1, \gamma_2, L, M_1, M_2) < +\infty$ and $0 < \varepsilon < \delta/4$, $\varepsilon \equiv \varepsilon(\gamma_1, \gamma_2, L)$ independent of R and v, such that whenever $0 < \rho < R/2$

$$\int_{B_\rho} |Dv|^{p_2}\, dx \leqq c\left(\frac{\rho}{R}\right)^n \int_{B_R} (1 + |Dv|^{p_2})\, dx, \tag{3.7}$$

$$\begin{aligned}
&\left(\fint_{B_R} |Dv|^{p_2(1+\varepsilon)}\, dx\right)^{1/(1+\varepsilon)} \\
&\qquad \leqq c\left(\fint_{B_R} |Dv|^{p_2}\, dx\right) + c\left(\fint_{B_{2R}} |D\bar{u}|^{p_2(1+\delta/4)}\, dx\right)^{1/(1+\delta/4)},
\end{aligned} \tag{3.8}$$

$$\int_{B_R} |Dv|^{p_2}\, dx \leqq c\int_{B_R} (|D\bar{u}|^{p_2} + 1)\, dx. \tag{3.9}$$

Now we compare $\bar{u}$ and v in B_R. Using Lemma 2.2 of [9] and the minimality of v we obtain by (3.7) and (3.9)

$$\begin{aligned}
&\int_{B_\rho} \left(\mu^2 + |D\bar{u}|^2\right)^{p_2/2}\, dx \\
&\qquad \leqq c\int_{B_\rho} \left(\mu^2 + |Dv|^2\right)^{p_2/2}\, dx \\
&\qquad\quad + c\int_{B_\rho} (\mu^2 + |D\bar{u}|^2 + |Dv|^2)^{(p_2-2)/2} |D\bar{u} - Dv|^2\, dx
\end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&\leqq c\left(\frac{\rho}{R}\right)^n \int_{B_R} (1+|D\bar{u}|^{p_2})\,dx \\
&\quad + c\int_{B_R} \left(\mu^2+|D\bar{u}|^2+|Dv|^2\right)^{(p_2-2)/2} |D\bar{u}-Dv|^2\,dx.
\end{aligned}
$$

Our main task now is to give an upper estimate for the last quantity appearing in (3.10). To this end, we observe that since the function g satisfies (2.4) with $p(x) \equiv p_2$ and it is of class C^2, then (see [13]) it also satisfies, for $\nu \equiv \nu(\gamma_1, \gamma_2, L) > 0$,

$$
D^2 g(z)\,\lambda\otimes\lambda \geqq \nu(\mu^2+|z|^2)^{(p_2-2)/2}|\lambda|^2,
$$

and using the minimality of v together with Lemma 2.1 from [3] (that still works for $p > 2$) we obtain

$$
\begin{aligned}
&\int_{B_R} [g(D\bar{u}) - g(Dv)]\,dx \qquad (3.11)\\
&\quad = \int_{B_R} \langle Dg(Dv), D\bar{u}-Dv\rangle\,dx \qquad [=0]\\
&\qquad + \int_{B_R} dx \int_0^1 (1-t)D^2 g(tD\bar{u}+(1-t)Dv)\\
&\qquad\qquad \cdot (D\bar{u}-Dv)\otimes(D\bar{u}-Dv)\,dt\\
&\quad \geqq \nu\int_{B_R} dx\int_0^1 (1-t)(\mu^2+|tD\bar{u}+(1-t)Dv|^2)^{(p_2-2)/2}|D\bar{u}-Dv|^2\,dt\\
&\quad \geqq c^{-1}\int_{B_R} \left(\mu^2+|D\bar{u}|^2+|Dv|^2\right)^{(p_2-2)/2}|D\bar{u}-Dv|^2\,dx,
\end{aligned}
$$

with $c \equiv c(L, \gamma_1, \gamma_2) < +\infty$. On the other hand, using now the minimality of $\bar{u}$,

$$
\begin{aligned}
&\int_{B_R} [g(D\bar{u}) - g(Dv)]\,dx \qquad (3.12)\\
&\quad = \int_{B_R} [\bar{f}(x_0, D\bar{u}) - \bar{f}(x, D\bar{u})]\,dx\\
&\qquad + \int_{B_R} [\bar{f}(x, D\bar{u}) - \bar{f}(x, Dv)]\,dx \qquad [\leqq 0]\\
&\qquad + \int_{B_R} [\bar{f}(x, Dv) - \bar{f}(x_0, Dv)]\,dx\\
&\quad \leqq \int_{B_R} [\bar{f}(x_0, D\bar{u}) - \bar{f}(x, D\bar{u})]\,dx\\
&\qquad + \int_{B_R} [\bar{f}(x, Dv) - \bar{f}(x_0, Dv)]\,dx\\
&\quad \overset{(2.5)}{\leqq} c\omega(R)\int_{B_R} \left(\left(\mu^2+|D\bar{u}|^2\right)^{p_2/2} + \left(\mu^2+|D\bar{u}|^2\right)^{p(x)/2}\right)\\
&\qquad\qquad \cdot (1+|\log(\mu^2+|D\bar{u}|^2)|)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+ c\omega(R)\int_{B_R}\Big((\mu^2+|Dv|^2)^{p_2/2}+(\mu^2+|Dv|^2)^{p(x)/2}\Big)\\
&\qquad\cdot(1+|\log(\mu^2+|Dv|^2)|)\,dx\\
&:= \mathrm{I}+\mathrm{II}.
\end{aligned}
$$

We begin by estimating the term I; keeping the notation employed in (3.2) for the norms,

$$
\begin{aligned}
\mathrm{I} \;&\leqq\; c\omega(R)\int_{B_R\cap\{|D\bar u|\geqq e\}}|D\bar u|^{p_2}\log|D\bar u|^{p_2}\,dx + c\omega(R)R^n\\
&\leqq\; c\omega(R)R^n \fint_{B_R}|D\bar u|^{p_2}\log\Big(e+\frac{|D\bar u|^{p_2}}{\||D\bar u|^{p_2}\|_1}\Big)\,dx\\
&\quad + c\omega(R)R^n\fint_{B_R}|D\bar u|^{p_2}\log(e+\||D\bar u|^{p_2}\|_1)\,dx + c\omega(R)R^n\\
&\overset{(3.3)}{\leqq}\; c(\delta)\omega(R)R^n\left(\fint_{B_R}|D\bar u|^{p_2(1+\delta/4)}\,dx\right)^{1/(1+\delta/4)}\\
&\quad + c\omega(R)\log\big(e+\||D\bar u|^{p_2}\|_1\big)\int_{B_R}|D\bar u|^{p_2}\,dx + c\omega(R)R^n\\
&\overset{(3.5)}{\leqq}\; c\omega(R)R^n\left(\fint_{B_R}|D\bar u|^{p(x)(1+\delta/4+\omega(8R))}\,dx\right)^{1/(1+\delta/4)}\\
&\quad + c(M_2)\omega(R)\log\Big(\frac{1}{R}\Big)\int_{B_R}|D\bar u|^{p_2}\,dx + c\omega(R)R^n\\
&\overset{(3.1)}{\leqq}\; c\omega(R)R^n\left(\fint_{B_{2R}}|D\bar u|^{p(x)}\,dx\right)^{(1+\delta/4+\omega(8R))/(1+\delta/4)}\\
&\quad + c\omega(R)\log\Big(\frac{1}{R}\Big)\int_{B_R}|D\bar u|^{p_2}\,dx + c\omega(R)R^n\\
&\leqq\; c\omega(R)R^{-n\omega(8R)/(1+\delta/4)}\left(\int_{B_{2R}}|D\bar u|^{p(x)}\,dx\right)^{\frac{\omega(8R)}{(1+\delta/4)}}\int_{B_{2R}}|D\bar u|^{p_2}\,dx\\
&\quad + c\omega(R)\log\Big(\frac{1}{R}\Big)\int_{B_R}|D\bar u|^{p_2}\,dx + c\omega(R)R^n\\
&\overset{(2.6)}{\leqq}\; c\omega(R)\log\Big(\frac{1}{R}\Big)\int_{B_{2R}}|D\bar u|^{p_2}\,dx + c\omega(R)R^n
\end{aligned}
$$

with $c \equiv c(\gamma_1,\gamma_2,L,M_1,M_2)$, since $\delta\equiv\delta(\gamma_1,\gamma_2,L,M_1)$. We observe that we used the boundedness of $R^{-\omega(8R)} \leqq c \equiv c(L)$ given by (2.6) to perform the last estimate and the elementary inequality

$$
\log(e+ab)\leqq \log(e+a)+\log(e+b)\qquad \forall\, a,b>0
$$

to perform the estimate on the second line.

The term II must be estimated in a different way; indeed,

$$\begin{aligned}
\mathrm{II} &\leqq c\omega(R)\int_{B_R\cap\{|Dv|\geqq e\}}|Dv|^{p_2}\log|Dv|^{p_2}\,dx + c\omega(R)R^n\\
&\leqq c\omega(R)R^n \fint_{B_R}|Dv|^{p_2}\log\Big(e+\frac{|Dv|^{p_2}}{\||Dv|^{p_2}\|_1}\Big)\,dx\\
&\quad + c\omega(R)R^n\fint_{B_R}|Dv|^{p_2}\log(e+\||Dv|^{p_2}\|_1)\,dx + c\omega(R)R^n\\
&\overset{(3.3)}{\leqq} c(\varepsilon)\omega(R)R^n\left(\fint_{B_R}|Dv|^{p_2(1+\varepsilon)}\,dx\right)^{1/(1+\varepsilon)}\\
&\quad + c\omega(R)\log\left(e+\||Dv|^{p_2}\|_1\right)\int_{B_R}|Dv|^{p_2}\,dx + c\omega(R)R^n\\
&\overset{(3.8)}{\leqq} c(\varepsilon)\omega(R)\int_{B_R}|Dv|^{p_2}\,dx\\
&\quad + c\omega(R)R^n\left(\fint_{B_{2R}}|D\bar u|^{p(x)(1+\omega(8R)+\delta/4)}\,dx\right)^{1/(1+\delta/4)}\\
&\quad + c\omega(R)\log\left(e+\||Dv|^{p_2}\|_1\right)\int_{B_R}|Dv|^{p_2}\,dx + c\omega(R)R^n\\
&\overset{(3.1)}{\leqq} c\left[\omega(R)+\omega(R)\log\left(e+\||Dv|^{p_2}\|_1\right)\right]\int_{B_R}|Dv|^{p_2}\,dx\\
&\quad + c\omega(R)R^{-n\omega(8R)/(1+\delta/4)}\left(\int_{B_{4R}}|D\bar u|^{p(x)}\,dx\right)^{\frac{\omega(8R)}{1+\delta/4}}\int_{B_{4R}}|D\bar u|^{p_2}\,dx\\
&\quad + c\omega(R)R^n\\
&\overset{(3.9)}{\leqq} c\omega(R)\log\left(\frac{1}{R}\right)\int_{B_{4R}}|D\bar u|^{p_2}\,dx + c\omega(R)R^n,
\end{aligned}$$

again with $c \equiv c(\gamma_1, \gamma_2, L, M_1, M_2)$. Connecting the estimates found for I and II to (3.11) we have

$$\begin{aligned}
&\int_{B_R}(\mu^2+|D\bar u|^2+|Dv|^2)^{(p_2-2)/2}|D\bar u - Dv|^2\,dx\\
&\qquad\leqq c\omega(R)\log\Big(\frac{1}{R}\Big)\int_{B_{4R}}|D\bar u|^{p_2}\,dx + c\omega(R)R^n \qquad (3.13)
\end{aligned}$$

and this last formula together with (3.10) finally gives, whenever $0<\rho<R/2$,

$$\int_{B_\rho}|D\bar u|^{p_2(R)}\,dx \leqq c\left[\Big(\frac{\rho}{R}\Big)^n + \omega(R)\log\Big(\frac{1}{R}\Big)\right]\int_{B_{4R}}|D\bar u|^{p_2(R)}\,dx + cR^n.$$

Now we define the function $\Phi : [0, R_0/16[\to \mathbb{R}$ as

$$\Phi(\rho) := \int_{B_\rho}(|D\bar u|^{p_2(\rho)}+1)\,dx.$$

This is a positive function and by (3.5) and Theorem 3.1 it is also bounded. Moreover we observe that, since the function $R \mapsto p_2(R)$ is nondecreasing, then it readily follows that $\Phi(s) \leqq 2\Phi(t)$ whenever $s \leqq t$ and

$$\int_{B_\rho} (|D\bar{u}|^{p_2(\rho)} + 1)\, dx \leqq 2 \int_{B_\rho} (|D\bar{u}|^{p_2(R)} + 1)\, dx.$$

With this notation we may write what we have proved as

$$\Phi(\rho) \leqq c\Big[\Big(\frac{\rho}{R}\Big)^n + \omega(R)\log\Big(\frac{1}{R}\Big)\Big]\Phi(R) + cR^n,$$

valid for any $\rho \leqq R/8$ and with $c \equiv c(\gamma_1, \gamma_2, L, M_1, M_2)$. So, fixing $0 < \tau < n$, if we apply Lemma 3.2, choosing $R_1 \equiv R_1(\gamma_1, \gamma_2, L, M_1, M_2, \omega, \tau) > 0$ such that $\omega(R)\log(1/R) \leqq \varepsilon_0$ whenever $0 < R < 16R_1$, we have:

$$\begin{aligned}
\int_{B_\rho} |D\bar{u}|^{p_2(\rho)}\, dx &\leqq c\left(\frac{\rho}{R_1}\right)^{n-\tau} \Big[\int_{B_{R_1}} |D\bar{u}|^{p_2(R_1)}\, dx + 1\Big] \\
&\leqq c\left(\frac{\rho}{R_1}\right)^{n-\tau} \Big[\int_{B_{R_0/4}} |D\bar{u}|^{p_m}\, dx + 1\Big] \\
&\leqq c\rho^{n-\tau}
\end{aligned}$$

whenever $0 < \rho < R_1$, which we may assume without loss of generality. But $\gamma_1 \leqq p_2(\rho)$, and this immediately gives (3.6) with $c \equiv c(\gamma_1, \gamma_2, L, M_1, M_2, \omega, \tau)$ since also $R_1^{-1} \equiv R_1^{-1}(\gamma_1, \gamma_2, L, M_1, M_2, \omega, \tau) < +\infty$. □

Remark 3.2. The last few lines are the only place where we used (2.7) instead of the weaker (2.6).

Proof of Theorem 2.1. We rely on an approximation argument. We take a smooth mollifier $\varphi \in C^\infty(B(0,1))$ such that $\int_{B(0,1)\setminus B(0,1/2)} \varphi(y)\, dy = 1/2$, and we define a sequence $f_m : \Omega \times \mathbb{R}^n \to \mathbb{R}$, $m \in \mathbb{N}$, of smooth approximations (in the variable z) of the energy density f:

$$f_m(x,z) := \int_{B(0,1)} f(x, z + y/m)\varphi(y)\, dy.$$

Following [13, Lemma 2.4], it is easy to check that the sequence f_m satisfies (2.3)–(2.5) uniformly with respect to $m \in \mathbb{N}$, with L replaced by $8^{\gamma_2}L$ and μ^2 by $\mu^2 + (1/m^2)$. Now, let $R_0 \equiv R_0(\gamma_1, \gamma_2, 8^{\gamma_2}L, M_1, \omega) > 0$ be as defined before Proposition 3.1, with M_1 as in (3.4). We define $u_m \in u + W_0^{1,\gamma_1}(B_{R_0})$ as the unique solution to the Dirichlet problem

$$\min\Big\{\int_{B_{R_0}} f_m(x, Dw)\, dx : w \in u + W_0^{1,\gamma_1}(B_{R_0})\Big\}.$$

By the minimality of u_m and the growth conditions satisfied by f_m, it easily follows that

$$\int_{B_{R_0}} |Du_m|^{p(x)}\, dx \leqq 8^{\gamma_2}L \int_{B_{R_0}} (1 + |Du|^2)^{p(x)/2}\, dx \overset{(3.4)}{\leqq} M_1. \tag{3.14}$$

Now we are in a position to apply Theorem 3.1 to $u_m \equiv \bar{u}$ with $\mathcal{O} = B_{R_0}$ and $f_m \equiv \bar{f}$, according to the choice of the constants made before Proposition 3.1. We then find:

$$\int_{B_{R_0/4}} |Du_m|^{p_m}\,dx \overset{(3.5)}{\leqq} \int_{B_{R_0/4}} |Du_m|^{p(x)(1+\delta)}\,dx + cR_0^n \tag{3.15}$$
$$\overset{(3.1)}{\leqq} cR_0^{-n\delta}\left(\int_{B_{R_0}} (1+|Du_m|^{p(x)})\,dx\right)^{(1+\delta)} \overset{(3.14)}{:\leqq} M_2$$

with $M_2 \equiv M_2(L, \gamma_1, \gamma_2, \||Du|^{p(x)}\|_{L^1(\Omega)})$; we may suppose that $M_2 \geqq M_1$. We apply Proposition 3.1 to $u_m \equiv \bar{u}$ and $f_m \equiv \bar{f}$; thus, using (3.15), for every fixed $0 < \tau < 1$ we find c, R_1, both depending on $(\gamma_1, \gamma_2, L, \||Du|^{p(x)}\|_{L^1(\Omega)}, \omega, \tau)$ but independent of m, and with $0 < R_1 < R_0$, such that

$$\int_{B_\rho} |Du_m|^{\gamma_1}\,dx \leqq c\rho^{n-\tau} \quad \text{for all } 0 < \rho < R_1. \tag{3.16}$$

By (3.14), (3.15) we note that, since $u_m = u$ on ∂B_{R_0}, up to taking a subsequence we may suppose

$$u_m \rightharpoonup w \ \text{ weakly in } \ W^{1,\gamma_1}(B_{R_0})$$

where $w \in u + W_0^{1,\gamma_1}(B_{R_0}) \cap W^{1,p_m}(B_{R_0/4})$. We claim that actually $w = u$; we will prove this later. With the claim accepted, the proof is finished, since letting $m \to \infty$ in (3.16) we get by lower semicontinuity

$$\int_{B_\rho} |Du|^{\gamma_1}\,dx \leqq c\rho^{n-\tau} \tag{3.17}$$

whenever $0 \leqq \rho \leqq R_1$ and with c and R_1 as in (3.16). At this point the result follows by Morrey-Campanato's integral characterization of Hölder continuity together with a standard covering argument. We conclude by proving our claim. By the minimality of u_m it follows that

$$\int_{B_{R_0}} f_m(x, Du_m)\,dx \leqq \int_{B_{R_0}} f_m(x, Du)\,dx. \tag{3.18}$$

Using also the elementary inequality

$$|f_m(x,z) - f(x,z)| \leqq \frac{c}{m}(|z|^{p(x)-1} + 1)$$

(which follows from the definition of f_m, the convexity and growth of f and Remark 2.1) and the fact that the sequence $|Du_m|^{p(x)}$ is bounded in $L^1(B_{R_0})$, and letting $m \to \infty$ in (3.18), again by semicontinuity it follows that

$$\int_{B_{R_0}} f(x, Dw)\,dx \leqq \liminf_m \int_{B_{R_0}} f(x, Du_m)\,dx$$
$$= \liminf_m \int_{B_{R_0}} f_m(x, Du_m)\,dx \leqq \int_{B_{R_0}} f(x, Du)\,dx.$$

This in turn, since u is a local minimizer, implies

$$\int_{B_{R_0}} f(x, Dw)\,dx = \int_{B_{R_0}} f(x, Du)\,dx.$$

At this point $w = u$ follows by the uniqueness of minimizers in $u + W_0^{1,\gamma_1}(B_{R_0})$, since by (2.4) it turns out (see [13]) that $\mathcal{F}$ is strictly convex. $\square$

Remark 3.3. An interesting observation is the following: if (2.6) holds for some L, then $u \in C^{0,\alpha}$ for some α, see [12,6]; at the other extreme, if (2.7) holds then $u \in C^{0,\alpha}$ for all α. In between, fix $0 < \bar{\alpha} < 1$, and consequently take $\tau = (1-\bar{\alpha})\gamma_1$: this value, together with an upper bound for $\int |Du|^{p(x)}\,dx$, determines the constant ε_0 in Lemma 3.2. Then $u \in C^{0,\bar{\alpha}}$ provided

$$\limsup_{R\to 0} \omega(R) \log\Big(\frac{1}{R}\Big) \leqq \varepsilon_0 :$$

thus we may also think of α as a function of

$$\lambda = \limsup_{R\to 0} \omega(R) \log\Big(\frac{1}{R}\Big),$$

and we see that

$$\lim_{\lambda\to 0} \alpha(\lambda) = 1.$$

Before proving Theorem 2.2 we still need a result that we adapt from [20]:

Proposition 3.2. *Let $g(z) : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^2 in $\mathbb{R}^n \setminus \{0\}$ with $D^2 g$ satisfying*

$$L^{-1}(\mu^2 + |z|^2)^{p/2} \leqq g(z) \leqq L(\mu^2 + |z|^2)^{p/2},$$
$$L^{-1}(\mu^2 + |z|^2)^{(p-2)/2}|\lambda|^2 \leqq D^2 g(z)\lambda \otimes \lambda \leqq L(\mu^2 + |z|^2)^{(p-2)/2}|\lambda|^2$$

for all $z \in \mathbb{R}^n \setminus \{0\}$ and $\lambda \in \mathbb{R}^n$, where $0 \leqq \mu \leqq 1$, $L \geqq 1$ and $\gamma_1 \leqq p \leqq \gamma_2$. Let $v \in W^{1,p}(B_R)$ be a local minimizer of the functional $w \mapsto \int_{B_R} g(Dw)\,dx$. Then there exist a constant $c \equiv c(\gamma_1, \gamma_2, L)$ and an exponent $0 < \beta \equiv \beta(L, \gamma_1, \gamma_2) < 1$ such that

$$\sup_{B_{R/2}} (\mu^2 + |Dv|^2)^{p/2} \leqq c \fint_{B_R} (\mu^2 + |Dv|^2)^{p/2}\,dx$$
$$\sup_{x,y\in B_\rho} |Dv(x) - Dv(y)| \leqq c\Big(\frac{\rho}{R}\Big)^{\beta} \sup_{B_{R/2}} |Dv|$$

for any $\rho \leqq R/2$.

Proof of Theorem 2.2. An approximation argument similar to the one employed in Theorem 2.1 allows us to suppose that f is of class C^2 with respect to z and that $\mu > 0$, with the same constants as before. All the following estimates will be independent of μ, so that the result will hold for the original function f. We shall define x_0 and $B(x_c, R)$ afterwards; for any $u \in W^{1,p(x_0)}\big(B(x_c, R)\big)$ the problem

$$\min\left\{\int_{B(x_c,R)} f(x_0, Dw)\,dx : \ w \in u + W_0^{1,p(x_0)}\big(B(x_c, R)\big)\right\} \tag{3.19}$$

admits for its unique solution v the estimate, easily obtained by Proposition 3.2 above by considering $g(z) := f(x_0, z)$,

$$\fint_{B(x_c,\rho)} |Dv - (Dv)_{x_c,\rho}|^{p(x_0)}\,dx \leqq c\left(\frac{\rho}{R}\right)^{\beta p(x_0)} \fint_{B(x_c,R)} (1 + |Dv|^{p(x_0)})\,dx \tag{3.20}$$

whenever $\rho \leqq R/2$, where $c > 0$ and β such that $0 < \beta < 1$ depend only on γ_1, γ_2, L (here we took into account that $1 < \gamma_1 \leqq p(x_0) \leqq \gamma_2$).

Keeping the notation employed in and immediately before Proposition 3.1, we consider balls $B(x_c, 4R) \subset\subset B_{R_0/4}$ and we set

$$p_2 := \max_{\overline{B_{4R}}} p(x) \equiv p_2(R).$$

(This is the last time we meet B_{R_0}. Henceforth all balls we consider will have center in x_c, which we shall omit.) Now let us fix $\tau = \alpha\beta/8(n + \beta)$. By the last inequality in Proposition 3.1, and reasoning as we did to obtain (3.17), we deduce the existence of a radius R_1 and a constant c, both depending on $\gamma_1, \gamma_2, L, |||Du|^{p(x)}||_{L^1(\Omega)}, \alpha$, such that whenever $0 < R < R_1$

$$\int_{B_R} |Du|^{p_2(R)}\,dx \leqq cR^{n-\tau}. \tag{3.21}$$

Take R such that $4R < R_1$ and let $x_0 \in \overline{B_{4R}}$ be such that $p(x_0) = p_2$; let $v \in u + W_0^{1,p_2}(B_R)$ be the solution to (3.19). Repeating the proof of Proposition 3.1, but using the fact that $\omega \leqq LR^\alpha$, we obtain

$$\int_{B_R} \left(\mu^2 + |Du|^2 + |Dv|^2\right)^{(p_2-2)/2} |Du - Dv|^2\,dx \leqq cR^{\alpha/2} \int_{B_{4R}} (|Du|^{p_2} + 1)\,dx.$$

Now, if $p_2 \geqq 2$, we readily have that

$$\int_{B_R} |Du - Dv|^{p_2}\,dx \leqq c\int_{B_R} \left(\mu^2 + |Du|^2 + |Dv|^2\right)^{(p_2-2)/2} |Du - Dv|^2\,dx \leqq cR^{\alpha/2} \int_{B_{4R}} (|Du|^{p_2} + 1)\,dx;$$

if $p_2 \leqq 2$, then Hölder inequality, the minimality of v and the bounds on f yield

$$\begin{aligned}
&\int_{B_R} |Du - Dv|^{p_2}\,dx \\
&\quad \leqq \left(\int_{B_R} \left(\mu^2 + |Du|^2 + |Dv|^2\right)^{(p_2-2)/2} |Du - Dv|^2\,dx\right)^{1/2} \\
&\qquad \cdot \left(\int_{B_R} \left(\mu^2 + |Du|^2 + |Dv|^2\right)^{(2-p_2)/2} |Du - Dv|^{2p_2-2}\,dx\right)^{1/2} \\
&\quad \leqq cR^{\alpha/4} \left(\int_{B_{4R}} (|Du|^{p_2} + 1)\,dx\right)^{1/2} \\
&\qquad \cdot \left(\int_{B_R} (1 + |Du|^{p_2} + |Dv|^{p_2})\,dx\right)^{1/2} \\
&\quad \leqq cR^{\alpha/4} \int_{B_{4R}} (|Du|^{p_2} + 1)\,dx,
\end{aligned}$$

so that, in any case, we come up with

$$\int_{B_R} |Du - Dv|^{p_2(R)}\,dx \leqq cR^{\alpha/4} \int_{B_{4R}} (|Du|^{p_2(R)} + 1)\,dx. \tag{3.22}$$

Now we compare u and v in B_R; indeed, since we chose $4R < R_1$ we may use (3.20)–(3.22), the minimality of v and the fact that the mapping $R \mapsto p_2(R)$ is nondecreasing and we have:

$$\begin{aligned}
&\int_{B_\rho} |Du - (Du)_{x_c,\rho}|^{p_2(R)}\,dx \\
&\quad \leqq c\int_{B_\rho} |Du - (Dv)_{x_c,\rho}|^{p_2(R)}\,dx \\
&\quad \leqq c\rho^n \fint_{B_\rho} |Dv - (Dv)_{x_c,\rho}|^{p_2(R)}\,dx + c\int_{B_R} |Du - Dv|^{p_2(R)}\,dx \\
&\quad \leqq c\left(\frac{\rho}{R}\right)^{\beta p_2(R)} \rho^n \fint_{B_R} (|Du|^{p_2(R)} + 1)\,dx \\
&\qquad + cR^{\frac{\alpha}{4}} \int_{B_{4R}} (|Du|^{p_2(4R)} + 1)\,dx \\
&\quad \leqq c\left(\frac{\rho}{R}\right)^{\beta} \rho^n R^{-\tau} + cR^{\frac{\alpha}{4}+n-\tau}.
\end{aligned}$$

We choose $\rho = \frac{1}{2}R^{1+\theta}$ where $\theta = \alpha/4(n+\beta)$: then in the previous inequality, written with ρ only, the two exponents are the same, and with the choice of τ we made at the beginning they are equal to $n + \lambda$ for some $\lambda \geqq \lambda_0 > 0$, with λ_0 depending on $\gamma_1, \gamma_2, L, \||Du|^{p(x)}\|_{L^1(\Omega)}, \alpha$. From this inequality it easily follows that

$$\int_{B(x_c,\rho)} |Du - (Du)_{x_c,\rho}|^{\gamma_1}\,dx \leqq c\rho^{n+(\lambda_0\gamma_1/\gamma_2)}.$$

The assertion follows by Campanato's characterization of Hölder continuity. □

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