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Regularity Results for Stationary Electro-Rheological Fluids

Emilio Acerbi & Giuseppe Mingione

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Abstract

We prove regularity results for weak solutions to systems modelling electro-rheological fluids in the stationary case, as proposed in [27,31]; a particular case of the system we consider is

$$\operatorname{div} u = 0, \qquad -\operatorname{div}\big((1+|\mathcal{E}(u)|^2)^{(p(x)-2)/2}\mathcal{E}(u)\big) + D\pi = f(x, u, Du),$$

where $\mathcal{E}(u)$ is the symmetric part of the gradient Du and the variable growth exponent $p(x)$ is a Hölder continuous function larger than $3n/(n+2)$.

1. Introduction

In recent years increasing attention has been paid to the study of electro-rheological fluids; these are particular fluids of high technological interest, possessing the ability to change, sometimes in a dramatic way, their mechanical properties when in the presence of an electromagnetic field $\mathbf{E}$ (their viscosity may vary by a factor of 1000). The mathematical modelling of such fluids was investigated by different authors adopting different points of view and involving various mathematical and numerical approaches (see the introduction of [31] and the references therein). In the context of continuum mechanics these fluids are seen as non-Newtonian fluids; very recently Růžička (following the ideas proposed by Rajagopal & Růžička in [27]) developed an interesting mathematical model for such fluids, taking into account the delicate interaction between the electromagnetic field $\mathbf{E}$ and the moving liquid. The resulting system (see [31] for a description of the building procedures and for a general analysis) arising from these studies is:

$$\begin{gathered} \operatorname{curl}\mathbf{E} = 0, \qquad \operatorname{div}\mathbf{E} = 0, \\ u_t - \operatorname{div} S\big(\mathbf{E}, \mathcal{E}(u)\big) + D\pi + [Du]u = f + \chi^{\mathbf{E}}[D\mathbf{E}]\mathbf{E}, \\ \operatorname{div} u = 0, \end{gathered} \tag{1.1}$$

where, according to the notation proposed in [29,31], $u : \Omega(\subset \mathbb{R}^3) \to \mathbb{R}^3$ is the velocity, $\mathbf{E}$ is the applied electromagnetic field, S the extra stress tensor, π the pressure and $\chi^{\mathbf{E}}$ the constant dielectric susceptibility; following a standard notation, $\mathcal{E}(u)$ denotes the symmetric part of the gradient.

The constitutive relation proposed in [27,29,31] for the extra stress S is

$$S(\mathbf{E}, z) := g(\mathbf{E})(1 + |z|^2)^{(p(\mathbf{E})-2)/2} z \\ + \text{terms with the same growth} \tag{1.2}$$

for any $z \in \mathcal{S}_3$, the space of symmetric 3×3 matrices; we remark that the other terms have a different shape, so system (1.1) has no overall special structure.

The main new feature of system (1.1) is that the monotonicity and the ellipticity properties of the vector field S are strongly influenced by $\mathbf{E}$ through a variable growth exponent dependence: indeed in (1.2) the exponent p is actually a function of the quantity $|\mathbf{E}|^2$. Since (1.1) is uncoupled, we may first obtain $\mathbf{E} = \mathbf{E}(x)$, thus the dependence of p and S on $\mathbf{E}$ is indeed a dependence on x. With a suitable choice of the parameters, it turns out that

$$\begin{aligned} &D_z S(x, z)\lambda \otimes \lambda \geqq \nu(1 + |z|^2)^{(p(x)-2)/2}|\lambda|^2, \\ &|D_z S(x, z)| \leqq L(1 + |z|^2)^{(p(x)-2)/2} \end{aligned} \tag{1.3}$$

for any symmetric 3×3 matrices z, λ, where the function $p : \mathbb{R}^+ \to (1, +\infty)$ reflects the physical properties of the fluid and has in general large oscillations when $|\mathbf{E}|$ changes (with $p < 2$ when $|\mathbf{E}|$ is large). The natural energy associated with this problem is thus given by

$$\int_\Omega |\mathcal{E}(u)|^{p(x)}\, dx.$$

The basic existence theory for the system (1.1) has been estabilished by RŮŽIČKA in [31], see also [29]; this theory is particularly satifactory in the steady case

$$-\operatorname{div} S + D\pi + [Du]u = f + \chi^{\mathbf{E}}[\mathbf{DE}]\mathbf{E}. \tag{1.4}$$

As is clear from (1.3), a major difficulty to be overcome is the fact that S exhibits a nonstandard growth (see [23–25,3] and the references therein), that is, its growth and coercivity exponents are different:

$$L^{-1}(|z|^{\gamma_1} - 1) \leqq S(x, z)z \leqq L(|z|^{\gamma_2} + 1),$$

where

$$\tfrac{9}{5} < \gamma_1 := \min p(x) \leqq \gamma_2 := \max p(x). \tag{1.5}$$

In this paper we are interested in the (interior) regularity properties of solutions to (1.1) in the stationary case (1.4). A first step in this direction has been performed in [31] where the author proves the existence of a $W^{2,2}$ solution to (1.4). Here we are in a more general setting and we analyse systems as

$$\operatorname{div} u = 0, \qquad \operatorname{div} A\big(x, \mathcal{E}(u)\big) + D\pi = B(x, u, Du), \tag{1.6}$$

where the vector field A exhibts an ellipticity property as in (1.3) for a fixed function $p(x)$. Under natural assumptions we shall prove that if u is a local weak solution to (1.6), then Du is Hölder continuous in an open subset of full measure, Ω_0, i.e., $Du \in C^{0,\alpha}(\Omega_0)$ for some α in the range $0 < \alpha < 1$ and with meas $(\Omega \setminus \Omega_0) = 0$.

To our knowledge this is, apart from the higher differentiability result obtained in [31] (see also [30] for periodic boundary conditions), the first regularity result for the model of electro-rheological fluids proposed in [31], and in any case the first in a pointwise sense. A further step, based on Theorem 2.1, will be the estimate of the Hausdorff dimension of the singular set $\Omega \setminus \Omega_0$, as well as regularity results for the pressure.

We add some comments; first, note that of special importance in the theory are the bounds (1.5) allowed for $p(x)$: these reflect the physical properties of a fluid. Of course, the larger the interval $[\gamma_1, \gamma_2]$, the larger the class of fluids the model is going to cover. In other words, the amplitude $\gamma_2 - \gamma_1$ relates to the possible excursions of the viscosity of the fluid when **E** changes, so it is important to prove results allowing for large values of $\gamma_2 - \gamma_1$.

In [31] the author proves existence of weak solutions for the stationary problem under the only hypothesis (1.5), and we remark that the same lower bound also appears when treating non-Newtonian fluids of standard type, that is when p is a constant (for these issues see the book [22]).

Subsequently different bounds (according to the type of problem under consideration) are introduced in [31] on γ_2 in order to prove existence of higher differentiable solutions, for which the single condition on γ_1 is no longer sufficient, in this way further restricting the class of fluids under consideration.

On the other hand, the hypotheses we consider here are consistent with, and in some respect weaker than, the ones considered by Růžička: in particular the lower bound (1.5) on γ_1 is the same as that found by Růžička, while there is no upper bound for γ_2, which is needed in [31] to prove existence of strong solutions: this allows us to treat a broad class of fluids for which higher excursions of the viscosity (and consequently of $p(x)$) are observed. In this way the existence theorem of Růžička, for which (1.5) suffices, has now a regularity counterpart.

The techniques used to obtain Theorem 2.1 are suitable for obtaining results for a more general class of systems, including the one of electro-rheological fluids we considered. This is of interest for several reasons; in the paper [38], following previous work by Baranger & Mikelić [5], Zhikov proposed a model for a class of fluids that are influenced in a similar way by the temperature T, rather than by an external electromagnetic field **E**. In this model, once again, the stress tensor satisfies growth conditions of the type (1.3), which we may call "$p(x)$ type", and the underlying energy is $\int |Du|^{p(x)}\,dx$. The exponent function $p(x) \equiv p\big(T(x)\big)$ turns out to be also an unknown of the system (which is highly coupled), thus minimal regularity assumptions must be considered on it: in Theorem 2.1 we allow $p(x)$ to be simply Hölder continuous rather than being Lipschitz continuous, as in [31]. Although this is not essential in the theory of electro-rheological fluids, the techniques developed here to treat this case could be useful when dealing with systems like the one proposed by Zhikov.

We remark that since this generalization involves no additional technical difficulties and is essentially no different to the three-dimensional physical case, we develop our results in any dimension n; this is due to the fact that our methods also allows us to prove regularity results for solutions to general elliptic systems with nonstandard growth conditions that were not covered by the available regularity theory, see [23–25]; this may be of interest in itself, judging by the large number of papers that recently appeared on the subject. For related results, see also [7, 9, 32–34].

Finally, we say a few words about the proofs. The starting point is proving a higher integrability result stating that actually $|Du|^{p(x)} \in L^{1+\delta_1}$ for some $\delta_1 > 0$, rather than just $|Du|^{p(x)} \in L^1$. This gives the manoeuvrability needed to adopt a blow-up procedure, which is a common tool when proving partial regularity; but the main point here is that the nonstandard growth conditions of the system force us to blow up solutions not in the whole Ω but in small open subsets depending on the solution itself, on the higher integrability exponent δ_1, and on the size of the oscillations of $p(x)$. At this stage various higher integrability results are important to overcome the lack of standard growth conditions of the system, and in particular a quantitative knowledge of the stability of certain higher integrability exponents arising from reverse Hölder inequalities will be crucial. Moreover, in order to treat the physically important case $\inf p(x) < 2$, we need to prove a certain form of Korn inequality for a two-parameter family of Orlicz spaces, paying attention to the stability of the constants appearing uniformly with respect to the parameters. The regularity of the solutions is then achieved via a quite delicate localization of the iteration arguments employed to get partial regularity.

2. Preliminaries and statements

In what follows, Ω denotes an open bounded domain in $\mathbb{R}^n$, and $B(x, R)$ the open ball $\{y \in \mathbb{R}^n : |x-y| < R\}$. If u is an integrable function defined on $B(x, R)$, we set

$$(u)_{x,R} = \fint_{B(x,R)} u(x)\,dx = \frac{1}{\omega_n R^n} \int_{B(x,R)} u(x)\,dx,$$

where ω_n is the Lebesgue measure of $B(0, 1)$. We also adopt the convention of writing B_R and $(u)_R$ instead of $B(x, R)$ and $(u)_{x,R}$ respectively, when the centre is not relevant, or is clear from the context; moreover, unless otherwise stated, all balls considered will have the same centre. We denote by $\mathcal{S}_n$ the set of all symmetric $n \times n$ matrices. Given two vectors $x, y \in \mathbb{R}^n$, we denote their tensor product by $x \otimes y := \{x_i y_j\}_{i,j} \in \mathbb{R}^{n^2}$ and their symmetric tensor product by $x \odot y := (1/2)(x \otimes y + y \otimes x) \in \mathcal{S}_n$. If $v : \Omega \to \mathbb{R}^n$ is an L^1 function, we denote by $\mathcal{E}(v)$ its symmetric distributional derivative:

$$\mathcal{E}(v) \equiv \{\mathcal{E}(v)\}_{ij} := \left(\partial_j v^i + \partial_i v^j\right)/2.$$

Abusing notation, if $z \in R^{n^2}$, we denote by $\mathcal{E}(z)$ its symmetric part:

$$\mathcal{E}(z) \equiv \{\mathcal{E}(z)\}_{ij} := (z_{ij} + z_{ji})/2.$$

If $s > 1$, then $s' := s/(s-1)$ is the conjugate exponent of s, while if $1 < s < n$, then $s^* := ns/(n-s)$ is the Sobolev conjugate exponent of s, whereas s^* is any real number if $s \geqq n$. Finally, the letter c freely denotes a constant, not necessarily the same in any two occurrences, while only the relevant dependences will be highlighted; if need be, we write, e.g., C_M or C_L or the like to stress that some constant depends on M, L etc., and we denote by $\hat{c}, \tilde{C}$ or the like any occurrence of some particular constant that we will later recall.

We are concerned with the following system:

$$\operatorname{div} u = 0, \qquad -\operatorname{div} A\big(x, \mathcal{E}(u)\big) + D\pi = B(x, u, Du), \tag{2.1}$$

where π is the pressure, as in (1.1), and the continuous vector fields $A : \Omega \times \mathcal{S}_n \to \mathbb{R}^{n^2}$ and $B : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}^n$ satisfy the following growth and ellipticity assumptions:

$$A(x, \cdot) \in C^1(\mathcal{S}_n), \tag{H1}$$

$$\begin{aligned} |D_z A(x,z)| &\leqq L(1+|z|^2)^{(p(x)-2)/2}, \\ D_z A(x,z)\lambda \otimes \lambda &\geqq L^{-1}(1+|z|^2)^{(p(x)-2)/2}|\lambda|^2, \end{aligned} \tag{H2}$$

$$\begin{aligned} |A(x,z) - A(x_0,z)| \leqq{} & L\omega(|x-x_0|) \\ & \times \Big[(1+|z|^2)^{(p(x)-1)/2} + (1+|z|^2)^{(p(x_0)-1)/2}\Big] \\ & \times \big(1+\log(1+|z|)\big), \end{aligned} \tag{H3}$$

$$|B(x,u,\tilde{z})| \leqq L\big(|u||\tilde{z}| + f(x)\big) \tag{H4}$$

for any $z, \lambda \in \mathcal{S}_n$, $x, x_0 \in \Omega$, $u \in \mathbb{R}^n$, $\tilde{z} \in \mathbb{R}^{n^2}$, where $L \geqq 1$, $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, $f : \Omega \to \mathbb{R}^+$ and $p : \Omega \to]1, +\infty[$ are functions such that

$$f \in L^{n+n\beta}_{\text{loc}}, \qquad p(x) \geqq \gamma_1 \geqq \frac{3n}{n+2} + 2\beta, \tag{H5}$$

for some $\beta > 0$. Moreover, $p(x)$ and ω are supposed to be continuous functions such that

$$|p(x) - p(x_0)| \leqq \omega(|x-x_0|) \leqq L|x-x_0|^{\alpha}, \tag{H6}$$

where $0 < \alpha < 1$. In view of the definition (2.5) below, it is not restrictive to assume that $A : \Omega \times \mathcal{S}_n \to \mathcal{S}_n$. We also remark that (H2) implies the following growth and coercivity properties for A:

$$|A(x,z)| \leqq c(|z|^{p(x)-1} + 1), \qquad A(x,z)z \geqq c^{-1}(|z|^{p(x)} - 1), \tag{2.2}$$

and also, through Lemma 2.1 in [2],

$$[A(x,z)-A(x,\xi)](z-\xi) \geqq c^{-1}(1+|z|^2+|\xi|^2)^{(p(x)-2)/2}|z-\xi|^2 \tag{2.3}$$

for a suitable constant $c \equiv c(n,L)$. Except at a single point, in place of (H6) we will need only the much weaker form of "logarithmic continuity"

$$R^{-\omega(R)} \leqq c \equiv c_{L,\alpha}. \tag{2.4}$$

If $p(x)$ is as above, we put

$$W^{1,p(x)} := \{u \in W^{1,1}(\Omega;\mathbb{R}^n) : \ |\mathcal{E}(u)|^{p(x)} \in L^1(\Omega)\},$$

with its local variant, $W^{1,p(x)}_{\mathrm{loc}}$, defined in a similar fashion; we remark that in the standard case $p(x) \equiv p$ this definition is equivalent to the usual one with $|\mathcal{E}(u)|^p$ replaced by $|Du|^p$, by Korn inequality, whereas the two notions might be different in this case. Moreover we set

$$C^\infty_{0,\mathrm{div}}(\Omega) := \{u \in C^\infty_0(\Omega;\mathbb{R}^n) : \ \operatorname{div} u = 0\}.$$

In our setting, according to [31], a function $u \in W^{1,p(x)}_{\mathrm{loc}}$ is a weak solution to system (2.1) if

$$\begin{aligned} &\operatorname{div} u = 0, \\ &\int_\Omega A\big(x,\mathcal{E}(u)\big)\mathcal{E}(\varphi)\,dx = \int_\Omega B(x,u,Du)\varphi\,dx \qquad \forall\varphi \in C^\infty_{0,\mathrm{div}}(\Omega). \end{aligned} \tag{2.5}$$

Our main result is the following:

Theorem 2.1. *Let $u \in W^{1,p(x)}_{\mathrm{loc}}$ be a weak solution to system* (2.1) *and assume* (H1)–(H6) *are satisfied. There exists an open set $\Omega_0 \subset \Omega$ such that $|\Omega \setminus \Omega_0| = 0$ and Du is Hölder continuous in Ω_0.*

Let us collect some auxiliary results. We shall widely use the function $V_p : \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$V_p(z) := (1+|z|^2)^{(p-2)/4}z \tag{2.6}$$

for each $z \in \mathbb{R}^k$ and for any $p > 1$. All the properties of V_p that we need may be found in [8], Lemma 2.1, and we restate them here in a way that suits our needs, together with some other properties that are a straightforward consequence of (2.6).

Lemma 2.2. *Let $p > 1$, and let $V \equiv V_p : \mathbb{R}^k \to \mathbb{R}^k$ be as in* (2.6); *then there exist $c, c(M)$ depending on p, k such that, for any $z, \eta \in \mathbb{R}^k$ and $t > 0$,*

(a) $|V(tz)| \leqq \max\{t, t^{p/2}\}|V(z)|$,

(b) $|V(z+\eta)| \leqq c\big(|V(z)| + |V(\eta)|\big)$,

(c) $|V(z) - V(\eta)| \leqq c(M)|V(z-\eta)|$ *if* $|\eta| \leqq M$ *and* $z \in \mathbb{R}^k$,

(d) $|V(z-\eta)| \leqq c(M)|V(z) - V(\eta)|$ *if* $|\eta| \leqq M$ *and* $z \in \mathbb{R}^k$,

(e) $\max\{|z|, |z|^{p/2}\} \leqq |V(z)| \leqq c \max\{|z|, |z|^{p/2}\}$ *if* $p \geqq 2$,
$c^{-1} \min\{|z|, |z|^{p/2}\} \leqq |V(z)| \leqq \min\{|z|, |z|^{p/2}\}$ *if* $1 < p < 2$,

(f) $c^{-1}|z-\eta| \leqq \frac{|V(z)-V(\eta)|}{(1+|z|^2+|\eta|^2)^{(p-2)/4}} \leqq c|z-\eta|$,

(g) *if* $p \geqq 3n/(n+2)$*, then the function* $z \mapsto |V_p(z)|^2$ *is convex,*

(h) *if* $p \geqq 2$*, then* $c^{-1}(|z|^2+|z|^p) \leqq |V_p(z)|^2 \leqq c(|z|^2+|z|^p)$.

Moreover, if $1 \leqq \gamma_1 \leqq p \leqq \gamma_2$*, all constants* $c, c(M)$ *above may be replaced by constants* $c(\gamma_1,\gamma_2), c(M,\gamma_1,\gamma_2)$ *independent of the particular value of* p.

As a consequence of (e) above, we deduce a frequently used bound on V_p: setting, with a little abuse of notation,

$$\mathbf{1}_{(p>2)} := \begin{cases} 1 \text{ if } p>2, \\ 0 \text{ otherwise,} \end{cases}$$

we have

$$|V_p(z)|^2 \leqq c(|z|^2 + \mathbf{1}_{(p>2)}|z|^p). \tag{2.7}$$

Young's inequality is quite a standard tool, but the dependence of the constants on the exponents is in general overlooked (and of course, we need to be precise just on this point). It is well known that if $A, B \geqq 0$ and $\alpha > 1$, then

$$AB \leqq \frac{1}{\alpha}A^\alpha + \left(1-\frac{1}{\alpha}\right)B^{\alpha/(\alpha-1)};$$

in particular, if $P > Q > 0$ and $a, b \geqq 0$, then for every ε in the range $0<\varepsilon<1$, setting $\lambda = [\varepsilon P/Q]^{Q/P}$ and applying Young's inequality with $A = \lambda a^Q$, $B = \lambda^{-1}b^{P-Q}$, $\alpha = P/Q$, we have

$$a^Q b^{P-Q} \leqq \varepsilon a^P + \frac{P-Q}{P}\left(\frac{Q}{P}\right)^{Q/(P-Q)}\left(\frac{1}{\varepsilon}\right)^{Q/(P-Q)} b^P,$$

and setting $H = (P/Q)-1 \geqq 0$, i.e., $P-Q = HQ$, we may write

$$\begin{aligned} a^Q b^{P-Q} &\leqq \varepsilon a^P + \frac{H}{1+H}\frac{1}{(1+H)^{1/H}}\left(\frac{1}{\varepsilon}\right)^{1/H} b^P \\ &\leqq \varepsilon a^P + \left(\frac{1}{\varepsilon}\right)^{1/H} b^P. \end{aligned} \tag{2.8}$$

In particular, if $H \geqq H_0 > 0$, then

$$a^Q b^{P-Q} \leqq \varepsilon a^P + \left(\frac{1}{\varepsilon}\right)^{1/H_0} b^P; \tag{2.9}$$

this is true as soon as, for some $H_0 > 0$,

$$P \geqq (1+H_0)Q > 0 \tag{2.10}$$

(remark that the condition $Q > 0$ may here be weakened into $Q \geqq 0$, since if $Q = 0$, then (2.9) is trivially true). We shall make frequent use of Young's inequality (2.8) with $Q = p(x) - 1$ and $P = p(x)$: since we will quickly reduce the problem to one on compact subsets, by the continuity of p we have

$$\frac{p(x)}{p(x)-1} \geqq 1 + \frac{1}{\max p(x) - 1},$$

i.e., (2.10) is satisfied and the coefficient of b^P in (2.8) depends only on ε.

The following Young-type inequality reveals a useful algebraic feature of the function V.

Lemma 2.3 (Young's inequatlity for V**).** *Let $a, b \in \mathbb{R}^k$, $1 \leqq p \leqq \gamma_2$, $\lambda \in \mathbb{R}$. For every $\varepsilon > 0$ there exists $c_\varepsilon \equiv c_\varepsilon(\varepsilon, \gamma_2)$ such that*

$$\begin{aligned}(1+|\lambda a|^2)^{(p-2)/2} a \cdot b &\leqq \varepsilon(1+|\lambda a|^2)^{(p-2)/2}|a|^2 + c_\varepsilon(1+|\lambda b|^2)^{(p-2)/2}|b|^2 \\ &= \varepsilon\lambda^{-2}|V(\lambda a)|^2 + c_\varepsilon\lambda^{-2}|V(\lambda b)|^2. \end{aligned} \tag{2.11}$$

Proof. If $\lambda = 0$ or $p = 2$, the inequality reduces to a mixture of the Cauchy-Schwarz and Young inequalities, and

$$a \cdot b \leqq |a||b| \leqq (\sqrt{2\varepsilon}|a|)(|b|/\sqrt{2\varepsilon}) \leqq \varepsilon|a|^2 + \frac{1}{4\varepsilon}|b|^2 \ : \tag{2.12}$$

from now on we take $\lambda \neq 0$ and $p \neq 2$. Multiplying both sides of (2.11) by λ^2 and applying the Cauchy-Schwarz inequality, we immediately see that it is enough to prove

$$(1+|\lambda a|^2)^{(p-2)/2}|\lambda a||\lambda b| \leqq \varepsilon(1+|\lambda a|^2)^{(p-2)/2}|\lambda a|^2 + c_\varepsilon(1+|\lambda b|^2)^{(p-2)/2}|\lambda b|^2,$$

that is, replacing $|\lambda a|$, $|\lambda b|$ by x, y, we have to prove

$$(1+x^2)^{(p-2)/2}xy \leqq \varepsilon(1+x^2)^{(p-2)/2}x^2 + c_\varepsilon(1+y^2)^{(p-2)/2}y^2 \tag{2.13}$$

for $x, y \geqq 0$. This is straightforward if

$$(1+x^2)^{(p-2)/2} \leqq (1+y^2)^{(p-2)/2},$$

because we may then just use (2.12) to deduce (2.13) again with $c_\varepsilon = 1/(4\varepsilon)$. We thus concentrate on the remaining case

$$(1+y^2)^{(p-2)/2} < (1+x^2)^{(p-2)/2},$$

which is equivalent to

$$p > 2,\ y < x \qquad \text{or} \qquad p < 2,\ x < y. \tag{2.14}$$

We remark that the function $x \mapsto (1+x^2)^\mu x$ is increasing on $\mathbb{R}^+$ if $\mu \geqq -1/2$, so this happens also for the function $x \mapsto [(1+x^2)^\mu x]^\nu$ for any $\nu > 0$. This would easily lead to the proof for $p = 1$, but see below. If $p > 1$, denoting by $q = p/(p-1)$ the conjugate exponent to p we have

$$\begin{aligned}(1+x^2)^{(p-2)/2}x &= [(1+x^2)^{(p-2)/2q}x^{2/q}]\,[(1+x^2)^{(p-2)/2p}x^{1-(2/q)}] \\ &= [(1+x^2)^{(p-2)/2}x^2]^{1/q}\,[(1+x^2)^{-1/2}x]^{(2-p)/p}.\end{aligned}$$

In both cases of (2.14) we have

$$[(1+x^2)^{-1/2}x]^{(2-p)/p} \leqq [(1+y^2)^{-1/2}y]^{(2-p)/p},$$

thus the last equality gives

$$\begin{aligned}(1+x^2)^{(p-2)/2}xy &\leqq [(1+x^2)^{(p-2)/2}x^2]^{1/q}\ [(1+y^2)^{(p-2)/2p}y^{1-(2/q)}]y \\ &= [(1+x^2)^{(p-2)/2}x^2]^{1/q}\ [(1+y^2)^{(p-2)/2}y^2]^{1/p}\end{aligned}$$

and applying Young's inequality

$$(1+x^2)^{(p-2)/2}xy \leqq \varepsilon(1+x^2)^{(p-2)/2}x^2 + \frac{(p-1)^{p-1}}{p^p\varepsilon^{p-1}}(1+y^2)^{(p-2)/2}y^2,$$

thus concluding the proof: the case $p = 1$ follows by taking the limit in the line above. □

The following lemma is more or less standard and its proof can be easily adapted from Lemma 2.3 in [1] and Lemma 3.3 in [8], using Lemma 2.1 and Lemma 2.2 from [2], which remain true also in the case $p \geqq 2$.

Lemma 2.4 (Scaling**).** *Let $M > 1$ and $x_0 \in \Omega$. Set, for every $\lambda > 0$ and $z, P \in \mathcal{S}_n$ with $|P| \leqq M$,*

$$A_{P,\lambda}(z) := \lambda^{-1}\,[A(x_0, P+\lambda z) - A(x_0, P)]\,,$$

where A satisfies (H1),(H2) *and $\gamma_1 \leqq p(x_0) \leqq \gamma_2$: then there exists a constant $\tilde{L}$ depending on $n, \gamma_1, \gamma_2, L, M$ such that, for any $z, \xi \in \mathcal{S}_n$,*

$$\begin{aligned}|A_{P,\lambda}(z)| &\leqq \tilde{L}(1+\lambda^2|z|^2)^{(p(x_0)-2)/2}|z|, \\ A_{P,\lambda}(z)z &\geqq \tilde{L}^{-1}(1+\lambda^2|z|^2)^{(p(x_0)-2)/2}|z|^2,\end{aligned} \tag{2.15}$$

$$\begin{aligned}|DA_{P,\lambda}(z)| &\leqq \tilde{L}(1+\lambda^2|z|^2)^{(p(x_0)-2)/2}, \\ DA_{P,\lambda}(z)\xi\otimes\xi &\geqq \tilde{L}^{-1}(1+\lambda^2|z|^2)^{(p(x_0)-2)/2}|\xi|^2.\end{aligned} \tag{2.16}$$

The following lemma is a well-known result (commonly referred to in the literature as Bogovskiĭ's theorem) which we restate in the form we need (see [6]; a proof may also be found in [16], Chapter 3, Section 3).

Lemma 2.5. *Let $B_R \subset \mathbb{R}^n$ and let $f \in L^q(B_R)$ with $1 < \gamma_1 \leqq q \leqq \gamma_2$ be such that $(f)_R = 0$. Then there exists $v \in W_0^{1,q}(B_R;\mathbb{R}^n)$ satisfying*

$$\operatorname{div} v = f$$

and such that

$$\int_{B_R} |Dv|^p\,dx \leqq c\int_{B_R} |f|^p\,dx$$

for every $p \in [\gamma_1, q]$, where $c \equiv c(n, \gamma_1, \gamma_2)$ is independent of $R > 0$; moreover, if the support of f is contained in B_r with $r < R$, the same holds for v.

The operator $f \in L^q(B_R) \mapsto v \in W^{1,q}(B_R;\mathbb{R}^n)$ defined in the previous lemma is linear and strongly continuous, so the fact that the constant c above is independent of q when $\gamma_1 \leqq q \leqq \gamma_2$ follows via a standard interpolation argument, while the independence of R is obtained via a rescaling argument.

We will need some specialized forms of Korn's inequality, especially as far as the stability of the constants is involved; we recall that the set of rigid motions in $\mathbb{R}^n$ is

$$\mathcal{R} := \{c + Sx : c \in \mathbb{R}^n, S \in \mathbb{R}^{n\times n}, {}^TS = -S\},$$

the set of affine functions with skew-symmetric gradient. It is easy to see that any distribution $u = (u^1, \dots, u^n) \in [\mathcal{D}'(\Omega)]^n$ in a connected open set $\Omega \subset \mathbb{R}^n$ satisfies

$$\mathcal{E}(u) \equiv 0 \qquad \Leftrightarrow \qquad u \in \mathcal{R}.$$

Let Ω be a fixed connected open subset of $\mathbb{R}^n$ and let $1 < p < +\infty$; for every $u \in L^p(\Omega;\mathbb{R}^n)$ we may define $R_{p,\Omega}(u)$ as the unique point of $\mathcal{R}$ (which is a subspace of finite dimension) of least L^p distance from u:

$$R_{p,\Omega}(u) \in \mathcal{R}, \qquad \|u - R_{p,\Omega}(u)\|_{L^p(\Omega;\mathbb{R}^n)} = \min\{\|u - r\|_{L^p(\Omega;\mathbb{R}^n)} : r \in \mathcal{R}\}\ ;$$

if $p = 1$ or $p = +\infty$ uniqueness may fail, and we may take R_1 and $R_{+\infty}$ to be any one of the minimum points. Then it is possible to deduce an appropriate form of Korn's inequality involving, for every $u \in W^{1,p}(\Omega;\mathbb{R}^n)$, the function $u - R_{p,\Omega}(u)$. Unfortunately this is unsuitable for our purposes, since p will vary with x, so it is not clear which function $R_{p,\Omega}$ is to be chosen; besides, in general $R_{p,\Omega}$ is not a linear mapping – except if $p = 2$ – although it is a projection (i.e., $R_{p,\Omega} \circ R_{p,\Omega} = R_{p,\Omega}$) and it exhibits some linear-like behaviour: for example, $R_{p,\Omega}(\lambda u) = \lambda R_{p,\Omega}(u)$ for all $\lambda \in \mathbb{R}$, and also

$$r \in \mathcal{R} \qquad \Leftrightarrow \qquad R_{p,\Omega}(u + r) = R_{p,\Omega}(u) + r.$$

Let us concentrate on the special case $p = 2$, for which an explicit representation of the rigid motion $R_{2,\Omega}(u)$ of least distance is available.

Let Ω be a bounded open subset of $\mathbb{R}^n$ and let x_0 be its barycentre; we define for every $u \in L^1(\Omega;\mathbb{R}^n)$ and $x \in \Omega$

$$(\mathcal{P}_\Omega u)_i(x) = c_i + S_{ij}(x - x_0)_j, \tag{2.17}$$

where

$$c_i := (u)_i = ⨍_\Omega u_i(x)\,dx,$$
$$S_{ij} := \frac{⨍_\Omega \big[[u - (u)]_i(x - x_0)_j - [u - (u)]_j(x - x_0)_i\big]\,dx}{⨍_\Omega \big[|(x - x_0)_i|^2 + |(x - x_0)_j|^2\big]\,dx}.$$

We collect the properties of $\mathcal{P}_\Omega$ in the following

Proposition 2.6. *Let* Ω, x_0 *and* $\mathcal{P}_\Omega$ *be as in* (2.17)*, and* $1 \leqq p \leqq +\infty$*; then*

(a) $\mathcal{P}_\Omega$ *is linear;*
(b) $\mathcal{P}_\Omega u \in \mathcal{R}$ *for all* u*;*

(c) *there exist functions* $c_{ik}(x), s_{ijk}(x) \in L^\infty(\Omega)$ *depending only on* Ω *such that*

$$(\mathcal{P}_\Omega u)_i(x) = \int_\Omega c_{ik} u_k \, dy + x_j \int_\Omega s_{ijk} u_k \, dy \ ;$$

(d) *for every* p *the operator* $\mathcal{P}_\Omega$ *maps* $L^p(\Omega; \mathbb{R}^n)$ *onto* $\mathcal{R}$*, and its norm is independent of* p*:*

$$\|\mathcal{P}_\Omega u\|_{L^p} \leqq \hat{C}(\Omega)\|u\|_{L^p} \ ;$$

(e) $\mathcal{P}$ *is invariant by rescaling, i.e.,* $\mathcal{P}_{t\Omega}[u(y/t)]$ *computed at* tx *is the same as* $\mathcal{P}_\Omega u(x)$*;*

(f) *for all* p*,*

$$\|u - R_{p,\Omega}(u)\|_{L^p} \leqq \|u - \mathcal{P}_\Omega u\|_{L^p} \leqq \big(1 + \hat{C}(\Omega)\big)\|u - R_{p,\Omega}(u)\|_{L^p} \ ; \tag{2.18}$$

(g) *if* $u \in L^2(\Omega; \mathbb{R}^n)$ *and* Ω *is symmetric enough so that* $\int_\Omega (x - x_0)_i (x - x_0)_j \, dx = 0$ *whenever* $i \neq j$ (*as, e.g., if* Ω *is a ball or a cube*) *then* $\mathcal{P}_\Omega u \equiv R_{2,\Omega}(u)$.

The first four statements are obvious, the fifth and sixth are easy and the last reduces to a tedious but simple computation of $R_{2,\Omega}$; by (2.18), we may solve the problem of choosing the projection by always taking the operator $\mathcal{P}_\Omega$, which enjoys good properties and is equivalent to every $R_{p,\Omega}$ as far as the norm of $u - R_{p,\Omega}(u)$ is concerned.

After these preliminary steps, we may state a collection of Korn-type inequalities, most of which we will need in what follows; most of these are reasonably easy consequences of the first, whose proof may be found, e.g., in [31], p. 197, and the dependence of the constants may be deduced as we explained after Lemma 2.5.

Proposition 2.7. *Let* Ω *be a bounded open set in* $\mathbb{R}^n$ *with appropriate boundary so that the Sobolev and Rellich theorems apply, let* $p \geqq 1$ *and let* $u \in L^1(\Omega; \mathbb{R}^n)$ *be such that* $\mathcal{E}(u) \in L^1(\Omega; \mathbb{R}^{n^2})$*. Then the following statements hold* (*unless otherwise specified all constants may depend on everything but the function* u)*:*

(a) *if* $p > 1$*, then*

$$\|Du\|_p \leqq k_0\|u - (u)_\Omega\|_1 + k_1'\|\mathcal{E}(u)\|_p \leqq k_1\|u - (u)_\Omega\|_p + k_1'\|\mathcal{E}(u)\|_p;$$

(b) *if* $p > 1$ *and* $u = 0$ *on* $\partial\Omega$*, then*

$$\|Du\|_p \leqq k_2\|\mathcal{E}(u)\|_p;$$

(c) *if* Ω *is connected and* $p \geqq 1$*, then*

$$\|u - \mathcal{P}_\Omega u\|_p \leqq k_3\|\mathcal{E}(u)\|_p \ ;$$

(d) *if Ω is connected and $p \geqq 1$, then for all $q \in \mathbb{R}$ with $1 \leqq q \leqq p^*$ we have*

$$q^{-1}\|u - \mathcal{P}_\Omega u\|_q \leqq k_4\|\mathcal{E}(u)\|_p;$$

(e) *as for the dependence of the constants on the diameter of Ω, we have:*

$$k_0(t\Omega) = \frac{1}{t^{n+1-n/p}} k_0(\Omega), \quad k_1(t\Omega) = \frac{1}{t} k_1(\Omega), \quad k_1'(t\Omega) = k_1'(\Omega),$$

$$k_2(t\Omega) = k_2(\Omega), \quad k_3(t\Omega) = t\, k_3(\Omega), \quad k_4(t\Omega) = t^{1+(n/q)-(n/p)}\, k_4(\Omega);$$

(f) *if $1 < \gamma_1 \leqq p \leqq \gamma_2 < +\infty$, the constants $k_0, k_1, k_1', k_2, k_3, k_4$ depend on p only through γ_1, γ_2.*

As a corollary, we write a version, valid on balls, of some of the inequalities above, in which $1 < \gamma_1 \leqq p \leqq \gamma_2$, $1 \leqq q \leqq p^*$ and the constants depend only on n, γ_1, γ_2:

$$\fint_{B_R} |Du|^p\, dx \leqq k \fint_{B_R} |\mathcal{E}(u)|^p\, dx + k\left(\fint_{B_R} \left|\frac{u - (u)_R}{R}\right|^q dx\right)^{p/q} \tag{2.19}$$

$$u = 0 \quad \text{on } \partial B_R \qquad \Rightarrow \qquad \frac{1}{q}\left(\fint_{B_R} \left|\frac{u}{R}\right|^q dx\right)^{1/q} \leqq k\left(\fint_{B_R} |\mathcal{E}(u)|^p\, dx\right)^{1/p} \tag{2.20}$$

$$\frac{1}{q}\left(\fint_{B_R} \left|\frac{u - \mathcal{P}_{B_R} u}{R}\right|^q dx\right)^{1/q} \leqq k\left(\fint_{B_R} |\mathcal{E}(u)|^p\, dx\right)^{1/p}. \tag{2.21}$$

3. Korn's inequalities in Orlicz spaces

Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be a Young function (or "N-function"), i.e., $G(0) = 0$, G is convex and increasing, $G(t)/t$ is increasing and $G(t)/t \to +\infty$ as $t \to +\infty$. We consider the Orlicz space generated by G, that is, the Banach space $L^G \equiv L^G(\mathbb{R}^n; \mathbb{R}^k)$, equipped with the following Luxemburg norm:

$$\|h\|_G := \inf\{\lambda > 0 : \int_{\mathbb{R}^n} G\Big(\frac{|h|}{\lambda}\Big)\, dx \leqq 1\}.$$

(We refer the reader to the monograph [28] for a complete account of the theory.) In the case $G(t) = t^p/p,\ p > 1$ the previous quantity is exactly the standard L^p norm. Beside L^G we shall also consider the Orlicz-Sobolev space $W^{1,G}(\mathbb{R}^n; \mathbb{R}^k)$, consisting of all the functions u such that both u and Du are in L^G (see again [28]). Let $0 < \lambda < 1$ and $p > 1$; we shall be particularily interested in the Young function

$$G \equiv G_{p,\lambda}(t) := (1 + \lambda t)^{p-2} t^2. \tag{3.1}$$

The main result of the section is the following:

Theorem 3.1. *Let* $\beta > 0$ *and* $3n/(n+2) + 2\beta \leqq \gamma_1 \leqq p \leqq 2$*, and let* $u \in W^{1,G_{p,\lambda}}(\mathbb{R}^n;\mathbb{R}^n)$*; there exists a constant* $c \equiv c(n,\gamma_1)$*, independent of* u, p, λ*, such that*

$$\|Du\|_{G_{p,\lambda}} \leqq c\|\mathcal{E}(u)\|_{G_{p,\lambda}}. \tag{3.2}$$

The previous Korn-type inequality will be derived via an interpolation theorem for singular integrals and will be applied in order to obtain certain strong convergences in the proof of Theorem 2.1 (see Step 3).

Remark 3.2. The main point in Theorem 3.1 is the fact that the constant c appearing in (3.2) is independent of λ (see Step 4). For this reason we will be careful when proving (3.2) and we will invoke Theorem 3.3 below.

Before proving Theorem 3.1 we recall some terminology that can be found in [37]. An operator T (called "operation", in [37]) acting from a subset of the class of measurable functions defined on $\mathbb{R}^n$ into itself, is said to be sublinear if the following three conditions are satisfied:

(i) if T is defined on f_1, f_2, then it is also defined on $f_1 + f_2$;
(ii) $|T(f_1+f_2)(x)| \leqq |T(f_1)(x)| + |T(f_2)(x)|$ for a.e. $x \in \mathbb{R}^n$;
(iii) $|T(cf)| = |c||T(f)|$ for any $c \in \mathbb{R}$.

An operator $T : L^q(\mathbb{R}^n;\mathbb{R}^k) \to L^q(\mathbb{R}^n;\mathbb{R}^h)$ is of type (q,q), with $1 \leqq q \leqq +\infty$, if there exists a constant M such that, for any $f \in L^q(\mathbb{R}^n;\mathbb{R}^k)$,

$$\|Tf\|_{L^q} \leqq M\|f\|_{L^q}.$$

We shall call the smallest of such numbers M, the q-norm of T. In the same way, given an Orlicz space $L^G(\mathbb{R}^n;\mathbb{R}^k)$, an operator T will be called of type (G,G) if

$$\|Tf\|_G \leqq M\|f\|_G$$

for any $f \in L^G(\mathbb{R}^n;\mathbb{R}^k)$ and for some $M < +\infty$, the G-norm of T being similarly defined.

We recall an interpolation result in Orlicz spaces, which is a particular case of a more general result that can be found, for instance, in [37], Theorem 2.3.

Theorem 3.3. *Let* $k = 1$ *and* $1 < s_1 < s_2 < +\infty$ *and suppose that a sublinear operator* T *is simultaneously of types* (s_1,s_1) *and* (s_2,s_2)*, with* s_1*-norm and* s_2*-norm equal to* M_1 *and* M_2 *respectively. Let* $G : \mathbb{R}^+ \to \mathbb{R}^+$ *be an Orlicz function such that*

$$G(t) = \int_0^t a(\xi)\, d\xi, \tag{3.3}$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ *is a monotone function. Moreover assume that the function* $G(t)/t^{s_1}$ *is increasing and the function* $G(t)/t^{s_2}$ *is decreasing, and:*

$$\int_0^t \frac{G(\xi)}{\xi^{s_1}} \frac{d\xi}{\xi} \leqq K_1 \frac{G(t)}{t^{s_1}}, \tag{3.4}$$

$$\int_t^{+\infty} \frac{G(\xi)}{\xi^{s_2}} \frac{d\xi}{\xi} \leqq K_2 \frac{G(t)}{t^{s_2}}, \tag{3.5}$$

for any $t > 0$*. Then the operator* T *is of type* (G, G) *with* G*-norm depending on* $s_1, s_2, K_1, K_2, M_1, M_2$.

Proof of Theorem 3.1. The beginning of the proof relies on an argument that allows us, roughly speaking, to represent Du trough $\mathcal{E}(u)$ via a singular integral operator. We follow the lines of [4], Theorem 7.4 (which, in turn, uses information from the second part of KOHN's thesis [19]). Let us first suppose that $u \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$. By the last formula of step 2 in the proof of Theorem 7.4 from [4] it follows that, for a.e. $x \in \mathbb{R}^n$,

$$|Du(x)| \leqq c(n)|\mathcal{E}(u)(x)| + c(n)T^*\big(\mathcal{E}(u)\big)(x), \tag{3.6}$$

where T^* is the singular integral operator, which is homogeneus and sublinear, defined as follows (according to the terminology of [35], Chapter 2): for any $v \in C^\infty(\mathbb{R}^n; \mathbb{R}^{n^2})$,

$$T^*(v)(x) := \sup_{\rho>0} |T_\rho(v)(x)|, \qquad T_\rho(v)(x) := \int_{\{|y-x|\geqq\rho\}} \frac{K(y-x)}{|y-x|^n} v(y)\, dy.$$

The function $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^{n^4}$ is a smooth, 0-homogeneus function with mean value zero on the unit sphere S^{n-1}. More precisely $K(x) = \Gamma(x)/|x|^2$ where $\Gamma(x) = \{\Gamma_{ij}^{lm}\}$ is the fourth-order tensor identified in the second part of [19], Section 5, see also [4], formula (6.2). See also [26] for similar representation formulas.

By the Calderón-Zygmund theory (see [35], Chapter 2, and [4]) the operator T^* is of type (q, q) for any $1 < q < +\infty$. In particular it is of types $(\frac{3n}{n+2}, \frac{3n}{n+2})$ and $(2 + \beta, 2 + \beta)$. As Theorem 3.3 is stated only in the scalar case, we may make a standard extension by considering for every $i = 1, \dots, n$ the operator $T_i(v) := T^*(v\mathbf{e}_i)$, where $\{\mathbf{e}_i\}_i$ is the canonical basis of $\mathbb{R}^n$; the facts that

$$\|T_i\| \leqq \|T^*\| \leqq \sum_i \|T_i\|, \qquad f \leqq g \quad \Rightarrow \quad \|f\|_{G_{p,\lambda}} \leqq \|g\|_{G_{p,\lambda}}$$

imply that we may apply Theorem 3.3 to each T_i and deduce the result for T^*. We want to use Theorem 3.3 with the following choice:

$$G \equiv G_{p,\lambda}, \qquad s_1 \equiv \frac{3n}{n+2}, \qquad s_2 \equiv 2 + \beta, \qquad T \equiv T^*.$$

Denote by $M_1 \equiv M_1(n)$ and $M_2 \equiv M_2(n)$ the s_1 and s_2 norms of T^*, respectively. We observe that, since the function $G_{p,\lambda}$ is convex, see (g) of Lemma 2.2, (3.3) is satisfied. Again, by the definition of $G_{p,\lambda}$, it follows that for each $\lambda \in (0, 1)$, the functions $G_{p,\lambda}(t)/t^{s_1}$ and $G_{p,\lambda}(t)/t^{s_2}$ are respectively increasing and decreasing. It remains to check inequalities (3.4), (3.5) uniformly for $\lambda \in (0, 1)$, i.e., the constants K_1 and K_2 must be chosen independent of λ. Since the argument is very elementary we shall only check (3.4), because (3.5) is similar. We first note that a simple change of variable allows us to consider only the case $\lambda = 1$: indeed, assuming (3.4) when

$\lambda = 1$ for a suitable K_1, then

$$\int_0^t \frac{G_{p,\lambda}(\xi)}{\xi^{s_1}} \frac{d\xi}{\xi} = \lambda^{s_1-2} \int_0^t \frac{G_{p,1}(\lambda\xi)}{(\lambda\xi)^{s_1}} \frac{\lambda d\xi}{\lambda\xi}$$
$$= \lambda^{s_1-2} \int_0^{\lambda t} \frac{G_{p,1}(\xi)}{\xi^{s_1}} \frac{d\xi}{\xi} \leqq K_1 \lambda^{s_1-2} \frac{G_{p,1}(\lambda t)}{(\lambda t)^{s_1}} = K_1 \frac{G_{p,\lambda}(t)}{t^{s_1}}.$$

Now we check (3.4) for $\lambda = 1$ and we show that it is possible to take $K_1 = 1/\beta$. Indeed, since $p \leqq 2$ the function $s \to (1 + 1/s)^{p-2}$ is increasing,

$$\int_0^t \frac{G_{p,1}(\xi)}{\xi^{s_1}} \frac{d\xi}{\xi} = \int_0^t (1 + 1/\xi)^{p-2} \xi^{p-s_1-1} \, d\xi$$
$$\leqq (1 + 1/t)^{p-2} \int_0^t \xi^{p-s_1-1} \, d\xi$$
$$= \frac{(1 + t)^{p-2} t^2}{(p - s_1) t^{s_1}} \leqq \frac{G_{p,1}(t)}{\beta t^{s_1}}.$$

As mentioned above, a similar argument, also showing that it is possible to choose $K_2 = 1/\beta$, applies in order to check (3.5). Now we apply the interpolation Theorem 3.3. It follows there exists a constant $c \equiv c(n, \gamma_1, \beta)$, but independent of v, $p \in [\gamma_1, 2]$ and λ, such that

$$\|T^*(v)\|_{G_{p,\lambda}} \leqq c \|v\|_{G_{p,\lambda}}$$

for any $v \in C^\infty(\mathbb{R}^n; \mathbb{R}^{n^2})$. This estimate, together with the pointwise inequality (3.6), gives (3.2) in the case $u \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$. The general case now follows by a density argument since the function $G_{p,\lambda}$ satisfies the $\triangle_2$ condition and hence the space $C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $L^{G_\lambda}(\mathbb{R}^n; \mathbb{R}^n)$, see [28]. $\square$

We conclude the section with a lemma from the theory of Orlicz spaces, which is standard when a single Young function is involved in the statement; since we are going to deal with a sequence of Young functions we include a slightly different proof.

Lemma 3.4. *Let $\lambda_h \to 0$ be a sequence of real numbers, and let $u_h \in L^{G_{p,\lambda_h}}(\mathbb{R}^n)$ for every h. Then $\|u_h\|_{G_{p,\lambda_h}} \to 0$ if and only if $\int_{\mathbb{R}^n} G_{p,\lambda_h}(|u_h|) \, dx \to 0$.*

Proof. Suppose that $s_h := \|u_h\|_{G_{p,\lambda_h}} \to 0$. Then by the convexity properties of G_{p,λ_h} we have

$$\int_{\mathbb{R}^n} G_{p,\lambda_h}(|u_h|) \, dx \leqq s_h \int_{\mathbb{R}^n} G_{p,\lambda_h}(|u_h|/s_h) \, dx \leqq s_h \to 0 \ ,$$

by definition of the Luxemburg norm. Conversely, suppose that $\int_{\mathbb{R}^n} G_{p,\lambda_h}(|u_h|) \, dx \to 0$: then for every $\varepsilon > 0$ there exists $\nu \equiv \nu(\varepsilon)$ such that for every $h \geqq \nu$ we have $\varepsilon^{-2} \int_{\mathbb{R}^n} G_{p,\lambda_h}(|u_h|) \, dx \leqq \varepsilon$. Then by the structure of $G_{p,\lambda}$, for $h \geqq \nu$,

$$\int_{\mathbb{R}^n} G_{p,\lambda_h}(|u_h|/\varepsilon) \, dx \leqq \varepsilon^{-2} \int_{\mathbb{R}^n} G_{p,\lambda_h}(|u_h|) \, dx \leqq \varepsilon \leqq 1 \ ,$$

which means that for $h \geqq \nu$ we have $\|u_h\|_{G_{p,\lambda_h}} \leqq \varepsilon$, that is, $\|u_h\|_{G_{p,\lambda_h}} \to 0$. $\square$

4. Higher integrability results

In this section we collect some higher integrability results that will be crucial for subsequent developments. Since our results are of a local nature, it is not restrictive (upon passing to open subsets, compactly contained in Ω and having an appropriate boundary, and possibly enlarging or reducing some constants) to assume that

$$|\mathcal{E}(u)|^{p(x)} \in L^1(\Omega), \qquad \gamma_1 \leqq p(x) \leqq \gamma_2 \quad \forall x \in \Omega \tag{4.1}$$

$$\gamma_1 < n\,, \qquad 0 < \beta < \alpha < \frac{1}{n+2}, \qquad \|f\|_{L^n(\Omega)} + \|f\|_{L^{n+n\beta}(\Omega)} \leqq L. \tag{4.2}$$

As a first application of the Korn inequalities stated in Proposition 2.7, and in particular of (a), we observe that from (4.1) and the Sobolev embedding theorem it follows that

$$|Du|^{\gamma_1} + |u|^{\gamma_1^*} \in L^1(\Omega). \tag{4.3}$$

Remark 4.1. The number β is going to be quite small in the applications, and in case the function f is, say, in L^∞, we shall treat f as a function in $L^{n+n\beta}$ only; we could have been more precise by introducing two different exponents, β_1 for the integrability of f and β_2 for the distance of γ_1 to $3n/(n+2)$, but we thought this was not worth the complication in the notation.

Theorem 4.2. *Let* $u \in W^{1,p(x)}_{\mathrm{loc}}$ *be a weak solution to system* (2.1) *and assume that the vector fields* A *and* B *satisfy* (H1)–(H5) *and* (2.4). *Then there exist* $c, \delta_1 > 0$, *both depending on* $n, \gamma_1, \gamma_2, L, c_{L,\alpha}, \beta$, *such that if* $B_{2R} \subset\subset \Omega$, *then*

$$\begin{aligned}\left(\fint_{B_R} |\mathcal{E}(u)|^{p(x)(1+\delta_1)}\,dx\right)^{1/(1+\delta_1)} \\ \leqq c \fint_{B_{2R}} |\mathcal{E}(u)|^{p(x)}\,dx + c \fint_{B_{2R}} \left(|Du|^{\gamma_1} + |u|^{\gamma_1^*} + 1\right)dx.\end{aligned} \tag{4.4}$$

Proof.
Step 1: Localization. Fix $\theta^2 := 1+1/n$, and let $R_0 > 0$ be such that $\omega(R_0) \leqq \theta - 1$, which is possible since $\omega(R) \to 0$ as $R \to 0$ by (2.4); we will prove the statement only for balls $B_{2R} \subset\subset \Omega$ with $R < R_0$, and the result for the remaining (larger) balls may then be obtained by covering each of them with at most a fixed number of the smaller balls.

Take $B_{2R} \subset\subset \Omega$ with $R < R_0$, and following Proposition 2.6 let us define $\mathcal{P} \equiv \mathcal{P}_{B_R} u$, the projection of u on the space of rigid displacements $\mathcal{R}$ with respect to the L^2 norm. Define

$$p_1 := \inf_{B_R} p(x), \qquad p_2 := \sup_{B_R} p(x).$$

By our choice of R_0 we have

$$p_2 \leqq p_1\left(\frac{n+1}{n}\right), \tag{4.5}$$

and the Sobolev-Korn inequality (2.21) easily gives

$$\fint_{B_R}\left|\frac{u-\mathcal{P}}{R}\right|^{p_2} dx \leqq c\left(\fint_{B_R}|\mathcal{E}(u)|^{p_1/\theta}\,dx\right)^{\theta p_2/p_1}. \tag{4.6}$$

Step 2: Caccioppoli-type inequality. Let $\eta \in C_0^\infty(B_R)$ be a cut-off function such that $\eta \equiv 1$ on $B_{R/2}$, $0 \leqq \eta \leqq 1$, $|D\eta| \leqq cR^{-1}$, and take

$$\varphi := \eta^{p_2}(u-\mathcal{P}) + w,$$

where the function w is defined according to Lemma 2.5 as a solution to

$$\operatorname{div} w = -\operatorname{div}\bigl(\eta^{p_2}(u-\mathcal{P})\bigr) = -(u-\mathcal{P})D(\eta^{p_2}). \tag{4.7}$$

We observe that such a w exists since

$$\operatorname{div} u = 0, \qquad \int_{B_R}(u-\mathcal{P})D(\eta^{p_2})\,dx = 0,$$

and we remark that $w \in W_0^{1,p_2}(B_R;\mathbb{R}^n)$ by (4.5) and the summability properties of u. Again by Lemma 2.5 we have the estimate

$$\fint_{B_R}|Dw|^{\tilde{p}}\,dx \leqq c\fint_{B_R}\left|\frac{u-\mathcal{P}}{R}\right|^{\tilde{p}} dx \tag{4.8}$$

for every exponent $\tilde{p}$ such that the right-hand side is finite, and with the constant c stable. We use φ as a test function in (2.5) and we obtain

$$\begin{aligned}
(\mathrm{I}) := &\int_{B_R}\eta^{p_2}A\bigl(x,\mathcal{E}(u)\bigr)\mathcal{E}(u)\,dx\\
= &-p_2\int_{B_R}\eta^{p_2-1}A\bigl(x,\mathcal{E}(u)\bigr)\bigl((u-\mathcal{P})\odot D\eta\bigr)\,dx\\
&-\int_{B_R}A\bigl(x,\mathcal{E}(u)\bigr)\mathcal{E}(w)\,dx + \int_{B_R}\eta^{p_2}B(x,u,Du)(u-\mathcal{P})\,dx\\
&+\int_{B_R}B(x,u,Du)w\,dx := (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}).
\end{aligned}$$

Now we estimate the terms introduced above. By (2.2),

$$\int_{B_R}\eta^{p_2}|\mathcal{E}(u)|^{p(x)}\,dx \leqq c(\mathrm{I}) + cR^n.$$

Since $p_2 \geqq p(x)$ for any $x \in B_R$, we have $\eta^{p_2} \geqq \eta^{(p_2-1)[(p(x))']}$; thus, for every $0 < \varepsilon < 1$,

$$(\mathrm{II}) \;\leqq\; \varepsilon\int_{B_R}\eta^{p_2}|\mathcal{E}(u)|^{p(x)}\,dx + C_\varepsilon\int_{B_R}\left|\frac{u-\mathcal{P}}{R}\right|^{p(x)} dx + cR^n,$$

where we used Young's inequality and (2.2). Again using Young's inequality together with (4.8), we have

$$
\begin{aligned}
\text{(III)} &\leqq c\int_{B_R} |\mathcal{E}(u)|^{p(x)-1}|Dw|\,dx + c\int_{B_R} |Dw|\,dx \\
&\leqq \varepsilon \int_{B_R} |\mathcal{E}(u)|^{p(x)}\,dx + C_\varepsilon \int_{B_R} |Dw|^{p(x)}\,dx + cR^n \\
&\leqq \varepsilon \int_{B_R} |\mathcal{E}(u)|^{p(x)}\,dx + C_\varepsilon \int_{B_R} \left|\frac{u-\mathcal{P}}{R}\right|^{p_2} dx + C_\varepsilon R^n.
\end{aligned}
$$

Finally, by (H4) we have

$$
\begin{aligned}
\text{(IV)} &\leqq c\int_{B_R} |u||Du||u-\mathcal{P}|\,dx + c\int_{B_R} f(x)|u-\mathcal{P}|\,dx, \\
\text{(V)} &\leqq c\int_{B_R} |u||Du||w|\,dx + c\int_{B_R} f(x)|w|\,dx.
\end{aligned}
$$

In order to estimate these last four quantities, we introduce the following exponents:

$$
\mu := 1 + \frac{\beta}{2}\left(\frac{n+2}{n}\right), \qquad q := \left[\frac{1}{2}\left(\frac{\gamma_1}{\mu}\right)^*\right]'. \tag{4.9}
$$

With such a choice, since $2\beta + 3n/(n+2) \leqq \gamma_1 < n < n\mu$ and using (4.2) we obtain

$$
1 \leqq \frac{\gamma_1}{\mu} < n, \qquad 1 < q < q\mu \leqq \gamma_1, \qquad \left(\frac{\gamma_1}{\mu}\right)^* \leqq \frac{\gamma_1^*}{\mu}. \tag{4.10}
$$

Using the Hölder and Young inequalities together with the definition of $\mathcal{P}$ and Proposition 2.7, we get (all norms are on B_R)

$$
\begin{aligned}
\int_{B_R} |u||Du||u-\mathcal{P}|\,dx &\leqq \|Du\|_q \|u-\mathcal{P}\|_{(\gamma_1/\mu)^*} \|u\|_{(\gamma_1/\mu)^*} \\
&\overset{(2.18)}{\leqq} c\|Du\|_q \; \|u\|^2_{(\gamma_1/\mu)^*} \\
&\leqq c\int_{B_R} |Du|^q\,dx + c\int_{B_R} |u|^{\left(\frac{\gamma_1}{\mu}\right)^*} dx.
\end{aligned}
$$

Again using the Hölder inequality and Proposition 2.6, and observing that 4.10 implies that $n/(n-1) = n' = 1^* \leqq (\gamma_1/\mu)^*$, we have

$$\begin{aligned}\int_{B_R} f(x)|u-\mathcal{P}|\,dx &= cR^n \fint_{B_R} f(x)|u-\mathcal{P}|\,dx \\ &\leqq cR^n \left(\fint_{B_R} |u-\mathcal{P}|^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}} \left(\fint_{B_R} [f(x)]^n\,dx\right)^{\frac{1}{n}} \\ &\leqq cR^{n-1}\left(\fint_{B_R} |u-\mathcal{P}|^{\gamma_1^*}\,dx\right)^{1/\gamma_1^*} \|f\|_n \\ &\leqq cR^n \|f\|_n \left(\fint_{B_R} |\mathcal{E}(u)|^{\gamma_1}\,dx\right)^{1/\gamma_1} \\ &\leqq \varepsilon \int_{B_R} |\mathcal{E}(u)|^{p(x)}\,dx + C_\varepsilon R^n,\end{aligned}$$

with $0<\varepsilon<1$. Next we estimate the terms bounding (V); keeping the previous notation, and recalling that $w \in W_0^{1,p_2}(B_R;\mathbb{R}^n)$,

$$\begin{aligned}\int_{B_R} |u||Du||w|\,dx &\leqq \|Du\|_q \|u\|_{(\gamma_1/\mu)^*} \|w\|_{(\gamma_1/\mu)^*} \\ &\leqq c\|Du\|_q \|u\|_{(\gamma_1/\mu)^*} \|Dw\|_{\gamma_1/\mu} \\ &\overset{(4.8)}{\leqq} c\|Du\|_q \|u\|_{(\gamma_1/\mu)^*} \|u-\mathcal{P}\|_{(\gamma_1/\mu)^*} \\ &\overset{(2.18)}{\leqq} c\|Du\|_q \|u\|^2_{(\gamma_1/\mu)^*} \\ &\leqq c\int_{B_R} |Du|^q\,dx + c\int_{B_R} |u|^{(\gamma_1/\mu)^*}\,dx.\end{aligned}$$

As we did for (IV) we estimate the remaining term using Korn's inequality (2.21):

$$\begin{aligned}\int_{B_R} f(x)|w|\,dx &\leqq cR^n \left(\fint_{B_R} |w|^{\gamma_1^*}\,dx\right)^{1/\gamma_1^*} \left(\fint_{B_R} [f(x)]^n\,dx\right)^{1/n} \\ &\leqq cR^{n+1}\left(\fint_{B_R} |Dw|^{\gamma_1}\,dx\right)^{1/\gamma_1} \left(\fint_{B_R} [f(x)]^n\,dx\right)^{1/n} \\ &\overset{(4.8)}{\leqq} cR^n \left(\fint_{B_R} \left|\frac{u-\mathcal{P}}{R}\right|^{\gamma_1}\,dx\right)^{1/\gamma_1} \|f\|_n \\ &\overset{(2.21)}{\leqq} cR^n \|f\|_{L^n} \left(\fint_{B_R} |\mathcal{E}(u)|^{\gamma_1}\,dx\right)^{1/\gamma_1} \\ &\leqq \varepsilon \int_{B_R} |\mathcal{E}(u)|^{p(x)}\,dx + C_\varepsilon R^n.\end{aligned}$$

Connecting the previous estimates and recalling (4.10), we find

$$\fint_{B_{R/2}} |\mathcal{E}(u)|^{p(x)}\,dx \leqq \varepsilon \fint_{B_R} |\mathcal{E}(u)|^{p(x)}\,dx + C_\varepsilon \fint_{B_R} \left|\frac{u-\mathcal{P}}{R}\right|^{p_2} dx$$
$$+C_\varepsilon \fint_{B_R} \big(|Du|^{\gamma_1/\mu} + |u|^{\gamma_1^*/\mu} + 1\big)\,dx \tag{4.11}$$

for any $0 < \varepsilon < 1$; if we set $g := |Du|^{\gamma_1/\mu} + |u|^{\gamma_1^*/\mu} + 1$, where $\mu \equiv \mu(n, \beta) > 1$ was defined in (4.9), then recalling (4.3), we find that $g \in L^\mu(B_R)$.

Step 3: Gehring lemma. Using (4.6) we obtain:

$$\begin{aligned}
&\fint_{B_R} \left|\frac{u-\mathcal{P}}{R}\right|^{p_2} dx \\
&\leqq c\left(\fint_{B_R} |\mathcal{E}(u)|^{p(x)/\theta}\,dx\right)^{\frac{\theta(p_2-p_1)}{p_1}} \left(\fint_{B_R} |\mathcal{E}(u)|^{p(x)/\theta}\,dx\right)^{\theta} + c \\
&\leqq cR^{-n\theta(p_2-p_1)/p_1} \|1 + |\mathcal{E}(u)|^{p(x)}\|_{L^1(\Omega)}^{\theta(p_2-p_1)/p_1} \left(\fint_{B_R} |\mathcal{E}(u)|^{p(x)/\theta}\,dx\right)^{\theta} + c \\
&\leqq cR^{-n\omega(R)\theta/p_1} \left(\fint_{B_R} |\mathcal{E}(u)|^{p(x)/\theta}\,dx\right)^{\theta} + c.
\end{aligned}$$

Combining the last estimate with (4.6) and (4.11) and recalling (2.4), we find that, for every $0 < \varepsilon < 1$ and every $B_{2R} \subset\subset \Omega$ such that $R \leqq R_0 \equiv R_0(n, c_{L,\alpha})$,

$$\begin{aligned}
&\fint_{B_{R/2}} |\mathcal{E}(u)|^{p(x)}\,dx \\
&\qquad \leqq \varepsilon \fint_{B_R} |\mathcal{E}(u)|^{p(x)}\,dx + C_\varepsilon \left(\fint_{B_R} |\mathcal{E}(u)|^{p(x)/\theta}\,dx\right)^{\theta} + C_\varepsilon \fint_{B_R} g\,dx,
\end{aligned}$$

with C_ε depending also on $n, \gamma_1, \gamma_2, L, \beta, c_{L,\alpha}$. At this point the conclusion follows by applying a variant of Gehring's lemma, see [36]. Again, the precise dependence of the constants can be deduced from [36], see also [18]. □

Remark 4.3. The possibility of having (4.10) is the main reason for assuming the lower bound $\gamma_1 \geqq 2\beta + 3n/(n+2)$. It is then clear that in (4.4) the higher integrability exponent δ_1 can be made smaller if need be. For related regularity results see also [31, 10].

The next lemma seems to be stated in a rather awkward form; indeed, we could have stated it for a vector field $A(z)$ independent of x and a fixed power p in a ball B_R, but we preferred to make it appear exactly in the form that will be used later.

Lemma 4.4. *Assume A satisfies* (H1) *and* (H2)*, let $\gamma_1 \leqq p_m \leqq \gamma_2$ and consider a ball $B(x_0, 2R)$; let $v \in W^{1,p_m}(B(x_0, 2R); \mathbb{R}^n)$ be a solution of*

$$\operatorname{div} v = 0, \qquad \int_{B(x_0,2R)} A\big(x_m, \mathcal{E}(v)\big)\mathcal{E}(\varphi)\,dx = 0 \qquad \forall \varphi \in C^\infty_{0,\operatorname{div}}(B_{2R}).$$

There exist a constant c *and an exponent* $\tilde{\delta} > 0$*, depending on* n, γ_1, γ_2, L *but independent of* R*,* υ*,* p_m*,* x_0*, such that*

$$\left(\fint_{B(x_0,R)} |\mathcal{E}(u)|^{p_m(1+\tilde{\delta})}\, dx \right)^{1/(1+\tilde{\delta})} \leqq c \fint_{B(x_0,2R)} |\mathcal{E}(u)|^{p_m} + 1\, dx\ . \quad (4.12)$$

We omit the proof of this result, which can be obtained as Theorem 4.2, being actually much simpler. Again, the dependence of the constants can be checked by looking at [36].

In the remainder of the section all the balls we consider are centred at the origin (except at the very end, but it will be specified), and we will omit the indication of the centre.

We will later apply the next lemma to a vector field $A_{P,\lambda}$ as was introduced in Lemma 2.4; to avoid the notation, we assume that $\tilde{A} : S_n \to S_n$ is of class C^1 and satisfies, for some $\tilde{L} > 1$, $\lambda > 0$ and $p > 1$,

$$\begin{aligned} |\tilde{A}(z)| &\leqq \tilde{L}(1+\lambda^2|z|^2)^{(p-2)/2}|z|, \\ \tilde{A}(z)z &\geqq \tilde{L}^{-1}(1+\lambda^2|z|^2)^{(p-2)/2}|z|^2, \end{aligned} \quad (4.13)$$

$$\begin{aligned} c|D\tilde{A}(z)| &\leqq \tilde{L}(1+\lambda^2|z|^2)^{(p-2)/2}, \\ D\tilde{A}(z)\xi\otimes\xi &\geqq \tilde{L}^{-1}(1+\lambda^2|z|^2)^{(p-2)/2}|\xi|^2. \end{aligned} \quad (4.14)$$

Lemma 4.5. *Let* $\gamma_1 \leqq p \leqq \gamma_2$ *be a fixed number and let* $\tilde{u} \in W^{1,p}(B_1; \mathbb{R}^n)$ *be a weak solution to the system*

$$\operatorname{div}\tilde{u} = 0, \int_{B_1} \tilde{A}\big(\mathcal{E}(\tilde{u})\big)\mathcal{E}(\varphi)\, dx = 0 \qquad \forall\varphi \in C^\infty_{0,\mathrm{div}}(B_1), \quad (4.15)$$

where the vector field $\tilde{A} : \mathcal{S}_n \to \mathcal{S}_n$ *satisfies the assumptions above. Then there exist* $c, \delta_2 > 0$*, both depending on* $n, \gamma_1, \gamma_2, \tilde{L}$ *but independent of* $\tilde{u}$ *and* $\lambda \in (0, 1)$*, such that*

$$\left(\fint_{B_{1/2}} \left| \frac{V_p(\lambda\mathcal{E}(\tilde{u}))}{\lambda} \right|^{2(1+\delta_2)} dy \right)^{\frac{1}{1+\delta_2}} \leqq c \fint_{B_1} \left| \frac{V_p(\lambda\mathcal{E}(\tilde{u}))}{\lambda} \right|^2 dy + c \fint_{B_1} |\tilde{u}|^2\, dy. \quad (4.16)$$

Proof. We need to distiguish the cases $1 < p < 2$ and $p \geqq 2$.

Case $\mathbf{1 < p < 2}$. This case is more involved, and the full proof will be given in three steps.

Step 1: Approximation. This approximation procedure will be based on Minty's argument and a suitable use of *a priori* estimates. Let $\{\Phi_\varepsilon\}_{\varepsilon>0}$ be a family of standard mollifiers; let us define, for $h \geqq 5$ and $y \in B_{3/4}$,

$$u_h := \tilde{u} * \Phi_{1/h}, \qquad \delta_h := (1 + h + \|Du_h\|^3_{L^2(B_{3/4})})^{-1}\ .$$

Moreover, we define a sequence of vector fields $A_h : \mathcal{S}_n \to \mathcal{S}_n$ by

$$A_h(z) := \tilde{A}(z) + \delta_h z\ .$$

The standard existence theory available (see [21, 31]) allows us to consider the unique solution $v_h \in u_h + W_0^{1,2}(B_{3/4};\mathbb{R}^n)$ to the following Stokes problem (remark that $\operatorname{div} u_h = 0$):

$$\operatorname{div} v_h = 0, \qquad \int_{B_{3/4}} A_h\big(\mathcal{E}(v_h)\big)\mathcal{E}(\varphi)\,dy = 0 \qquad \forall \varphi \in C^\infty_{0,\operatorname{div}}(B_{3/4}). \tag{4.17}$$

Plugging the test function $\varphi := v_h - u_h$ into (4.17) and using (4.13) together with the Young-type inequality of Lemma 2.3, we deduce:

$$\begin{aligned}
&\tilde{\nu}(\tilde{L})\int_{B_{3/4}}\left(\left|\frac{V_p(\lambda\mathcal{E}(v_h))}{\lambda}\right|^2 + \delta_h|\mathcal{E}(v_h)|^2\right)dy\\
&\quad\leqq c\int_{B_{3/4}} A_h\big(\mathcal{E}(v_h)\big)\mathcal{E}(v_h)\,dy \leqq c\int_{B_{3/4}} |A_h\big(\mathcal{E}(v_h)\big)\mathcal{E}(u_h)|\,dy\\
&\quad\leqq c(\tilde{L})\int_{B_{3/4}}\big[(1+\lambda^2|\mathcal{E}(v_h)|^2)^{\frac{p-2}{2}}|\mathcal{E}(v_h)||\mathcal{E}(u_h)| + \delta_h|\mathcal{E}(v_h)||\mathcal{E}(u_h)|\big]\,dy\\
&\quad\leqq \tilde{\nu}(\tilde{L})/4\int_{B_{3/4}}\left(\left|\frac{V_p(\lambda\mathcal{E}(v_h))}{\lambda}\right|^2 + \delta_h|\mathcal{E}(v_h)|^2\right)dy\\
&\qquad + c(\tilde{L})\int_{B_{3/4}}\left(\left|\frac{V_p(\lambda\mathcal{E}(u_h))}{\lambda}\right|^2 + \delta_h|\mathcal{E}(u_h)|^2\right)dy.
\end{aligned} \tag{4.18}$$

Now we use Jensen's inequality, recalling that by (g) of Lemma 2.2 the function $z \mapsto |V_p(z)|^2$ is convex, and from the definition of u_h we estimate

$$\int_{B_{3/4}}\left|\frac{V_p(\lambda\mathcal{E}(u_h))}{\lambda}\right|^2 dy \leqq \int_{B_{(3/4)+(1/h)}}\left|\frac{V_p(\lambda\mathcal{E}(\tilde{u}))}{\lambda}\right|^2 dy.$$

Finally, connecting this estimate to (4.18) and remembering the value of δ_h, we obtain

$$\int_{B_{3/4}}\left(\left|\frac{V_p(\lambda\mathcal{E}(v_h))}{\lambda}\right|^2 + \delta_h|\mathcal{E}(v_h)|^2\right)dy \leqq c\int_{B_{3/4+1/h}}\left|\frac{V_p(\lambda\mathcal{E}(\tilde{u}))}{\lambda}\right|^2 dy + o_h, \tag{4.19}$$

where o_h denotes any quantity vanishing when $h \to +\infty$ and the constant $c \equiv c(\tilde{L})$ is independent of h and of $\lambda \in (0,1)$. By (e) of Lemma 2.2 the sequence $\mathcal{E}(v_h)$ is bounded in L^p, thus, up to (not relabelled) subsequences, $v_h \rightharpoonup w \in \tilde{u} + W_0^{1,p}(B_{3/4};\mathbb{R}^n)$ weakly in $W^{1,p}(B_{3/4};\mathbb{R}^n)$. Our aim now is to prove that actually $w \equiv \tilde{u}$. We preliminarily observe that a standard monotonicity argument ("Minty's trick") shows that v_h is a solution to (4.17) if and only if the function $w_h := v_h - u_h$ is such that

$$\begin{aligned}
(\mathrm{I})_h + (\mathrm{II})_h &:= \int_{B_{3/4}} A\big(\mathcal{E}(u_h)+\mathcal{E}(\varphi)\big)\big(\mathcal{E}(\varphi)-\mathcal{E}(w_h)\big)\,dy\\
&\quad + \delta_h\int_{B_{3/4}}\big(\mathcal{E}(u_h)+\mathcal{E}(\varphi)\big)\big(\mathcal{E}(\varphi)-\mathcal{E}(w_h)\big)\,dy \geqq 0
\end{aligned} \tag{4.20}$$

for any $\varphi \in C^\infty_{0,\mathrm{div}}(B_{3/4})$. Now, φ being a fixed smooth test function, using the Cauchy-Schwarz inequality we have the following estimate for $(\mathrm{II})_h$:

$$|(\mathrm{II})_h| \leqq c\sqrt{\delta_h \int_{B_{3/4}} \big(1+|\mathcal{E}(u_h)|^2\big)\,dy} \times \\ \times \left[1+\sqrt{\delta_h \int_{B_{3/4}} |\mathcal{E}(u_h)|^2\,dy}+\sqrt{\delta_h \int_{B_{3/4}} |\mathcal{E}(v_h)|^2\,dy}\right];$$

by (4.19) and by our choice of δ_h it follows that $(\mathrm{II})_h \to 0$. Since $w_h \rightharpoonup w-\tilde{u}$ and $u_h \to \tilde{u}$, letting $h \to +\infty$ in (4.20) yields

$$\int_{B_{3/4}} A\big(\mathcal{E}(\tilde{u})+\mathcal{E}(\varphi)\big)\big(\mathcal{E}(\varphi)-\mathcal{E}(w-\tilde{u})\big)\,dy \geqq 0$$

for any $\varphi \in C^\infty_{0,\mathrm{div}}(\Omega)$. Then, again by Minty's trick, this implies that $w \in \tilde{u} + W^{1,p}_0(B_{3/4};\mathbb{R}^n)$ is a weak solution to (4.15), and the uniqueness of solutions to (4.15) yields $w \equiv \tilde{u}$.

Step 2: A priori estimates. Here we find uniform higher integrability estimates for the functions v_h. We shall benefit from [13]. Let us put

$$\sigma_h := A_h\big(\mathcal{E}(v_h)\big)\ ;$$

we note that the vector field A_h has quadratic growth, and from our assumptions it follows that $v_h \in W^{2,2}_{\mathrm{loc}}(B_{3/4})$ and $\sigma_h \in W^{1,2}_{\mathrm{loc}}(B_{3/4};\mathcal{S}_n)$: for a proof, see for instance [12], Theorem 3.1. Moreover there exists a "pressure" function p_h such that

$$p_h \in L^2(B_{3/4}), \qquad (p_h)_{B_{3/4}} = 0,$$

and if we set

$$\tau_h := \sigma_h - p_h\mathbb{I}_n, \tag{4.21}$$

then

$$\int_{B_{3/4}} \tau_h \mathcal{E}(\varphi)\,dy = 0 \qquad \forall \varphi \in C^\infty_0(B_{3/4}); \tag{4.22}$$

thus $Dp_h = \operatorname{div}\sigma_h \in L^2_{\mathrm{loc}}(B_{3/4})$ and $\operatorname{div}\tau_h = 0$ (see, e.g., [20]). By (4.22) we have, for every index α,

$$\int_{B_{3/4}} \partial_\alpha \tau_h\ \mathcal{E}(\varphi)\,dy = 0 \qquad \forall \varphi \in C^\infty_0(B_{3/4}). \tag{4.23}$$

From now on we shall omit the subscript h, simply writing σ, τ, p, v instead of $\sigma_h, \tau_h, p_h, v_h$ respectively; the full notation will be recovered later. Recalling that

$v \in W^{2,2}_{\mathrm{loc}}$, in (4.23) we use as a test function $\varphi := \eta^6 \partial_\alpha v$, where $\eta \in C^\infty_0(B_{3/4})$ is a cut-off function, equal to 1 on $B_{1/2}$. Since $\operatorname{div} v = 0$ we easily get

$$\begin{aligned}
&\int_{B_{3/4}} \eta^6 \partial_\alpha \sigma \; \mathcal{E}(\partial_\alpha v)\, dy \\
&\quad = -\int_{B_{3/4}} \partial_\alpha \tau_{ij} \partial_\alpha v^i \partial_j \eta^6 \, dy \\
&\quad = -2\int_{B_{3/4}} \partial_\alpha \tau_{ij} \; \mathcal{E}_{i\alpha}(v) \partial_j \eta^6 \, dy + \int_{B_{3/4}} \partial_\alpha \tau_{ij} \; \partial_i v^\alpha \partial_j \eta^6 \, dy \\
&\quad := -2(\mathrm{I})_1 + (\mathrm{I})_2 .
\end{aligned} \tag{4.24}$$

We estimate the last two terms: using the fact that $Dp = \operatorname{div} \sigma$ we obtain

$$\begin{aligned}
(\mathrm{I})_1 &= \int_{B_{3/4}} \left[\partial_\alpha \sigma_{ij} \; \mathcal{E}_{i\alpha}(v) \partial_j \eta^6 - \partial_j \sigma_{j\alpha} \; \mathcal{E}_{i\alpha}(v) \partial_j \eta^6 \right] dy \\
&= \int_{B_{3/4}} \left[\partial_\alpha \sigma_{ij} \; \mathcal{E}_{i\alpha}(v) \partial_j \eta^6 - \partial_\alpha \sigma_{j\alpha} \; \mathcal{E}_{kj}(v) \partial_k \eta^6 \right] dy \\
&= 6\int_{B_{3/4}} \eta^5 \partial_\alpha \sigma \, S^{(\alpha)} \, dy \; ,
\end{aligned}$$

where we used permutation of indices and we introduced the tensors Q and S as

$$Q^{(\alpha)}_{ij} := \mathcal{E}_{i\alpha}(v) \partial_j \eta - \mathbb{K}_{i\alpha} \mathcal{E}_{lj}(v) \partial_l \eta \; , \qquad S^{(\alpha)} := \tfrac{1}{2}\left[Q^{(\alpha)} + {}^T(Q^{(\alpha)}) \right] :$$

we denoted (this is the only spot where it appears) by $\mathbb{K}$ the Kronecker symbol, since the standard δ is used quite often in this paper. Using Cauchy-Schwarz inequality we obtain:

$$\begin{aligned}
|(\mathrm{I})_1| &\leqq c\,\delta \int_{B_{3/4}} \eta^5 |D\mathcal{E}(v)| |\mathcal{E}(v)| |D\eta| \, dy \\
&\quad + c\int_{B_{3/4}} \eta^5 D\tilde{A}\big(\mathcal{E}(v)\big) \mathcal{E}(\partial_\alpha v) \otimes S^{(\alpha)} \, dy \\
&\leqq c\sqrt{\int_{B_{3/4}} \delta \eta^6 |D\mathcal{E}(v)|^2 \, dy} \sqrt{\int_{B_{3/4}} \delta \eta^4 |\mathcal{E}(v)|^2 \, dy} \\
&\quad + c\sqrt{\int_{B_{3/4}} \eta^6 D\tilde{A}\big(\mathcal{E}(v)\big) \mathcal{E}(\partial_\alpha v) \otimes \mathcal{E}(\partial_\alpha v) \, dy} \\
&\quad \times \sqrt{\int_{B_{3/4}} \eta^4 D\tilde{A}\big(\mathcal{E}(v)\big) S^{(\alpha)} \otimes S^{(\alpha)} \, dy} \\
&\overset{(4.14)}{\leqq} c\sqrt{\int_{B_{3/4}} \eta^6 \partial_\alpha \sigma \; \mathcal{E}(\partial_\alpha v) \, dy} \\
&\quad \times \sqrt{\int_{B_{3/4}} \left[\delta |\mathcal{E}(v)|^2 + (1+\lambda^2 |\mathcal{E}(v)|^2)^{\frac{p-2}{2}} |\mathcal{E}(v)|^2 \right] dy},
\end{aligned} \tag{4.25}$$

with c independent of h. For $(\mathrm{I})_2$, since $\operatorname{div}\tau = \operatorname{div} v = 0$ we have

$$\begin{aligned}
|(\mathrm{I})_2| &\leqq \left|\int_{B_{3/4}} \tau_{ij}\,\partial_\alpha(\partial_i v^\alpha\ \partial_j\eta^6)\,dy\right| = \left|\int_{B_{3/4}} \tau_{ij}\,\partial_i v^\alpha\ \partial_{\alpha j}\eta^6\,dy\right| \\
&= \left|\int_{B_{3/4}} \tau_{ij}\,v^\alpha\,\partial_{\alpha ij}\eta^6\,dy\right| \leqq c\int_{B_{3/4}}\big(|\tau|^2+|v|^2\big)\,dy \qquad (4.26)\\
&\overset{(4.21)}{\leqq} c\int_{B_{3/4}}\big(|\sigma|^2+|p|^2+|v|^2\big)\,dy \leqq c\int_{B_{3/4}}\big(|\sigma|^2+|v|^2\big)\,dy \\
&\overset{(4.13),\,p\leqq 2}{\leqq} c\int_{B_{3/4}}\big[(1+\lambda^2|\mathcal{E}(v)|^2)^{(p-2)/2}|\mathcal{E}(v)|^2+\delta|\mathcal{E}(v)|^2+|v|^2\big]\,dy,
\end{aligned}$$

where we used the estimates of LADYZHENSKAYA for the pressure function (see [20] Chapter 3 and the remark at the end of this proof) in order to perform the second-last estimate. Connecting (4.25) and (4.26) to (4.24) and using Young's inequality, we obtain

$$\begin{aligned}
&\int_{B_{3/4}} \eta^6(1+\lambda^2|\mathcal{E}(v)|^2)^{(p-2)/2}|\mathcal{E}(\partial_\alpha v)|^2\,dy \\
&\overset{(4.14)}{\leqq} c\int_{B_{3/4}} \eta^6\partial_\alpha\sigma\ \mathcal{E}(\partial_\alpha v)\ dy \qquad (4.27)\\
&\leqq c\int_{B_{3/4}}\big[\delta|\mathcal{E}(v)|^2+|v|^2\big]\,dy + c\int_{B_{3/4}}(1+\lambda^2|\mathcal{E}(v)|^2)^{(p-2)/2}|\mathcal{E}(v)|^2\,dy,
\end{aligned}$$

where c is independent of h. Now we set $\chi := 2^*/2$ if $n>2$ and $\chi>1$ if $n=2$. Using the Sobolev inequality together with (4.27) we obtain:

$$\begin{aligned}
&\left(\int_{B_{3/4}}\Big[\eta^6(1+\lambda^2|\mathcal{E}(v)|^2)^{(p-2)/2}|\mathcal{E}(v)|^2\Big]^\chi\ dy\right)^{1/\chi} \\
&\leqq c\int_{B_{3/4}}\Big|D[\eta^3(1+\lambda^2|\mathcal{E}(v)|^2)^{(p-2)/4}\mathcal{E}(v)]\Big|^2\ dy \qquad (4.28)\\
&\leqq c\int_{B_{3/4}}(1+\lambda^2|\mathcal{E}(v)|^2)^{(p-2)/2}\Big[\eta^4|\mathcal{E}(v)|^2+\eta^6\sum_\alpha|\mathcal{E}(\partial_\alpha v)|^2\Big]\,dy \\
&\overset{(4.27)}{\leqq} c\int_{B_{3/4}}\big[\delta|\mathcal{E}(v)|^2+|v|^2+(1+\lambda^2|\mathcal{E}(v)|^2)^{(p-2)/2}|\mathcal{E}(v)|^2\big]\,dy.
\end{aligned}$$

Step 3: Conclusion. Recovering the full notation and letting $\delta_2 := \chi-1>0$, we get

$$\begin{aligned}
&\left(\int_{B_{1/2}}\left|\frac{V_p(\lambda\mathcal{E}(v_h))}{\lambda}\right|^{2(1+\delta_2)} dy\right)^{1/(1+\delta_2)} \\
&\overset{(4.19),(4.28)}{\leqq} c\int_{B_{(3/4)+(1/h)}}\left(\left|\frac{V_p(\lambda\mathcal{E}(\tilde u))}{\lambda}\right|^2+|v_h|^2\right)\,dy+o_h\ ,
\end{aligned}$$

with c independent of h. Since $2 < p^*$ by (H5), the weak convergence of v_h to $\tilde{u}$ in $W^{1,p}$ implies $v_h \to \tilde{u}$ in L^2; moreover the first integral above is lower semicontinuous with respect to weak $W^{1,p}$ convergence, since the function $z \mapsto |V_p(z)|^{2(1+\delta_2)}$ is convex, again by (H5) and by (g) of Lemma 2.2, so we get (4.16). We remark that in this case, since the Gehring lemma is not involved, the exponent δ_2 is explicitely determined, independent of $\tilde{L}$.

Case p $\geqq$ 2. This case is simpler and no approximation procedure is required; we closely follow the proof of Theorem 4.2.

Let $B_{2R} \subset\subset B_1$ be any ball, thus (beware!) not necessarily concentric with B_1, and let $\eta \in C_0^\infty(B_R)$ be a cut-off function such that $\eta \equiv 1$ on $B_{R/2}$, $0 \leqq \eta \leqq 1$, $|D\eta| \leqq cR^{-1}$. We again define, as for Theorem 4.2, $\varphi := \eta^p(\tilde{u} - \mathcal{P}) + w$, where the function $w \in W_0^{1,p}(B_R; \mathbb{R}^n)$ is defined as in (4.7) with p_2 replaced by p: in particular, according to Lemma 2.5, it satisfies

$$\int_{B_R} |Dw|^2 \, dy \leqq c \int_{B_R} \left|\frac{\tilde{u} - \mathcal{P}}{R}\right|^2 dx, \quad \int_{B_R} |Dw|^p \, dy \leqq c \int_{B_R} \left|\frac{\tilde{u} - \mathcal{P}}{R}\right|^p dx \tag{4.29}$$

with $c \equiv c(n, \gamma_1, \gamma_2)$. Using φ as a test function in the system, we obtain

$$\begin{aligned} \int_{B_R} \eta^p \tilde{A}(\mathcal{E}(\tilde{u}))\mathcal{E}(\tilde{u}) \, dy = & -p \int_{B_R} \eta^{p-1} \tilde{A}(\mathcal{E}(\tilde{u}))((\tilde{u} - \mathcal{P}) \odot D\eta) \, dy \\ & - \int_{B_R} \tilde{A}(\mathcal{E}(\tilde{u}))\mathcal{E}(w) \, dy. \end{aligned} \tag{4.30}$$

For any $0 < \varepsilon < 1$, using (4.13), (4.29) and Young's inequality we may bound the last integral above:

$$\begin{aligned} & \left| \int_{B_R} \tilde{A}(\mathcal{E}(\tilde{u}))\mathcal{E}(w) \, dy \right| \\ & \quad \leqq c \int_{B_R} |\mathcal{E}(\tilde{u})||Dw| \, dy + c\lambda^{p-2} \int_{B_R} |\mathcal{E}(\tilde{u})|^{p-1}|Dw| \, dy \\ & \quad \leqq \varepsilon \int_{B_R} \left(|\mathcal{E}(\tilde{u})|^2 + \lambda^{p-2}|\mathcal{E}(\tilde{u})|^p\right) dy \\ & \qquad + C_\varepsilon \int_{B_R} \left(\left|\frac{u - \mathcal{P}}{R}\right|^2 + \lambda^{p-2}\left|\frac{u - \mathcal{P}}{R}\right|^p\right) dy \, . \end{aligned} \tag{4.31}$$

Connecting (4.31) to (4.30) and using (4.13) and Young's inequality to deal with the first two terms in (4.30), we obtain

$$\begin{aligned} & \fint_{B_{R/2}} \left(|\mathcal{E}(\tilde{u})|^2 + \lambda^{p-2}|\mathcal{E}(\tilde{u})|^p\right) dy \\ & \quad \leqq \varepsilon \fint_{B_R} \left(|\mathcal{E}(\tilde{u})|^2 + \lambda^{p-2}|\mathcal{E}(\tilde{u})|^p\right) dy \\ & \qquad + C_\varepsilon \fint_{B_R} \left(\left|\frac{u - \mathcal{P}}{R}\right|^2 + \lambda^{p-2}\left|\frac{u - \mathcal{P}}{R}\right|^p\right) dy. \end{aligned}$$

Now we may proceed exactly as in Theorem 4.2: using the Sobolev-Korn inequality as we did to get (4.6), we arrive at

$$\begin{aligned}\fint_{B_{R/2}} &\left(|\mathcal{E}(\tilde{u})|^2 + \lambda^{p-2}|\mathcal{E}(\tilde{u})|^p\right) dy \\ &\leqq \varepsilon \fint_{B_R} \left(|\mathcal{E}(\tilde{u})|^2 + \lambda^{p-2}|\mathcal{E}(\tilde{u})|^p\right) dy \\ &\quad + C_\varepsilon \left(\fint_{B_{R/2}} \left(|\mathcal{E}(\tilde{u})|^2 + \lambda^{p-2}|\mathcal{E}(\tilde{u})|^p\right)^{1/\theta} dy\right)^\theta ,\end{aligned}$$

with the same θ as in Theorem 4.2. The assertion again follows by applying Gehring's lemma, but this time to the function $|\mathcal{E}(\tilde{u})|^2 + \lambda^{p-2}|\mathcal{E}(\tilde{u})|^p$, and recalling that by (h) of Lemma 2.2 this function is equivalent to $\lambda^{-2}|V_p(\tilde{u})|^2$. We see that in this case the last term appearing in (4.16) is not present. Lower bounds for the exponent δ_2 are available also in this case, but they show, in contrast to the first case, that $\delta_2 \equiv \delta_2(n, \gamma_1, \gamma_2, \tilde{L})$ depends also on $\tilde{L}$, see [18, 36]. □

Remark 4.6. In (4.26) we used the estimate

$$\int_B |p_h|^2 \, dx \leqq c \int_B |\sigma_h|^2 \, dx,$$

where $B = B_{3/4}$ and the constant c depends on n (and the radius of the ball). Since the proof of this pressure estimate is scattered through the literature, we sketch here how it may be obtained: denote by $L : H_0^1(B; \mathbb{R}^n) \to L^2(B)$ the divergence operator $Lu = \operatorname{div} u$, so that its adjoint operator $L^* : L^2(B) \to H^{-1}(B; \mathbb{R}^n)$ is given by $L^* f = Df$ in the sense of distributions. We remark that the range of L^* is closed, thus there exists a constant c such that for every $f \in (\operatorname{Ker} L^*)^\perp$

$$\|f\|_{L^2} \leqq c\|L^* f\|_{H^{-1}}. \tag{4.32}$$

Since the perturbed vector field A_h has linear growth at infinity (recall that $p < 2$) and since $v_h \in H^1(B; \mathbb{R}^n)$, we deduce that $D\sigma_h \in H^{-1}(B; \mathbb{R}^n)$; also, from (4.17) and the fact that σ_h is symmetric we deduce

$$\int_B \sigma_h D\varphi \, dx = 0 \qquad \forall \varphi \in H_0^1(B; \mathbb{R}^n) \text{ such that } \operatorname{div} \varphi = 0.$$

Then $D\sigma_h \in (\operatorname{Ker} L)^\perp$. Thus there exists $p_h \in L^2(B)$, which we may take with average zero, such that $D\sigma_h = L^* p_h$: then $p_h \in (\operatorname{Ker} L^*)^\perp$ and, using (4.32),

$$\|p_h\|_{L^2} \leqq c\|D\sigma_h\|_{H^{-1}} \leqq c\|\sigma_h\|_{L^2}.$$

Note that in the previous argument the boundary value of v_h is irrelevant.

5. Decay estimates

Here we make some preliminary reductions and fix some quantities that will be important in the remainder of this paper. As we remarked at the beginning of the previous section, the local nature of our results allows us to suppose, in view of Theorem 4.2, that

$$|\mathcal{E}(u)|^{p(x)(1+\delta_1)} \in L^1(\Omega),$$

where δ_1 is the exponent introduced in Theorem 4.2.

Remark 5.1. This point deserves an explanation; in the previous section, and in particular in Theorem 4.2, we proved a higher integrability result which is only local, thus a standard covering argument shows that for any $\Omega' \subset\subset \Omega$ there exists δ_1, depending also on Ω', for which $|\mathcal{E}(u)|^{p(x)(1+\delta_1)} \in L^1(\Omega')$. It may well happen that $\delta(\Omega') \to 0$ when $\Omega' \nearrow \Omega$, so the assumption above has to be read as a simplification of the following procedure: take a sequence $\Omega_h \nearrow \Omega$ and the corresponding exponents δ_1^h and continue with the proof; then you will actually get partial regularity (i.e., up to a set of null measure) only in Ω_h, but this leads to partial regularity in Ω. Since we have already too many indices around, we preferred to drop this h altogether.

A simple argument based on a local application of the classical Korn inequality (a) of Proposition 2.7, together with a standard covering procedure and the fact that the exponent δ_1 can be made smaller at will (see Remark 4.3), allow us to suppose also that

$$\int_\Omega |Du|^{p(x)(1+\delta_1)}\, dx < +\infty, \qquad 0 < \delta_1 \leqq \min\{\gamma_1 - 1, 1/n, \tilde{\delta}\}\,, \tag{5.1}$$

where $\tilde{\delta}$ is the exponent defined in Lemma 4.4. Fix $M > 1$ and denote by $\tilde{L} \equiv \tilde{L}(M)$ the constant given by Lemma 2.4: we apply Lemma 4.5 in a situation where the vector field $\tilde{A}$ considered is exactly as in Lemma 2.4, that is $\tilde{A}(z) \equiv A_{P,\lambda}(z)$ with $|P| \leqq M$. Recalling that the statement of Lemma 4.5 is also independent of the particular solution v, we come up with a further higher integrability exponent $\delta_2 \equiv \delta_2(M)$. Also this δ_2 can be made smaller if need be.

We remark that there is a crucial difference in the quantitave behaviour of the two exponents δ_1 and δ_2: indeed, if $M \to +\infty$, we have in general $\tilde{L} \to +\infty$, and so it may well happen that $\delta_2 \to 0$, see the second case of Lemma 4.5; on the contrary, δ_1 is a fixed quantity, independent of the value of M and only depending on the fixed data $n, \gamma_1, \gamma_2, L, \beta, c_{L,\alpha}$, thus it remains bounded away from 0 when $M \to +\infty$. For this reason, without loss of generality we shall also assume that $\delta_2 \leqq \delta_1$.

After fixing M, we select a radius $R_M > 0$ in such a way that $\omega(R_M) \leqq \delta_2/4$: from now on, $\mathcal{O} \subset\subset \Omega$ will denote an open subset whose diameter does not exceed R_M. In this way if, similarly to what we did in the previous section, we set

$$p_1 := \inf_{\mathcal{O}} p(x), \qquad p_2 := \sup_{\mathcal{O}} p(x), \tag{5.2}$$

then

$$p_2(1+\delta_1/4) \leqq p_1(1+\delta_1) \leqq p(x)(1+\delta_1),$$
$$p_2(1+\delta_2/4) \leqq p_1(1+\delta_2) \leqq p(x)(1+\delta_2) \tag{5.3}$$

whenever $x \in \mathcal{O}$. We shall often consider balls $B(x_0, 4R) \subset\subset \mathcal{O}$, denoting by p_m a number such that:

$$p_m := \sup_{B_{4R}} p(x) = p(x_m), \qquad \text{for some } x_m \in \overline{B_{4R}}\ .$$

According to (5.3),

$$p_2(1+\delta_1/4) \leqq p_m(1+\delta_1)\ , \qquad p_2(1+\delta_2/4) \leqq p_m(1+\delta_2). \tag{5.4}$$

We remark that the numbers p_1, p_2 are fixed with $\mathcal{O}$, while p_m changes when $B(x_0, 4R) \subset\subset \mathcal{O}$ moves. Finally, $u \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^n)$ is the solution appearing in Theorem 2.1; without loss of generality we assume all constants c, C_M to be no smaller than 1. We start with a technical lemma.

Lemma 5.2. *Assume* (H1)–(H6) *and let $B(x_0, 4R) \subset\subset \mathcal{O}$, where $\mathcal{O} \subset\subset \Omega$ is an open subset as described above. For every $C_M \geqq 1$, there exist $\bar{\beta}$, depending on $n, \gamma_1, \gamma_2, L, \beta$ but independent of M, R and $x_0 \in \mathcal{O}$, and a constant $\check{C}_M$, depending also on C_M, such that if $v \in u + W^{1,p_m}_0(B(x_0, 2R); \mathbb{R}^N)$ is the (unique) solution to the system,*

$$\operatorname{div} v = 0, \qquad \int_{B(x_0,2R)} A(x_m, \mathcal{E}(v))\mathcal{E}(\varphi)\, dx = 0 \quad \forall \varphi \in C^\infty_{0,\text{div}}(B_{2R}) \tag{5.5}$$

and, if

$$(|Du|^{p_2})_{x_0,4R} + (|u|^{\gamma_1^*})_{x_0,4R} \leqq C_M, \tag{5.6}$$

then $v \in W^{1,p_2}(B(x_0, 2R); \mathbb{R}^N)$ and

$$\fint_{B(x_0,R)} |Du - Dv|^{p_2}\, dx \leqq \check{C}_M R^{\bar{\beta}}. \tag{5.7}$$

The reader may get slightly confused at this point. Indeed we are freezing the vector field $A(x, z)$ at a point x_m which is different from the centre x_0 and may even lie outside the ball B_{2R}, although $x_m \in \overline{B(x_0, 4R)}$.

Proof. Throughout the proof, all balls we consider will be centred at x_0; also, instead of introducing a new symbol we will denote by C_M any constant depending through known, immaterial quantities on the actual C_M in the statement. Since by our choice of $\mathcal{O}$ we have $u \in W^{1,p_m}(B_{4R}; \mathbb{R}^N)$, we may test the weak form (5.5) with the function $v - u$, and using the monotonicity properties (2.2) of A and Young's inequality we deduce

$$\fint_{B_{2R}} |\mathcal{E}(v)|^{p_m}\, dx \leqq c \fint_{B_{2R}} |\mathcal{E}(u)|^{p_m}\, dx + c \overset{(5.6)}{\leqq} C_M. \tag{5.8}$$

Using Lemma 4.4 and the previous inequality, and keeping (5.1) in mind, we find

$$\fint_{B_R} |\mathcal{E}(v)|^{p_m(1+\delta_1)}\,dx \overset{(4.12)}{\leqq} c\left(\fint_{B_{2R}} |\mathcal{E}(v)|^{p_m}\,dx\right)^{(1+\delta_1)} + c \overset{(5.8)}{\leqq} C_M. \tag{5.9}$$

Moreover, applying Theorem 4.2 we have:

$$\begin{aligned}\fint_{B_{2R}} |\mathcal{E}(u)|^{p_2(1+\delta_1/4)}\,dx &\overset{(5.3)}{\leqq} c\fint_{B_{2R}} |\mathcal{E}(u)|^{p(x)(1+\delta_1)}\,dx + c \\ &\overset{(4.4)}{\leqq} c\left(\fint_{B_{4R}} |\mathcal{E}(u)|^{p(x)}\,dx\right)^{1+\delta_1} + c\left(\fint_{B_{4R}} \big(|Du|^{\gamma_1} + |u|^{\gamma_1^*} + 1\big)\,dx\right)^{1+\delta_1} \\ &\overset{(5.6)}{\leqq} C_M.\end{aligned} \tag{5.10}$$

Now we observe that, if $\varphi \in W_0^{1,p_m}(B_{2R};\mathbb{R}^n)$ and $\operatorname{div}\varphi = 0$, then

$$\begin{aligned}\text{(I)} &:= \fint_{B_{2R}} [A(x_m,\mathcal{E}(v)) - A(x_m,\mathcal{E}(u))]\,\mathcal{E}(\varphi)\,dx \\ &= \fint_{B_{2R}} [A(x,\mathcal{E}(u)) - A(x_m,\mathcal{E}(u))]\,\mathcal{E}(\varphi)\,dx - \fint_{B_{2R}} B(x,u,Du)\varphi\,dx \\ &:= \text{(II)} + \text{(III)}.\end{aligned}$$

Letting $\varphi := v - u$ in the previous formula and using (2.3) we obtain

$$\text{(I)} \geqq c^{-1}\fint_{B_{2R}} (1 + |\mathcal{E}(u)|^2 + |\mathcal{E}(v)|^2)^{(p_m-2)/2}|\mathcal{E}(v) - \mathcal{E}(u)|^2\,dx, \tag{5.11}$$

while using (H3) and Young's inequality yields:

$$\begin{aligned}&\text{(II)} \\ &\leqq c\,\omega(R)\fint_{B_{2R}} (1 + |\mathcal{E}(u)|^2)^{\frac{p_m-1}{2}}\,(\log(1 + |\mathcal{E}(u)|) + 1)\,(|\mathcal{E}(v)| + |\mathcal{E}(u)|)\,dx \\ &\leqq c\,\omega(R)\fint_{B_{2R}} \left[(1 + |\mathcal{E}(u)|)^{p_m}\left(\log^{\frac{p_m}{p_m-1}}(1 + |\mathcal{E}(u)|) + 1\right) + |\mathcal{E}(v)|^{p_m}\right]dx \\ &\overset{(H6)}{\leqq} cR^{\alpha}\fint_{B_{2R}} (1 + |\mathcal{E}(u)|^{p_m(1+\delta_1/4)} + |\mathcal{E}(v)|^{p_m})\,dx \overset{(5.8),(5.10)}{\leqq} C_M R^{\alpha},\end{aligned} \tag{5.12}$$

and

$$|\text{(III)}| \leqq c\fint_{B_{2R}} |Du||u||u - v|\,dx + c\fint_{B_{2R}} |f(x)||u - v|\,dx\ .$$

Remark 5.3. The chain of inequalities (5.12) is the only point in the whole paper where we need the full strength of (H6) instead of the weaker form (2.4) of continuity of p; the fact that in the statement of the lemma the constants do not seem to depend on α is due to the condition $\alpha > \beta$ added in (4.2).

We keep the notation introduced for Theorem 4.2, but this time we take $\mu = 1$ and $q = (\gamma_1^*/2)'$, so that again $q \leqq \gamma_1$, as for (4.10). We apply the Sobolev-Korn inequality (2.20) and we estimate as in Theorem 3.1:

$$
\begin{aligned}
&\fint_{B_{2R}} |u||Du||u-v|\,dx \\
&\quad \leqq \left(\fint_{B_{2R}} |Du|^q\,dx\right)^{\frac{1}{q}} \left(\fint_{B_{2R}} |u|^{\gamma_1^*}\,dx\right)^{1/\gamma_1^*} \left(\fint_{B_{2R}} |u-v|^{\gamma_1^*}\,dx\right)^{1/\gamma_1^*} \\
&\quad \overset{(5.6)}{\leqq} C_M \left(\fint_{B_{2R}} |u-v|^{\gamma_1^*}\,dx\right)^{1/\gamma_1^*} \leqq C_M R \left(\fint_{B_{2R}} |\mathcal{E}(u)-\mathcal{E}(v)|^{\gamma_1}\,dx\right)^{1/\gamma_1} \\
&\quad \overset{(5.8)}{\leqq} C_M R
\end{aligned}
$$

and

$$
\begin{aligned}
&\fint_{B_{2R}} |f(x)||u-v|\,dx \\
&\quad \leqq C_M \left(\fint_{B_{2R}} |u-v|^{\gamma_1^*}\,dx\right)^{1/\gamma_1^*} \left(\fint_{B_{2R}} |f(x)|^{n(1+\beta)}\,dx\right)^{1/n(1+\beta)} \\
&\quad \leqq C_M R^{\beta/(\beta+1)}.
\end{aligned}
$$

Connecting the previous estimates with (5.11) and (5.12), in the case $p_m \geqq 2$ we immediately find that

$$
\fint_{B_R} |Du-Dv|^{p_m}\,dx \leqq c \fint_{B_{2R}} |\mathcal{E}(u)-\mathcal{E}(v)|^{p_m}\,dx \leqq C_M R^{\beta/(\beta+1)}, \tag{5.13}
$$

whereas if $p_m \leqq 2$ we have, by the Hölder inequality and (5.8),

$$
\begin{aligned}
&\fint_{B_R} |Du-Dv|^{p_m}\,dx \\
&\quad \leqq c \fint_{B_{2R}} |\mathcal{E}(u)-\mathcal{E}(v)|^{p_m}\,dx \qquad (5.14)\\
&\quad \leqq c \sqrt{\fint_{B_{2R}} (1+|\mathcal{E}(u)|^2+|\mathcal{E}(v)|^2)^{(p_m-2)/2}|\mathcal{E}(v)-\mathcal{E}(u)|^2\,dx} \\
&\qquad \times \sqrt{\fint_{B_{2R}} \left(|\mathcal{E}(u)|^{p_m}+|\mathcal{E}(v)|^{p_m}\right)dx} \leqq C_M R^{\beta/2(\beta+1)}\ .
\end{aligned}
$$

This estimate is not yet satisfactory, since p_m changes as x_0 moves (as we already remarked). We are almost ready to prove (5.7). Note that (5.1) and Korn-Poincaré inequalities (2.19),(2.20) imply

$$\begin{aligned}
&\fint_{B_R} |Dv|^{p_m(1+\delta_1)}\,dx \\
&\leqq \quad c\fint_{B_R} |\mathcal{E}(v)|^{p_m(1+\delta_1)}\,dx + c\biggl(\fint_{B_{2R}} |v-(v)_{2R}|\,dx\biggr)^{p_m(1+\delta_1)} \\
&\overset{(5.1),(5.9)}{\leqq} C_M + c\biggl(\fint_{B_{2R}} |u-(u)_{2R}|\,dx\biggr)^{p_2(1+1/n)} \\
&\qquad + c\biggl(\fint_{B_{2R}} |u-v|\,dx\biggr)^{p_m(1+\delta_1)} \\
&\overset{(5.6)}{\leqq} \quad C_M + c\biggl(\fint_{B_{2R}} |\mathcal{E}(u)-\mathcal{E}(v)|^{p_m}\,dx\biggr)^{1+\delta_1} \overset{(5.6),(5.8)}{\leqq} C_M.
\end{aligned} \tag{5.15}$$

In much the same way, from (5.10) and (5.4) it also follows that

$$\fint_{B_R} |Dv|^{p_2(1+\delta_1/4)}\,dx + \fint_{B_R} |Du|^{p_2(1+\delta_1/4)}\,dx \leqq C_M. \tag{5.16}$$

Using this last information we finally interpolate as follows: if

$$\vartheta := \left(\frac{1}{p_m} - \frac{1}{p_2(1+\delta_1/4)}\right)^{-1}\left(\frac{1}{p_2} - \frac{1}{p_2(1+\delta_1/4)}\right)$$

and thus by $(5.4)_1$

$$\frac{p_2\vartheta}{p_m} = \frac{\delta_1/4}{1+(\delta_1/4)-(p_m/p_2)} \geqq \frac{1+\delta_1}{4+\delta_1} \geqq \frac{1}{4},$$

then by (5.13)–(5.16)

$$\begin{aligned}
&\fint_{B_R} |Du-Dv|^{p_2}\,dx \\
&\leqq \left(\fint_{B_R} |Du-Dv|^{p_m}\,dx\right)^{(p_2\vartheta)/p_m}\left(\fint_{B_R} |Du-Dv|^{p_2(1+\delta_1/4)}\,dx\right)^{\frac{1-\vartheta}{1+\delta_1/4}} \\
&\leqq C_M R^{\frac{\beta}{2(\beta+1)}\frac{p_2\vartheta}{p_m}} \leqq C_M R^{\beta/8(\beta+1)} := R^{\bar\beta},
\end{aligned}$$

and (5.7) follows with $\bar\beta := \beta/8(\beta+1)$, since δ_1 depends on n, γ_1, γ_2, L. □

Now we introduce the numbers $\underline{q}$, $\hat\beta$, depending on n, γ_1, γ_2, L, by:

$$\underline{q} := \min\{2, p_2\}, \qquad \hat\beta := \bar\beta/\gamma_2, \tag{5.17}$$

and we define the fundamental quantity

$$E(x_0,R) := \fint_{B(x_0,R)} |V_{p_2}(Du) - V_{p_2}((Du)_{x_0,R})|^2 dx + R^{\hat\beta}$$

whenever $B(x_0,4R) \subset\subset \mathcal{O}$. Roughly speaking, E (usually called excess) provides an integral measure of the oscillations of the gradient Du in a ball B_R. The next decay estimate for E is the keystone in the proof of Theorem 2.1.

Proposition 5.4. *Under the assumptions of Theorem* 2.1*, let* $M > 1$ *and let* $\mathcal{O} \subset\subset \Omega$ *be an open subset related to* M *in the way described above. There exists a constant* $C(M)$ *such that for every* $0 < \tau < 1/4$ *there exists* $\varepsilon \equiv \varepsilon(\tau, M)$ *such that if* $B(x_0, 4R) \subset\subset \mathcal{O}$ *and*

$$\begin{gathered} |(Du)_{x_0,\tau R}|, |(Du)_{x_0,R}|, |(Du)_{x_0,4R}|, |(u)_{x_0,R}| \leqq M, \\ E(x_0, R) < \varepsilon, \qquad E(x_0, 4R) \leqq 1, \end{gathered} \tag{5.18}$$

then

$$E(x_0, \tau R) \leqq C(M)\tau^{\hat{\beta}} E(x_0, R) \tag{5.19}$$

with $\hat{\beta}$ *defined in* (5.17).

Proof.
Step 1: Blow-up and limit system. Arguing by contradiction, we suppose that for a certain τ there exists a sequence of balls $B(x_h, 4R_h) \subset\subset \mathcal{O}$ such that

$$\begin{gathered} |(Du)_{x_h,\tau R_h}|, |(Du)_{x_h,R_h}|, |(Du)_{x_h,4R_h}|, |(u)_{x_h,R_h}| \leqq M, \\ \mu_h^2 := E(x_h, R_h) \to 0, \qquad E(x_h, 4R_h) \leqq 1, \end{gathered} \tag{5.20}$$

but

$$E(x_h, \tau R_h) \geqq C(M)\tau^{\hat{\beta}} E(x_h, R_h) \tag{5.21}$$

for some constant $C(M)$ whose value (independent of τ) will be defined later; without loss of generality we may assume that $R_h \to 0$ and $\mu_h > 0$. Using (d) and (e) of Lemma 2.2 we immediately find that there exists C_M such that

$$(|Du|^{p_2})_{x_h,4R_h} + (|u|^{\gamma_1^*})_{x_h,4R_h} \leqq C_M. \tag{5.22}$$

Indeed

$$\begin{aligned} (|Du|^{p_2})_{x_h,4R_h} &\leqq c \fint_{B(x_h,4R_h)} |Du - (Du)_{x_h,4R_h}|^{p_2}\, dx + C_M \\ &\leqq C_M E(x_h, 4R_h) + C_M \leqq C_M, \end{aligned}$$

and the bound on $(|u|^{\gamma_1^*})_{x_h,4R_h}$ follows by (5.20). Now we define (obviously not for $h = 1, 2$) the numbers p_h as follows:

$$p_h := \sup_{B(x_h,4R_h)} p(x) = p(x_{h,m}), \qquad x_{h,m} \in \overline{B(x_h, 4R_h)}\,. \tag{5.23}$$

Remark that in general $p_h \neq p(x_h)$. Let $u_h \in u + W_0^{1,p_h}(B(x_h, 2R_h); \mathbb{R}^n)$ be the unique solution to the system

$$\operatorname{div} u_h = 0, \qquad \int_{B_{2R_h}} A(x_{h,m}, \mathcal{E}(u_h))\mathcal{E}(\varphi)\, dx = 0 \quad \forall \varphi \in C^\infty_{0,\mathrm{div}}(B_{2R_h}).$$

By Lemma 5.2 the sequence $u_h \in u + W_0^{1,p_2}(B(x_h, 2R_h); \mathbb{R}^n)$ satisfies

$$\fint_{B(x_h,R_h)} |Du - Du_h|^{p_2}\, dx \leqq \check{C}_M R_h^{\bar\beta}, \tag{5.24}$$

with $\check{C}_M, \bar\beta$ independent of h. Define

$$P_h := (Du)_{x_h,R_h}, \qquad \lambda_h^2 := \fint_{B(x_h,R_h)} |V_{p_2}(Du_h) - V_{p_2}(P_h)|^2 dx + R_h^{\hat\beta}, \tag{5.25}$$

where $\hat\beta$ is the exponent introduced in (5.17); we remark that P_h is not the average of Du_h. We rescale each function u_h in the ball $B(x_h, R_h)$ in order to have a sequence of functions defined on $B(0,1) \equiv B_1$:

$$v_h(y) := (\lambda_h R_h)^{-1}[u_h(x_h + R_h y) - (u_h)_{x_h,R_h} - R_h P_h y]$$

for $y \in B(0,1)$. Applying (d) of Lemma 2.2 yields

$$\begin{aligned}\lambda_h^{-2} \fint_{B(0,1)} |V_{p_2}(\lambda_h Dv_h(y))|^2\, dy &= \lambda_h^{-2} \fint_{B(x_h,R_h)} |V_{p_2}(Du_h(x) - P_h)|^2\, dx \\ &\leqq C_M \lambda_h^{-2} \fint_{B(x_h,R_h)} |V_{p_2}(Du_h(x)) - V_{p_2}(P_h)|^2\, dx \leqq C_M \end{aligned}\tag{5.26}$$

by (5.25); so, by (5.17) and (e) from Lemma 2.2

$$\||Dv_h|^{\underline{q}}\|_{L^1(B_1)} + \mathbf{1}_{(p_2>2)} \|\lambda_h^{p_2-2}|Dv_h|^{p_2}\|_{L^1(B_1)} \leqq C_M$$

uniformly in h. Remarking that we also have $(v_h)_{0,1} = 0$, by eventually selecting a subsequence we show that there exists $v \in W^{1,\underline{q}}(B_1; \mathbb{R}^n)$ such that as $h \to +\infty$

$$\begin{array}{rll} |v_h - v|^2 \to 0 & & \text{strongly in } L^1(B_1), \\ \lambda_h^{p_2-2}|v_h - v|^{p_2} \to 0 & & \text{strongly in } L^1(B_1) \text{ if } p_2 > 2, \\ Dv_h \rightharpoonup Dv & & \text{weakly in } L^{\underline{q}}(B_1; \mathbb{R}^n), \\ x_{h,m} \to x_\infty & & \text{in } \mathbb{R}^n, \text{ with } x_\infty \in \overline{\mathcal{O}}, \\ P_h \to P & & \text{in } \mathbb{R}^{n^2}, \text{ with } |P| \leqq M. \end{array} \tag{5.27}$$

Let us just remark that $(5.27)_1$ follows by the Sobolev embedding theorem since $2 < \gamma_1^* \leqq p_2^*$ by (H5). Finally, we prove that

$$\lambda_h^2 \leqq C_M \mu_h^2, \tag{5.28}$$

a relation that will be useful in what follows: in particular it implies that $\lambda_h^2 \to 0$. Using Lemma 2.2 and Jensen's inequality we get

$$\begin{aligned}
\lambda_h^2 &\overset{(c)}{\leqq} C_M \fint_{B(x_h,R_h)} |V_{p_2}(Du_h - P_h)|^2\,dx + R_h^{\hat\beta} \\
&\overset{(b)}{\leqq} C_M \fint_{B(x_h,R_h)} \big(|V_{p_2}(Du_h - Du)|^2 + |V_{p_2}(Du - P_h)|^2\big)\,dx + R_h^{\hat\beta} \\
&\overset{(d),(e)}{\leqq} C_M \fint_{B(x_h,R_h)} |Du_h - Du|^{p_2}\,dx \\
&\qquad + C_M \mathbf{1}_{(p_2>2)} \left(\fint_{B(x_h,R_h)} |Du_h - Du|^{p_2}\,dx\right)^{2/p_2} + C_M\mu_h^2 \\
&\overset{(5.24)}{\leqq} C_M[R_h^{\hat\beta} + R_h^{(2\bar\beta)/p_2} + \mu_h^2] \overset{(5.17)}{\leqq} C_M\mu_h^2.
\end{aligned}$$

As we introduced the rescaled functions v_h, it is natural to introduced a sequence of rescaled vector fields:

$$A_h(z) \equiv A_{P_h,\lambda_h}(z) := \lambda_h^{-1}\left[A(x_{h,m}, \mathcal{E}(P_h) + \lambda_h z) - A(x_{h,m}, \mathcal{E}(P_h))\right]$$

for any $z \in \mathcal{S}_n$. It is clear that each A_h is of the type considered in Lemma 2.4 and consequently satisfies the growth and the ellipticity conditions (2.15), (2.16) with $\tilde L \equiv \tilde L(C_M) \equiv \tilde L(M)$ independently of h as was chosen at the beginning of this section. By the definitions of A_h and v_h it follows that each rescaled function v_h is a solution to the following rescaled system in B_1:

$$\operatorname{div} v_h = 0, \qquad \int_{B_1} A_h(\mathcal{E}(v_h))\mathcal{E}(\varphi)\,dx = 0 \quad \forall \varphi \in C^\infty_{0,\mathrm{div}}(B_1). \tag{5.29}$$

Now, using the fact that $p_h \leqq p_2$ by (5.2), (5.23), and using the information in (5.27), we deduce as $h \to \infty$ that the limit function v satisfies the following limit system with constant coefficients:

$$\operatorname{div} v = 0, \qquad \int_{B_1} D_z A(x_\infty, \mathcal{E}(P))\mathcal{E}(v) \otimes \mathcal{E}(\varphi)\,dy = 0 \quad \forall \varphi \in C^\infty_{0,\mathrm{div}}(B_1). \tag{5.30}$$

The uniform condition in (H2) implies that the matrix $D_z A(x, \mathcal{E}(P))$ satisfies the following strong Legendre-Hadamard condition, see e.g. [17]:

$$C_M^{-1}|\lambda|^2|\mu|^2 \leqq \langle D_z A(x_\infty, \mathcal{E}(P))\lambda \otimes \mu, \lambda \otimes \mu\rangle \leqq C_M|\lambda|^2|\mu|^2$$

for any $\lambda, \mu \in \mathbb{R}^n$ and for some constant C_M. Therefore, from the standard regularity theory available for such systems (see [14], Lemma 3.0.5) it follows that v is smooth and

$$\fint_{B_\tau} |Dv - (Dv)_\tau|^2\,dy \leqq C_M\tau^2 \tag{5.31}$$

for any $\tau \leqq 1/4$. Moreover,

$$\sup_{B_{1/2}} |Dv| \leqq c(n, \gamma_1, \gamma_2, L) \int_{B_1} |Dv| \, dy \leqq C_M. \tag{5.32}$$

Actually, when $p_2 \geqq 2$ the previous estimates are an easy consequence of the arguments developed in [14]; in the case $p_2 \leqq 2$ it is possible to get (5.31), (5.32) by combining the arguments in [14] with those in [8], Proposition 2.10. We remark that using (5.31), (5.32) and the estimate (2.7) we have, for every p,

$$\begin{aligned} &\fint_{B_\tau} |V_p(Dv - (Dv)_\tau)|^2 \, dy \\ &\leqq c \fint_{B_\tau} |Dv - (Dv)_\tau|^2 + \mathbf{1}_{(p>2)} |Dv - (Dv)_\tau|^p \, dy \\ &\leqq \fint_{B_\tau} (1 + C_M \mathbf{1}_{(p>2)}) |Dv - (Dv)_\tau|^2 \, dy \leqq C_M \tau^2. \end{aligned} \tag{5.33}$$

Step 2: Strong convergence (I). Here we are going to prove that, up to not-relabelled subsequences:

$$\lim_h \lambda_h^{-2} \int_{B_{1/2}} |V_{p_2}(\lambda_h(\mathcal{E}(v) - \mathcal{E}(v_h)))|^2 \, dy = 0. \tag{5.34}$$

We shall do this by first proving that, again up to not-relabelled subsequences,

$$\lim_h \lambda_h^{-2} \int_{B_{1/2}} |V_{p_h}(\lambda_h(\mathcal{E}(v) - \mathcal{E}(v_h)))|^2 \, dy = 0; \tag{5.35}$$

then we shall improve (5.35) into (5.34), using the higher integrability estimates of Lemma 4.5. Let $\eta \in C_0^\infty(B_1)$ be a cut-off function such that $\eta = 1$ on $B_{1/2}$, and consider the test functions $\varphi_h := \eta^{p_h}(v_h - v) + w_h$ where, by Lemma 2.5, $w_h \in W_0^{1,p_h}(B_1; \mathbb{R}^N)$ is such that $\operatorname{div} \varphi_h = 0$, $\operatorname{spt} w_h \subset K \subset\subset B_1$ for a fixed compact set K and

$$\int_{B_1} |Dw_h|^{p_h} \, dy \leqq c \int_{B_1} |v_h - v|^{p_h} \, dy, \quad \int_{B_1} |Dw_h|^2 \, dy \leqq c \int_{B_1} |v_h - v|^2 \, dy \tag{5.36}$$

with c independent of h. By (5.29), (5.30) we have

$$\begin{aligned} (\mathrm{I})_h &:= \int_{B_1} [A_h(\mathcal{E}(v_h)) - A_h(\mathcal{E}(v))] \, \mathcal{E}(\varphi_h) \, dy \\ &= \int_{B_1} \left[D_z A(x_\infty, \mathcal{E}(P)) \mathcal{E}(v) - A_h(\mathcal{E}(v)) \right] \mathcal{E}(\varphi_h) \, dy \ := (\mathrm{II})_h. \end{aligned}$$

Since v is smooth $|D_z A(x_\infty, \mathcal{E}(P))\mathcal{E}(v) - A_h(\mathcal{E}(v))| \to 0$ uniformly on compact subsets of B_1, we have $(\mathrm{II})_h \to 0$. To deal with $(\mathrm{I})_h$ we write:

$$
\begin{aligned}
(\mathrm{I})_h := & \int_{B_1} [A_h(\mathcal{E}(v_h)) - A_h(\mathcal{E}(v))]\, \eta^{p_h} \mathcal{E}(v_h - v)\, dy \\
& + p_h \int_{B_1} [A_h(\mathcal{E}(v_h)) - A_h(\mathcal{E}(v))]\, \eta^{p_h - 1}((v_h - v) \odot D\eta)\, dy \\
& + \int_{B_1} [A_h(\mathcal{E}(v_h)) - A_h(\mathcal{E}(v))]\, \mathcal{E}(w_h)\, dy \\
:= & \ (\mathrm{III})_h + (\mathrm{IV})_h + (\mathrm{V})_h.
\end{aligned}
$$

By the ellipticity property in (2.3) it follows that

$$
\begin{aligned}
& (\mathrm{III})_h \\
& \geqq c^{-1} \int_{B_1} \eta^{p_h} (1 + |\mathcal{E}(P_h) + \lambda_h \mathcal{E}(v_h)|^2 + |\mathcal{E}(P_h) + \lambda_h \mathcal{E}(v)|^2)^{\frac{p_h - 2}{2}} \times \\
& \quad \times |\mathcal{E}(v_h - v)|^2\, dy \\
& \geqq C_M^{-1} \int_{B_{1/2}} (|1 + |\mathcal{E}(P_h) + \lambda_h \mathcal{E}(v)|^2 + |\lambda_h \mathcal{E}(v_h - v)|^2)^{\frac{p_h - 2}{2}} |\mathcal{E}(v_h - v)|^2\, dy \\
& \geqq C_M^{-1} \lambda_h^{-2} \int_{B_{1/2}} |V_{p_h}(\lambda_h(\mathcal{E}(v) - \mathcal{E}(v_h)))|^2\, dy,
\end{aligned}
$$

where to perform the last inequality we used (5.32) and the fact that $|P_h| \leqq M$ by (5.20), (5.25). We estimate the remaining terms:

$$
\begin{aligned}
(\mathrm{IV})_h & \leqq c \int_{B_1} [|A_h(\mathcal{E}(v_h))| + |A_h(\mathcal{E}(v))|]\, |v - v_h||D\eta|\, dy \\
& \leqq c \int_{B_1} (1 + \lambda_h^2 |\mathcal{E}(v_h)|^2)^{\frac{p_h - 2}{2}} |\mathcal{E}(v_h)||v_h - v|\, dy + C_M \int_{B_1} |v - v_h|\, dy \\
& \leqq \varepsilon \int_{B_1} (1 + \lambda_h^2 |\mathcal{E}(v_h)|^2)^{\frac{p_h - 2}{2}} |\mathcal{E}(v_h)|^2\, dy \qquad (5.37) \\
& \quad + C_{\varepsilon, M} \int_{B_1} \left((1 + \lambda_h^2 |v - v_h|^2)^{\frac{p_h - 2}{2}} |v - v_h|^2 + |v - v_h|\right) dy,
\end{aligned}
$$

where to perform the last estimate we used the Young-type inequality (2.11); in the same way we also get

$$
\begin{aligned}
(\mathrm{V})_h & \leqq \varepsilon \int_{B_1} (1 + \lambda_h^2 |\mathcal{E}(v_h)|^2)^{\frac{p_h - 2}{2}} |\mathcal{E}(v_h)|^2\, dy \qquad (5.38) \\
& \quad + C_{\varepsilon, M} \int_{B_1} \left((1 + \lambda_h^2 |\mathcal{E}(w_h)|^2)^{\frac{p_h - 2}{2}} |\mathcal{E}(w_h)|^2 + |\mathcal{E}(w_h)|\right) dy.
\end{aligned}
$$

Now we use (2.7) and both inequalities (5.36) to show that the integrals on the last lines in (5.37) and (5.38) can be bounded by

$$
C_{\varepsilon, M} \left(\sqrt{\int_{B_1} |v - v_h|^2\, dy} + \int_{B_1} |v - v_h|^2 + \mathbf{1}_{(p_2 > 2)} \lambda_h^{p_2 - 2} |v - v_h|^{p_2}\, dy \right) \to 0
$$

as $h \to \infty$, by $(5.27)_{1,2}$. Connecting the estimates for the terms $(\mathrm{I})_h, \dots, (\mathrm{V})_h$ we obtain

$$\begin{aligned}\limsup_h \lambda_h^{-2} \int_{B_{1/2}} |V_{p_h}(\lambda_h(\mathcal{E}(v) - \mathcal{E}(v_h)))|^2\, dy \\ \leqq c\varepsilon \limsup_h \lambda_h^{-2} \int_{B_1} |V_{p_2}(\lambda_h \mathcal{E}(v_h))|^2\, dy \\ \leqq c\varepsilon \limsup_h \lambda_h^{-2} \int_{B_1} |V_{p_2}(\lambda_h D v_h)|^2\, dy \stackrel{(5.26)}{\leqq} \varepsilon C_M,\end{aligned}$$

and (5.35) follows by letting $\varepsilon \to 0$. We are now going to prove (5.34): for this we rely on Lemma 4.5. We observe that when $|z| \geqq 1$ then (5.3) and the elementary properties of the function V_p imply that $|V_{p_2}(z)|^2 \leqq c|V_{p_h}(z)|^{2(1+\delta_2)}$ with c independent of h, and when $|z| \leqq 1$ then $|V_{p_2}(z)| \leqq c|V_{p_h}(z)|$, thus we have, using also (b) of Lemma 2.2 and denoting as usual by o_h a quantity that vanishes when $h \to \infty$,

$$\begin{aligned}&\lambda_h^{-2} \int_{B_{1/2}} |V_{p_2}(\lambda_h(\mathcal{E}(v) - \mathcal{E}(v_h)))|^2\, dy \\ &\leqq \quad c\lambda_h^{-2}\Big[\int_{\{\lambda_h|\mathcal{E}(v)-\mathcal{E}(v_h)|<1\}\cap B_{1/2}} |V_{p_h}(\lambda_h(\mathcal{E}(v) - \mathcal{E}(v_h)))|^2\, dy \\ &\qquad + \int_{\{\lambda_h|\mathcal{E}(v)-\mathcal{E}(v_h)|\geqq 1\}\cap B_{1/2}} \big|V_{p_h}(\lambda_h(\mathcal{E}(v) - \mathcal{E}(v_h)))\big|^{2(1+\delta_2)}\, dy\Big] \\ &\stackrel{(4.16),(5.35)}{\leqq} \quad o_h + c\lambda_h^{2\delta_2} \int_{B_{1/2}} \Big|\frac{V_{p_h}(\lambda_h D v)}{\lambda_h}\Big|^{2(1+\delta_2)} dy \\ &\qquad + c\lambda_h^{2\delta_2}\left(\int_{B_1} \Big|\frac{V_{p_h}(\lambda_h \mathcal{E}(v_h))}{\lambda_h}\Big|^2 dy\right)^{1+\delta_2} \\ &\qquad + c\lambda_h^{2\delta_2}\left(\int_{B_1} |v_h|^2\, dy\right)^{1+\delta_2} \\ &\leqq \quad o_h + C_M \lambda_h^{2\delta_2} \to 0,\end{aligned}$$

where we used (5.27), (5.32) and (5.26) to perform the last estimate, and (5.34) is completely proved.

Remark 5.5. The previous estimate, and consequently the proof of (5.34), relies on the possibility of estimating $p_2 \leqq p_h(1 + \delta_2)$, that is, the function $p(x)$ must have small oscillations when blowing up the solution. This is the main reason to blow up minimizers in open subset like $\mathcal{O}$, rather that directly in the whole Ω. Since the choice of $\mathcal{O}$ depends on M and thus on the solution u itself, this will force us in the next section to adopt a delicate localization argument in the iteration procedure when proving Theorem 2.1.

Step 3: Strong convergence (II). Our aim here is to establish the stronger statement

$$\limsup_h \lambda_h^{-2} \int_{B_{1/4}} |V_{p_2}(\lambda_h(Dv - Dv_h))|^2\, dy = 0. \tag{5.39}$$

We preliminarily observe that in the case $p_2 \geqq 2$ this property is an easy consequence of (5.34) via the standard Korn inequality (2.19) and by (5.27): indeed, keeping into account (h) of Lemma 2.2, we see that in this case, using the strong convergence in (5.27) and (5.34),

$$\begin{aligned}\lambda_h^{-2}\int_{B_{1/2}}|V_{p_2}(\lambda_h(Dv-Dv_h))|^2\,dy \\ &\leqq c\int_{B_{1/2}}\left(|Dv-Dv_h|^2+\lambda_h^{p_2-2}|Dv-Dv_h|^{p_2}\right)dy \\ &\leqq c\lambda_h^{-2}\int_{B_{1/2}}|V_{p_2}(\lambda_h(\mathcal{E}(v)-\mathcal{E}(v_h)))|^2\,dy \\ &\quad+c\int_{B_{1/2}}\left(|v-v_h|^2+\lambda_h^{p_2-2}|v-v_h|^{p_2}\right)dy\to 0.\end{aligned}$$

The case $p_2 < 2$ is more delicate and needs the full strength of the arguments developed in Section 3. We start by defining a new sequence of functions $\tilde{w}_h \in W^{1,p_2}(\mathbb{R}^n;\mathbb{R}^n)$: fix a cut-off function $\eta \in C_0^\infty(B_{1/2})$ such that $\eta \equiv 1$ on $B_{1/4}$, and set $\tilde{w}_h := \eta(v_h - v)$, extended as zero outside $B_{1/2}$. Then clearly

$$\lambda_h^{-2}\int_{B_{1/4}}|V_{p_2}(\lambda_h(Dv_h-Dv))|^2\,dy \leqq \lambda_h^{-2}\int_{\mathbb{R}^n}|V_{p_2}(\lambda_h D\tilde{w}_h)|^2\,dy. \tag{5.40}$$

From Lemma 2.2 we have:

$$\begin{aligned}\lambda_h^{-2}\int_{\mathbb{R}^n}|V_{p_2}(\lambda_h\mathcal{E}(\tilde{w}_h))|^2\,dy \;&\leqq\; c\lambda_h^{-2}\int_{B_{1/2}}|V_{p_2}(\lambda_h(\mathcal{E}(v_h)-\mathcal{E}(v)))|^2\,dy \\ &\quad+c\lambda_h^{-2}\int_{B_1}|V_{p_2}(\lambda_h(v_h-v))|^2\,dy \\ &:= (\mathrm{V})_h+(\mathrm{VI})_h\to 0.\end{aligned}\tag{5.41}$$

Indeed, since $p_2 < 2$ we immediately have, from (e) of Lemma 2.2,

$$(\mathrm{VI})_h \leqq c\int_{B_1}|v_h-v|^2\,dy \overset{(5.27)_1}{\to} 0,$$

while $(\mathrm{V})_h \to 0$ is just (5.34). Now we observe that as $p_2 < 2$

$$G_{p,\lambda}(|z|) \leqq \lambda^{-2}|V_p(\lambda z)|^2 \leqq 2G_{p,\lambda}(|z|), \tag{5.42}$$

where $G_{p,\lambda}$ is the Young function defined in (3.1). Using Theorem 3.1 and a couple of times each (5.42) and Lemma 3.4, we find that (5.41) implies

$$\lambda_h^{-2}\int_{\mathbb{R}^n}|V_{p_2}(\lambda_h D\tilde{w}_h)|^2\,dy \to 0.$$

This together with (5.40) finally gives (5.39) also in the case $p_2 < 2$.

Step 4: Comparison and conclusion. We preliminarily observe that, using (2.7),

$$
\begin{aligned}
\mu_h^{-2} & \fint_{B(x_h,\tau R_h)} |V_{p_2}(Du_h - Du)|^2\,dx \\
&\leqq c\tau^{-n}\,\mu_h^{-2} \fint_{B(x_h,R_h)} \big(|Du_h - Du|^{p_2}\,dx \\
&\qquad + \mathbf{1}_{(p_2>2)} c\tau^{-n}\,\mu_h^{-2}\left(\fint_{B(x_h,R_h)} |Du_h - Du|^{p_2}\,dx\right)^{2/p_2} \\
&\overset{(5.24)}{\leqq} C_M \mu_h^{-2}[R_h^{\bar\beta} + \mathbf{1}_{(p_2>2)} R_h^{2\bar\beta/p_2}] \\
&\leqq C_M[R_h^{\bar\beta-\hat\beta} + \mathbf{1}_{(p_2>2)} R_h^{(2\bar\beta/p_2)-\hat\beta}] \overset{(5.17)}{\to} 0,
\end{aligned}
\tag{5.43}
$$

and also, in a similar way,

$$
\mu_h^{-2} \fint_{B(x_h,\tau R_h)} |V_{p_2}\big((Du_h)_{x_h,\tau R_h} - (Du)_{x_h,\tau R_h}\big)|^2\,dx \to 0. \tag{5.44}
$$

Since $|(Du)_{x_h,\tau R_h}| \leqq M$ by our assumption (5.20), we may use (b) and (c) of Lemma 2.2 together with the previous estimates to obtain

$$
\begin{aligned}
&\limsup_h \mu_h^{-2} E(x_h,\tau R_h) \\
&\overset{(c)}{\leqq} C_M \limsup_h \mu_h^{-2} \fint_{B(x_h,\tau R_h)} |V_{p_2}(Du - (Du)_{x_h,\tau R_h})|^2\,dx \\
&\qquad + C_M \tau^{\hat\beta} \limsup_h \mu_h^{-2} R_h^{\hat\beta} \\
&\overset{(b)}{\leqq} C_M\tau^{\hat\beta} + C_M \limsup_h \mu_h^{-2} \fint_{B(x_h,\tau R_h)} |V_{p_2}(Du - Du_h)|^2\,dx \\
&\qquad + C_M \limsup_h \mu_h^{-2} \fint_{B(x_h,\tau R_h)} |V_{p_2}(Du_h - (Du_h)_{x_h,\tau R_h})|^2\,dx \\
&\qquad + C_M \limsup_h \mu_h^{-2} \fint_{B(x_h,\tau R_h)} |V_{p_2}((Du_h - Du)_{x_h,\tau R_h})|^2\,dx \\
&\overset{*}{\leqq} C_M\tau^{\hat\beta} + C_M \limsup_h \lambda_h^{-2} \fint_{B_\tau} |V_{p_2}(\lambda_h(Dv_h - (Dv_h)_\tau))|^2\,dy \\
&\overset{(b)}{\leqq} C_M\tau^{\hat\beta} + C_M \limsup_h \lambda_h^{-2} \fint_{B_\tau} |V_{p_2}(\lambda_h(Dv_h - Dv))|^2\,dy \\
&\qquad + C_M \limsup_h \lambda_h^{-2} \fint_{B_\tau} |V_{p_2}(\lambda_h(Dv - (Dv)_\tau))|^2\,dy \\
&\qquad + C_M \limsup_h \lambda_h^{-2} \fint_{B_\tau} |V_{p_2}(\lambda_h((Dv)_\tau - (Dv_h)_\tau))|^2\,dy \\
&\overset{(5.33),(5.39)}{\leqq} C_M(\tau^2 + \tau^{\hat\beta}) \leqq \hat C_M \tau^{\hat\beta},
\end{aligned}
$$

where the estimate denoted by $*$ was performed using (5.28), (5.43), (5.44). Now the contradiction to (5.21) follows if we choose, for instance, $C(M) := 2\hat C_M$. □

6. Regularity

In the case of standard p-growth, i.e., when $p(x)$ is constant, once the decay estimate in Lemma 4.5 is attained the Hölder continuity of Du on an open subset of full measure follows via a standard iteration argument, see e.g. [11]. In our case the situation is different, and a delicate localization argument will be worked out.

Proof of Theorem 2.1.
Step 1: Construction of Ω_0. Let $R_0 > 0$ be a radius such that $\omega(R_0) \leqq \delta_1/4$, where δ_1 is the higher integrability exponent introduced in the higher integrability Theorem 4.2 and eventually reduced in (5.1); it is possible to cover Ω with a finite number k of balls $B_i \equiv B(x_i, R_0)$. We let:

$$p^i := \sup_{B_i \cap \Omega} p(x),$$

$$\Omega_0^i := \{x_0 \in B_i \cap \Omega : \limsup_{\rho \to 0} \left[(|Du|^{p^i})_{x_0,\rho} + (|u|)_{x_0,\rho} \right] < +\infty \text{ and}$$
$$\lim_{\rho \to 0} \fint_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^{p^i}\, dx = 0\},$$

and finally

$$\Omega_0 := \bigcup_{i=1}^{k} \Omega_0^i.$$

We claim that Ω_0 is the set we are looking for in order to prove Theorem 2.1. We begin by remarking that Ω_0 has full measure: indeed with the choice of R_0 made above it follows that, analogously to what we found in (5.3),

$$p^i(1 + \delta_1/4) \leqq p(x)(1 + \delta_1) \quad \text{for every } x \in B_i \cap \Omega. \tag{6.1}$$

From (6.1), the higher integrability result of Theorem 4.2 and the Lebesgue differentiation theorem, it immediately follows that Ω_0^i is a set of full measure in $\Omega \cap B_i$ for each i, and consequently so is Ω_0. We must now prove that Ω_0 is open and that Du is Hölder continuous in Ω_0.

Step 2: Localization. From now on we shall work on a single Ω_0^i. Fix $x_0 \in \Omega_0^i$: by the definition of Ω_0^i it is possible to find $M > 64$ such that

$$\limsup_{\rho \to 0} \left[(|Du|^{p^i})_{x_0,\rho} + (|u|)_{x_0,\rho} \right] < \frac{M}{64}.$$

As at the beginning of Section 5 we deduce, via Lemma 4.5, a higher integrability exponent $\delta_2 \equiv \delta_2(M)$, we determine a radius R_M such that $\omega(R_M) \leqq \delta_2(M)/4$ and we consider the ball $B(x_0, R_M)$; without loss of generality we may assume that $B(x_0, R_M) \subset B_i \cap \Omega$. Following the notation introduced at the beginning of Section 5 we put

$$p_2 := \sup_{B(x_0,R_M)} p(x).$$

Step 3: Iteration. Fix M as above and let $B(x_0, 16R) \subset\subset \mathcal{O} \subset\subset \Omega$ where $\mathcal{O} \equiv B(x_0, R_M) \equiv \mathcal{O}_M$ is as in Proposition 5.4; if C_M is the constant appearing in (5.19) and $0 < \tau < 1/4$ is such that $C_M \tau^{\hat{\beta}/2} < 1/4$, then a minor modification of the iteration scheme developed in [15] shows that there exists $\eta \equiv \eta(M, \tau) \equiv \eta(M) \leqq \varepsilon \leqq 1$, with ε as in (5.18), such that if

$$\begin{gathered} |(Du)_{x_0,\tau R}| + |(Du)_{x_0,R}| + |(Du)_{x_0,4R}| + |(u)_{x_0,R}| \leqq M/4, \\ E(x_0, R) \leqq \eta, \qquad E(x_0, 4R) \leqq 1 \end{gathered} \tag{6.2}$$

then a standard iteration procedure built upon Lemma 4.5 starts and leads to

$$|(u)_{x_0,\tau^k R}|, |(Du)_{x_0,\tau^k R}| \leqq M, \qquad E(x_0, \tau^k R) \leqq \tau^{k\hat{\beta}/2} \tag{6.3}$$

for every $k \geqq 1$. With this choice of $\mathcal{O}$, we find the two numbers τ and η, which both depend on M. Again by the definition of Ω_0^i, it is possible to determine $\tilde{R}_M < R_M/1000$ such that if ρ is any of the numbers $\tau\tilde{R}_M, \tilde{R}_M, 4\tilde{R}_M$, we have

$$\begin{gathered} [(|Du|^{p^i})_{x_0,\rho} + (|u|)_{x_0,\rho}] < \frac{M}{32}, \\ \tilde{c} \fint_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^{p^i}\, dx + \rho^{\hat{\beta}} < \min\left\{\left(\frac{\eta}{8}\right)^{p^i/p_2}, \left(\frac{\eta}{8}\right)^{p^i/2}, \frac{\eta}{4}\right\}, \end{gathered}$$

where $\tilde{c} \equiv \tilde{c}(M) > 1$ is a constant coming up in the next estimate and depending on the ones in (c) and (e) of Lemma 2.2. We now remark that these inequalities hold also in a neighbourhood of x_0, i.e., there exists an open set $A \subset B(x_0, R_M)$ such that $x_0 \in A$ and that, for every $x_* \in A$,

$$[(|Du|^{p^i})_{x_*,\rho} + (|u|)_{x_*,\rho}] < \frac{M}{32}, \tag{6.4}$$

$$\tilde{c} \fint_{B(x_*,\rho)} |Du - (Du)_{x_*,\rho}|^{p^i}\, dx + \rho^{\hat{\beta}} < \min\left\{\left(\frac{\eta}{8}\right)^{p^i/p_2}, \left(\frac{\eta}{8}\right)^{p^i/2}, \frac{\eta}{4}\right\} \tag{6.5}$$

whenever ρ is any of the numbers $\tau\tilde{R}_M, \tilde{R}_M, 4\tilde{R}_M$.

Our goal now is to check that the inequalities (6.2) are satisfied (at the point x_*, not only at x_0), in order to make the iteration work and obtain

$$|(u)_{x_*,\tau^k R}|, |(Du)_{x_*,\tau^k R}| \leqq M, \qquad E(x_*, \tau^k R) \leqq \tau^{k\hat{\beta}/2}. \tag{6.6}$$

Clearly $(6.2)_1$ is satisfied by (6.4), while in order to prove $(6.2)_2$ we use (c) and (e) of Lemma 2.2 and the Hölder inequality, recalling that $p^i \geqq p_2$, to get, again for

$\rho = \tau \tilde{R}_M, \tilde{R}_M, 4\tilde{R}_M,$

$$
\begin{aligned}
E(x_*, \rho) &= \fint_{B(x_*,\rho)} |V_{p_2}(Du) - V_{p_2}((Du)_{x_*,\rho})|^2\, dx + \rho^{\hat{\beta}} \\
&\leqq c(M) \fint_{B(x_*,\rho)} |Du - (Du)_{x_*,\rho}|^{p_2}\, dx \\
&\quad + c(M)\mathbf{1}_{(p_2>2)} \left(\fint_{B(x_*,\rho)} |Du - (Du)_{x_*,\rho}|^{p_2}\, dx \right)^{2/p_2} + \rho^{\hat{\beta}} \\
&\leqq \left(\tilde{c}(M) \fint_{B(x_*,\rho)} |Du - (Du)_{x_*,\rho}|^{p^i}\, dx \right)^{p_2/p^i} \\
&\quad + \mathbf{1}_{(p_2>2)} \left(\tilde{c}(M) \fint_{B(x_*,\rho)} |Du - (Du)_{x_*,\rho}|^{p^i}\, dx \right)^{2/p^i} + \rho^{\hat{\beta}} \\
&\overset{(6.5)}{\leqq} \eta/2 < \eta < 1,
\end{aligned}
$$

thus all the inequalities in (6.2) are satisfied with x_* in place of x_0 and by iteration we get (6.6).

Step 4: Conclusion. Now we want to prove that (6.6) implies

$$
\fint_{B(x_*,\rho)} |Du - (Du)_{x_*,\rho}|^{p_2}\, dx \leqq C_M \rho^{\hat{\beta}/4} \tag{6.7}
$$

for any $0 < \rho \leqq \tilde{R}_M$. A simple interpolation shows that it suffices to prove (6.7) only for the numbers ρ of the type $\rho = \tau^k \tilde{R}_M$, to which case we restrict our attention henceforth. Starting from (6.6), if $p_2 \geqq 2$ then by (e) of Lemma 2.2

$$
\begin{aligned}
&\fint_{B(x_*,\rho)} |Du - (Du)_{x_*,\rho}|^{p_2}\, dx \\
&\qquad \leqq \fint_{B(x_*,\rho)} |V_{p_2}\big(Du - (Du)_{x_*,\rho}\big)|^2\, dx \\
&\qquad \leqq C_M E(x_*, \rho) \leqq C_M E(x_*, \rho)^{1/2} \overset{(6.6)}{\leqq} C_M \rho^{\hat{\beta}/4}
\end{aligned}
$$

(now, as customary, C_M denotes any constant depending on M in a harmless way). If $1 < p_2 < 2$, again using (e) of Lemma 2.2 and setting $S := \{|Du - (Du)_{x_*,\rho}| \geqq 1\}$, we estimate

$$
\begin{aligned}
&\fint_{B(x_*,\rho)} |Du-(Du)_{x_*,\rho}|^{p_2}\,dx \\
&\quad = \omega_n \rho^{-n}\left[\int_{B_\rho\cap S}\cdots\,dx + \int_{B_\rho\setminus S}\cdots\,dx\right] \\
&\quad \leqq \omega_n \rho^{-n}\int_{B_\rho\cap S} |Du-(Du)_{x_*,\rho}|^{p_2}\,dx + c\fint_{B_\rho} |V_{p_2}(Du-(Du)_{x_*,\rho})|^{p_2}\,dx \\
&\quad \leqq c\fint_{B_\rho} |V_{p_2}(Du-(Du)_{x_*,\rho})|^{2}\,dx + \left(\fint_{B_\rho} |V_{p_2}(Du-(Du)_{x_*,\rho})|^{2}\,dx\right)^{p_2/2} \\
&\quad \leqq C_M\left[E(x_*,\rho)+E(x_*,\rho)^{1/2}\right] \leqq C_M E(x_*,\rho)^{1/2} \overset{(6.6)}{\leqq} C_M \rho^{\hat\beta/4},
\end{aligned}
$$

and (6.7) is proved. Next we have to prove that

$$
\lim_{\rho\to 0}\fint_{B(x_*,\rho)} |Du-(Du)_{x_*,\rho}|^{p^i}\,dx = 0, \tag{6.8}
$$

and again we restrict our attention to the numbers ρ of the type $\rho = \tau^k \tilde R_M$. If $p^i = p_2$, there is nothing else to do by (6.7); if instead $p_2 < p^i$, we interpolate

$$
p_2 < p^i < p^i(1+\delta_1/4),
$$

thus for some numbers θ_1, θ_2

$$
\begin{aligned}
&\fint_{B(x_*,\rho)} |Du-(Du)_{x_*,\rho}|^{p^i}\,dx \\
&\leqq \left(\fint_{B(x_*,\rho)} |Du-(Du)_{x_*,\rho}|^{p_2}\,dx\right)^{\theta_1} \\
&\quad \times \left(\fint_{B(x_*,\rho)} |Du-(Du)_{x_*,\rho}|^{p^i(1+\delta_1/4)}\,dx\right)^{\theta_2}.
\end{aligned}
$$

Since the first factor tends to zero as $\rho \to 0$ by (6.7), we only need to prove that for a suitable constant C

$$
\fint_{B(x_*,\rho)} |Du-(Du)_{x_*,\rho}|^{p^i(1+\delta_1/4)}\,dx \leqq C. \tag{6.9}
$$

We first remark that for all $0 < \rho < \tilde R_M$

$$
(|Du|^{p_2})_{x_*,\rho} \leqq c\fint_{B(x_*,\rho)} |Du-(Du)_{x_*,\rho}|^{p_2}\,dx + c|(Du)_{x_*,\rho}|^{p_2} \leqq c \tag{6.10}
$$

for some constant c depending only on those in (6.6) and (6.7), and thus only on those in (6.4), (6.5). We use (2.19) with $p = p^i(1+\delta_1/4)$ and Poincaré inequality to get

$$\begin{aligned}
&\fint_{B(x_*,\rho)} |Du - (Du)_{x_*,\rho}|^{p^i(1+\delta_1/4)}\,dx \\
&\quad \leqq c \fint_{B(x_*,\rho)} |Du|^{p^i(1+\delta_1/4)}\,dx \\
&\quad \leqq c \fint_{B(x_*,\rho)} |\mathcal{E}(u)|^{p^i(1+\delta_1/4)}\,dx + c\left(\fint_{B(x_*,\rho)} |Du|\,dx\right)^{p^i(1+\delta_1/4)} \\
&\quad \overset{(6.10)}{\leqq} c\left(1 + \fint_{B(x_*,\rho)} |\mathcal{E}(u)|^{p(x)(1+\delta_1)}\,dx\right).
\end{aligned}$$

Now by (4.4)

$$\begin{aligned}
&\left(\fint_{B(x_*,\rho)} |\mathcal{E}(u)|^{p(x)(1+\delta_1)}\,dx\right)^{\frac{1}{1+\delta_1}} \\
&\quad \leqq c \fint_{B(x_*,2\rho)} |\mathcal{E}(u)|^{p(x)}\,dx \\
&\qquad + c \fint_{B(x_*,2\rho)} \left(|Du|^{\gamma_1} + |u - (u)_{x_*,2\rho}|^{\gamma_1^*} + |(u)_{x_*,2\rho}|^{\gamma_1^*} + 1\right) dx \\
&\quad \overset{(6.6)}{\leqq} c + c \fint_{B(x_*,2\rho)} |Du|^{p_2}\,dx + \rho^{\gamma_1^*}\left(\fint_{B(x_*,2\rho)} |Du|^{p_2}\,dx\right)^{\gamma_1^*/p_2} \\
&\quad \leqq C
\end{aligned}$$

by (6.10), and the proof of (6.9) is finished. We conclude by remarking that from what we proved we have $A \subset \Omega_0^i$, thus Ω_0^i is open.

From these facts and Campanato integral characterization of Hölder continuity, it finally follows that Du is Hölder continuous in Ω_0. □

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Dipartimento di Matematica
Università di Parma
V. D'Azeglio 85/a
43100 Parma, Italy
e-mail: emilio.acerbi@unipr.it
e-mail: giuseppe.mingione@unipr.it

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