J. reine angew. Math. **584** (2005), 117—148

Journal für die reine und angewandte Mathematik

Gradient estimates for the $p(x)$-Laplacean system

By *Emilio Acerbi* and *Giuseppe Mingione* at Parma

Abstract. We prove Calderón and Zygmund type estimates for a class of elliptic problems whose model is the non-homogeneous $p(x)$-Laplacean system

$$-\mathrm{div}(|Du|^{p(x)-2}Du) = -\mathrm{div}(|F|^{p(x)-2}F).$$

Under optimal continuity assumptions on the function $p(x) > 1$ we prove that

$$|F|^{p(x)} \in L^q_{\mathrm{loc}} \Rightarrow |Du|^{p(x)} \in L^q_{\mathrm{loc}} \quad \forall q > 1.$$

Our estimates are motivated by recent developments in non-Newtonian fluidmechanics and elliptic problems with non-standard growth conditions, and are the natural, "non-linear" counterpart of those obtained by Diening and Růžička [12] in the linear case.

1. Introduction

In recent years, increasing attention has been paid to the study of the so called *generalized Lebesgue spaces* $L^{p(x)}(\Omega; \mathbb{R}^N)$, that is

$$(1) \qquad L^{p(x)}(\Omega; \mathbb{R}^N) := \left\{ f : \Omega \to \mathbb{R}^N \colon f \text{ is measurable and } \int_\Omega |f|^{p(x)}\,dx < \infty \right\}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $p : \Omega \to (1, +\infty)$ is in general taken to be a continuous function (there is no obstruction in taking a more general $p(x)$, but the resulting space has very few properties if no geometric condition on p is imposed). The Luxemburg type norm

$$\|f\|_{L^{p(x)}(\Omega;\mathbb{R}^N)} := \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{f}{\lambda}\right|^{p(x)} dx \leqq 1 \right\}$$

makes $L^{p(x)}$ a Banach space. Accordingly, the *generalized* $W^{1,p(x)}(\Omega; \mathbb{R}^N)$ *spaces* are defined by

$$W^{1,p(x)}(\Omega; \mathbb{R}^N) := \{u \in L^{p(x)}(\Omega; \mathbb{R}^N) : Du \in L^{p(x)}(\Omega; \mathbb{R}^{nN})\},$$

where Du denotes the gradient of the function u. These also become Banach spaces with

$$\|u\|_{W^{1,p(x)}(\Omega;\mathbb{R}^N)} := \|u\|_{L^{p(x)}(\Omega;\mathbb{R}^N)} + \|Du\|_{L^{p(x)}(\Omega;\mathbb{R}^{nN})},$$

see [30], [17], [10], [14] for more details and references. Apart from the basic theoretical issues, such spaces are relevant in the study of non-Newtonian fluids. Indeed, the underlying integral energy appearing in the modelling of the so called electrorheological fluids, as conceived by Růžička, and Rajagopal and Růžička [28], [30], [31] in the contest of continuum mechanics, is

$$\int_\Omega |Du|^{p(x)}\,dx. \tag{2}$$

The analysis of such fluids is performed in the space $W^{1,p(x)}(\Omega;\mathbb{R}^N)$. Moreover, energies of the previous type occur in Homogenization [35], Image Restoration [24] and, more generally, in the modelling of strongly inhomogeneous physical behaviours. Therefore a great deal of work has been developed around variational and elliptic problems with "$p(x)$-growth" that is, involving the energy (2); see also [3], [4] and related references. Very recently, Diening and Růžička [12] established estimates of Calderón and Zygmund type for Singular integrals in the spaces $L^{p(x)}$. This eventually led them to give $L^{p(x)}$ versions of a class of results that can be obtained through the use of singular integrals, as e.g. the classical Korn's inequality and some classical estimates for linear problems as, for instance, the so called Bogowsky's lemma: for the equation $\operatorname{div} u = f$, they are able to prove the existence of a solution $u \in W^{1,p(x)}$ provided $f \in L^{p(x)}$ has null mean value. Such analysis allows to give estimates for problems involving second order linear operator with constant coefficients as, for instance, $\triangle u = f$.

In this paper we are going to treat another basic issue concerning the integrability of the gradient, adopting a viewpoint which is "dual" to that in [12]: instead of seeking estimates in the spaces $L^{p(x)}$ for solutions to linear elliptic problems with constant coefficients, we consider classical Lebesgue spaces but we look at the differential operator coming up when considering the energy (2), and therefore the model of electrorheological fluids. Our investigation will involve the non-homogeneous $p(x)$-Laplacean system

$$-\operatorname{div}(|Du|^{p(x)-2}Du) = -\operatorname{div}(|F|^{p(x)-2}F) \quad \text{in } \Omega, \tag{3}$$

whose weak solutions (see Section 2 for precise definitions) are taken in the natural space $W^{1,p(x)}(\Omega;\mathbb{R}^N)$; the vector field F is initially taken in the natural space $L^{p(x)}(\Omega;\mathbb{R}^{nN})$ and the function $p(x)$ is supposed to be continuous and to satisfy (which is not restrictive for local results)

$$1 < \gamma_1 \leqq p(x) \leqq \gamma_2 < \infty. \tag{4}$$

We remark that, even in the case $p(x) \equiv$ constant, the approach via singular integrals cannot be used to prove L^q-estimates for solutions; our results and techniques are indeed in the framework of *nonlinear potential theory*. If $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ denotes the modulus of continuity of the function $p(x)$

$$|p(x) - p(y)| \leqq \omega(|x - y|) \tag{5}$$

then the main assumption on the function $p(x)$ will be

$$\lim_{R\to 0}\omega(R)\log\left(\frac{1}{R}\right)=0. \tag{6}$$

This slightly reinforces the condition considered by Diening and Růžička [12], where the right hand side of (6) is supposed to be a finite number rather than zero, see (18), and plays a central role in the regularity analysis of solutions of this kind of problems; it is essentially optimal in order to obtain the results we are going to present, see Remark 2 where our results are restated in the weaker version that can be deduced when (6) is weakened into the condition used by Diening and Růžička. Both conditions have become customary when dealing with the energy (2), see [2], [36]. Here we shall prove integrability results for a class of elliptic problems, that in the model case (3) will lead to establish that for all $q > 1$

$$|F|^{p(x)}\in L^q_{\mathrm{loc}}(\Omega)\Rightarrow |Du|^{p(x)}\in L^q_{\mathrm{loc}}(\Omega) \tag{7}$$

(Theorems 1 and 3). In the case $p(x)\equiv$ constant, this type of result has been established, in the case of the p-Laplacean equation, in the fundamental paper by T. Iwaniec [19]. As far as we know, our result is the first of Calderón and Zygmund type valid for elliptic operators under non-standard growth conditions, see Remark 1 below. Let us remark that such kind of estimate is relevant for the numerical treatment of problems modelled by energies like (2), as e.g. electrorheological fluids: the a priori knowledge of higher integrability of the gradient allows to implement better finite element schemes.

Finally, about the techniques. The main difficulty is the interplay of the nonlinearity and the fact that the system we consider exhibits the so called non-standard growth conditions, see Remark 1 below. In order to deal with such a peculiarity, we shall rely on a new and beautiful method to prove L^q estimates introduced by Caffarelli and Peral [7], [8], and based on Calderón and Zygmund type covering arguments and iteration of level sets; this will be combined with a careful localization technique tailored to the non-standard structure of the $p(x)$-Laplacean operator, fine estimates in $L\log^\beta L$ spaces and the use of certain restricted Maximal Operators. We explicitly observe that a consistent part of our efforts here is put in the task of deriving *natural* local estimates for the gradient of solutions, as similar as possible to those available for the case $p(x)\equiv$ constant: what we shall come up with is a sort of reverse Hölder inequality for Du (14), that keeps into account the non-standard growth conditions exhibited by the $p(x)$-Laplacean system. As mentioned above, we shall prove gradient estimates for more general elliptic operators whose degenerate structure is similar to (3). We finally remark that we confined our analysis to right-hand side structures as in (3), in order to have the possibility to formulate the regularizing properties of the $p(x)$-Laplacean system in the neat way (7), as customary in the case when $p(x)$ is a constant function [19], [23]. Anyway the arguments presented in this paper allow the treatment of different equations and systems such as

$$-\operatorname{div}(|Du|^{p(x)-2}Du)=-\operatorname{div}F,\quad -\operatorname{div}(|Du|^{p(x)-2}Du)=F,$$

provided suitable integrability assumptions are made on F; the proofs have to be suitably modified according to the different structure coming into the play.

Acknowledgment. The research of the authors has been partially supported by MIUR/Cofin via the national project "Calcolo delle Variazioni" (2002). Part of the work was done while G.M. was visiting the Universities of Madrid (Autonoma) and Seville: both institutions are acknowledged for their support. Special thanks to Magdalena Walias for her precious assistance.

2. Results

General notation. In the sequel $\Omega \subset \mathbb{R}^n$ will be a bounded domain; by "cube" we will always mean an open cube with edges parallel to the coordinate axes; when relevant, we will mention the side length, denoting e.g. by Q_R a cube with side length equal to $2R$: with a slight abuse, we will call R the *radius* of such cube. Moreover, for $\gamma > 0$, we will adopt the convention that γQ or $Q_{\gamma R}$ denote cubes *with the same centre* as Q or Q_R, and radius multiplied by γ. Adopting a usual convention, c will denote a constant whose value may change in any two occurrences, and only the relevant dependences will be specified, as e.g. in $c(\gamma, p)$; particular constants will be denoted by c_1, $\tilde{c}$ and the like. For the Lebesgue measure of a measurable set A we shall employ either of the notations

$$|A| = \operatorname{meas}(A);$$

then we define the mean value on a cube $Q_R \subset \Omega$ of a locally integrable function $v \in L^1_{\mathrm{loc}}(\Omega)$ by

$$(v)_{Q_R} \equiv (v)_R \equiv \fint_{Q_R} v\,dx := \frac{1}{|Q_R|} \int_{Q_R} v\,dx.$$

Structure conditions. For the case of equations ($N = 1$) we shall consider a vector field $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ such that $z \mapsto a(\cdot, z)$ belongs to $C^0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ and the following growth, ellipticity and continuity assumptions, inspired by the energy density $(\mu^2 + |Du|^2)^{p(x)/2}$, are satisfied:

$$\nu(\mu^2 + |z|^2)^{(p(x)-2)/2}|\lambda|^2 \leqq D_z a(x, z)\lambda \otimes \lambda \leqq L(\mu^2 + |z|^2)^{(p(x)-2)/2}|\lambda|^2, \tag{8}$$

$$|a(x, z) - a(y, z)| \leqq L\omega(|x - y|)|\log(\mu^2 + |z|^2)|(\mu^2 + |z|^2)^{(p(x)-1)/2} \tag{9}$$

for every $x, y \in \Omega$, $z, \lambda \in \mathbb{R}^n$, where $\nu^{-1}, L \in [1, \infty)$ and the parameter $\mu \in [0, 1]$ appears to deal simultaneously with the degenerate and the non-degenerate cases (and will be only briefly seen in the remainder of the paper). The function $p : \Omega \to (1, \infty)$ is supposed to satisfy (4), (5), where the modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies (6); without loss of generality, we assume $\omega(\cdot)$ to be non-decreasing. Observe also that, eventually enlarging L and decreasing ν, by (8) we can suppose that

$$|a(x, z)| \leqq L(1 + |z|^2)^{(p(x)-1)/2} \tag{10}$$

and

$$\nu(\mu^2 + |z|^2)^{p(x)/2} - L \leqq \langle a(x, z), z \rangle \quad \forall x \in \Omega,\ z \in \mathbb{R}^n. \tag{11}$$

Weak solutions. Let $F \in L^{p(x)}(\Omega; \mathbb{R}^{nN})$ be a vector field; if $N = 1$, we define as a weak solution to the equation

$$-\operatorname{div} a(x, Du) = -\operatorname{div}\big(|F(x)|^{p(x)-2}F(x)\big) \tag{12}$$

a function $u \in W^{1,p(x)}(\Omega)$ such that

$$\int_\Omega a(x, Du)D\varphi\, dx = \int_\Omega |F(x)|^{p(x)-2}F(x)D\varphi\, dx \tag{13}$$

for every test function $\varphi \in W^{1,p(x)}(\Omega)$ with compact support in Ω. In the same way, in $N \geqq 1$, a function $u \in W^{1,p(x)}(\Omega; \mathbb{R}^N)$ is defined to be a weak solution to the "$p(x)$-Laplacean system" (3) with $F \in L^{p(x)}(\Omega; \mathbb{R}^{nN})$, if and only if

$$\int_\Omega |Du|^{p(x)-2}DuD\varphi\, dx = \int_\Omega |F(x)|^{p(x)-2}F(x)D\varphi\, dx$$

holds for every test function $\varphi \in W^{1,p(x)}(\Omega; \mathbb{R}^N)$ with compact support in Ω. For existence results concerning weak solutions we refer to [30].

Theorem 1. *Let $u \in W^{1,p(x)}(\Omega)$ be a weak solution to* (12) *under the assumptions* (4), (6), (8), (9) *and let $|F|^{p(x)} \in L^q_{\mathrm{loc}}(\Omega)$ for some $q > 1$. Then*

$$|Du|^{p(x)} \in L^q_{\mathrm{loc}}(\Omega).$$

This result is necessarily complemented by the following estimate (which is indeed the proof of Theorem 1).

Theorem 2. *Under the assumptions of Theorem* 1, *if $\Omega' \subset\subset \Omega$ is an open subset and $|F|^{p(x)} \in L^q(\Omega')$ then for every $\varepsilon \in (0, q-1)$ there exists a positive radius $R_0 > 0$, depending on*

$$n, \gamma_1, \gamma_2, \nu, L, \varepsilon, q, \omega(\cdot), \|\, |Du(\cdot)|^{p(\cdot)}\|_{L^1(\Omega)}, \|\, |F(\cdot)|^{p(\cdot)}\|_{L^q(\Omega')}$$

such that, whenever $Q_{4R} \subset\subset \Omega'$ and $R \leqq R_0$,

$$\left(\fint_{Q_R} |Du|^{p(x)q}\, dx\right)^{\frac{1}{q}} \leqq cK^\varepsilon \fint_{Q_{4R}} |Du|^{p(x)}\, dx + cK^\varepsilon\left(\fint_{Q_{4R}} |F|^{p(x)q}\, dx + 1\right)^{\frac{1}{q}} \tag{14}$$

where $c \equiv c(n, \gamma_1, \gamma_2, \nu, L, q)$ and

$$K := \int_{Q_{4R}} |Du|^{p(x)} + |F|^{p(x)(1+\varepsilon)}\, dx + 1. \tag{15}$$

The previous results extend to weak solutions to the $p(x)$-Laplacean system (3):

Theorem 3. *Assume* (4), (6) *hold and let* $u \in W^{1,p(x)}(\Omega;\mathbb{R}^N)$ *be a weak solution to the* $p(x)$*-Laplacean system* (3) *such that* $|F|^{p(x)} \in L^q_{\mathrm{loc}}(\Omega)$ *for some* $q > 1$; *then*

$$|Du|^{p(x)} \in L^q_{\mathrm{loc}}(\Omega).$$

Moreover, *estimate* (14) *holds as for Theorem* 1, *with* K *as in* (15) *and* R_0 *and* c *depending also on* N.

Now that the statements have been given with all the appropriate localizations, we make an assumption to improve the readability of the proofs: we suppose rightaway that

$$|F|^{p(x)} \in L^q(\Omega):$$

it would be a boring but very easy task to rewrite the proofs without this assumption.

Remark 1. *Comments on the estimate* (14). The appearance of K^ε, which prevents (14) to be a classical reverse Hölder inequality with increasing support, is essentially due to the fact that the operators we consider are anisotropic with respect to growth and ellipticity exponents; this fact is best framed when set in the following more general context. Let us recall that a vector field $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies non-standard growth conditions of (p,q) type when the ellipticity and growth conditions

$$\nu(\mu^2 + |z|^2)^{(p-2)/2}|\lambda|^2 \leqq D_z a(x,z)\lambda \otimes \lambda \leqq L(\mu^2 + |z|^2)^{(q-2)/2}|\lambda|^2, \tag{16}$$

hold for all $z, \lambda \in \mathbb{R}^n$, $x \in \Omega$, where $1 < p < q$ and $\nu^{-1}, L \in [1,\infty)$. Under a suitable smallness assumption on the ratio q/p, solutions to the elliptic equation $\operatorname{div} a(x,Du) = 0$ satisfy an estimate of the type

$$\|Du\|_{L^q(Q_R)} \leqq c\|Du\|^{1+\varepsilon}_{L^p(Q_{\gamma R})}, \quad \gamma > 1.$$

In this case ε is a fixed, positive quantity depending on p, q in such a way that $\varepsilon \to 0$ when $q/p \to 1$. The reader may look at [15], [26] and related references. The estimates we find here obey this general principle: indeed the vector field a we consider in (12) satisfies locally (p,q)-growth conditions where $p := \min_{Q_{4R}} p(x)$ and $q := \max_{Q_{4R}} p(x)$. Since $p(x)$ is a continuous function and the results in Theorems 1, 2 and 3 are local in nature, taking R small enough we may take ε small at will. We shall give an asymptotic estimate on R in Remark 5 below. □

Remark 2. *On the sharpness of* (6). Assumption (6) is essentially optimal. Indeed the occurrence of

$$\lim_{R\to 0} \omega(R) \log\left(\frac{1}{R}\right) = \infty \tag{17}$$

rules out the possibility to prove that $|Du|^{p(x)} \in L^s$ for any $s > 1$, even in the case $F \equiv 0$; this fact can be inferred from the counterexamples in [36], [27], [18]. On the other hand, just supposing that

$$\limsup_{R\to 0} \omega(R) \log\left(\frac{1}{R}\right) \leqq M < \infty \tag{18}$$

leads to establishing that there exists a $\tilde{q} \equiv \tilde{q}(M) > 1$ such that Theorems 1 and 2 hold whenever $q \leqq \tilde{q}$; this is essentially the content of Theorem 5 below: a purposeful inspection of the proof reveals that everything works just assuming (18) instead of (6), and that the use of (6) may be clarified as follows: for any $q > 1$ there exists $\delta \equiv \delta(q) > 0$ such that if

$$\limsup_{R \to 0} \omega(R) \log\left(\frac{1}{R}\right) \leqq \delta \tag{19}$$

then Theorems 1 and 2 hold for the chosen q. This fact can be deduced from the choice of the quantities made in (85). More precisely, once n, γ_1, γ_2, ν, L and also the norms $\| |Du(\cdot)|^{p(\cdot)} \|_{L^1(\Omega)}$, $\| |F(\cdot)|^{p(\cdot)} \|_{L^q(\Omega)}$, that is *the data* of the problem, are fixed, the quantity δ depends on q. As a consequence, the possibility of getting L^q estimates essentially depends on the smallness assumption in (19). Clearly (6) ensures that (19) is satisfied for any choice of $q > 1$. □

3. Preliminary material

In this section we are going to collect a list of preliminary results for later use. Let us start from a restatement of the classical Calderón and Zygmund covering argument; at the same time we shall take the opportunity to add more notation about cubes.

Calderón and Zygmund coverings. Let $Q_0 \subset \mathbb{R}^n$ be a cube; we shall denote with $\mathscr{D}(Q_0)$ the class of all dyadic cubes obtained from Q_0, that is the class of those cubes, with sides parallel to those of Q_0, that have been obtained by a positive, finite number of dyadic subdivisions of the cube Q_0; therefore in particular $Q_0 \notin \mathscr{D}(Q_0)$. Let us recall a few simple properties of the class $\mathscr{D}(Q_0)$. If $Q_1, Q_2 \in \mathscr{D}(Q_0)$ then either the two cubes are disjoint: $Q_1 \cap Q_2 = \emptyset$, or one of the cubes contains the other: $Q_1 \subseteqq Q_2$ or $Q_2 \subseteqq Q_1$. We shall call Q_p "a" predecessor of Q if Q has been obtained from the cube Q_p through a finite number of subsequent dyadic subdivision; we shall call $\tilde{Q} \in \mathscr{D}(Q_0)$ "the" predecessor of Q if Q has been obtained by *exactly one* dyadic subdivision from the original cube $\tilde{Q}$.

Proposition 1. *Let $Q_0 \subset \mathbb{R}^n$ be a cube. Assume that $X \subset Y \subset Q_0$ are measurable sets satisfying the following conditions*: (i) *there exists $\delta > 0$ such that*

$$|X| < \delta |Q_0|$$

and: (ii) *if $Q \in \mathscr{D}(Q_0)$ then*

$$|X \cap Q| > \delta |Q| \Rightarrow \tilde{Q} \subset Y$$

where $\tilde{Q}$ denotes the predecessor of Q. Then

$$|X| < \delta |Y|.$$

The (simple) proof of the previous lemma is a consequence of a Calderón and Zygmund type covering argument and its proof can be found, for instance, in [8].

Maximal operators. Let $Q_0 \subset \mathbb{R}^n$ be a cube. We shall consider, in the following, the Restricted Maximal Function Operator relative to Q_0. This is defined as

$$M^*_{Q_0}(f)(x) := \sup_{Q \subseteqq Q_0,\, x \in Q} \fint_Q |f(y)|\,dy,$$

whenever $f \in L^1(Q_0)$, where Q denotes any cube contained in Q_0, not necessarily with the same centre, as long as it contains the point x. In the same way, if $s > 1$ we define

$$M^*_{s,Q_0}(f)(x) := \sup_{Q \subseteqq Q_0,\, x \in Q} \left(\fint_Q |f(y)|^s\,dy \right)^{1/s}$$

whenever $f \in L^s(Q_0)$. We recall the following weak type $(1,1)$ estimate for $M^*_{Q_0}$:

$$|\{x \in Q_0 : |M^*_{Q_0}(f)(x)| \geqq \lambda\}| \leqq \frac{c_W}{\lambda} \int_{Q_0} |f(y)|\,dy \quad \forall \lambda > 0, \tag{20}$$

which is valid for any $f \in L^1(Q_0)$; the constant c_W depends only on n; for this and related issues we refer to [32]. A standard consequence of the previous inequality is then

$$\int_{Q_0} |M^*_{Q_0}(f)(y)|^q\,dy \leqq \frac{c(n,q)}{q-1} \int_{Q_0} |f(y)|^q\,dy, \quad q > 1. \tag{21}$$

The similar estimate for the M^*_{s,Q_0} operator is

$$\int_{Q_0} |M^*_{s,Q_0}(f)(y)|^q\,dy \leqq \frac{c(n)q^2}{s(q-s)} \int_{Q_0} |f(y)|^q\,dy, \quad q > s, \tag{22}$$

which can be deduced from (21), compare [20], Section 7.

The spaces $L\log^\beta L$. The Orlicz space $L\log^\beta L(\Omega;\mathbb{R}^n)$ is defined via

$$L\log^\beta L(\Omega;\mathbb{R}^n) := \left\{ f \in L^1(\Omega;\mathbb{R}^n) : \int_\Omega |f| \log^\beta(e + |f|)\,dx < \infty \right\}$$

and it becomes a Banach space with the Luxemburg norm

$$\|f\|_{L\log^\beta L(\Omega)} := \inf\left\{ \lambda > 0 : \fint_\Omega \left|\frac{f}{\lambda}\right| \log^\beta\left(e + \left|\frac{f}{\lambda}\right|\right) dx \leqq 1 \right\}.$$

This space embeds in any $L^p(\Omega;\mathbb{R}^n)$, for $p > 1$; more precisely, for any $p > 1$ the following inequality takes place:

$$\|f\|_{L\log^\beta L(\Omega)} \leqq c\left(\fint_\Omega |f|^p\,dx \right)^{\frac{1}{p}} \quad \forall f \in L\log^\beta L(\Omega;\mathbb{R}^n) \tag{23}$$

where the constant c only depends on p, and blows up when $p \to 1$ (see (30) below). Here we want to recall a fact, basically due to T. Iwaniec [20], [22], [5]; let us put

$$[f]_{L\log^\beta L(\Omega)} := \fint_\Omega |f| \log^\beta\left(e + \frac{|f|}{\|f\|_1} \right) dx \tag{24}$$

where, here and in the following, we adopt the notation

$$\|f\|_1 := \fint_\Omega |f|\,dx. \tag{25}$$

The quantity $[f]_{L\log^\beta L(\Omega)}$ is comparable to the Luxemburg norm in $L\log^\beta L(\Omega;\mathbb{R}^n)$ in the sense that there exists a constant $c \equiv c(\beta) \geqq 1$, independent of Ω and f, such that

$$c^{-1}\|f\|_{L\log^\beta L(\Omega)} \leqq [f]_{L\log^\beta L(\Omega)} \leqq c\|f\|_{L\log^\beta L(\Omega)} \tag{26}$$

for all $f \in L\log^\beta L(\Omega;\mathbb{R}^n)$. We shall need these inequalities for the range

$$\frac{\gamma_2}{\gamma_2 - 1} \leqq \beta \leqq \frac{\gamma_1}{\gamma_1 - 1}, \tag{27}$$

therefore, since the constant appearing in (26) is continuous with respect to $\beta > 0$ [22], [5], we shall assume that the constant c appearing in (26) only depends on γ_1 and γ_2, and is valid for the full range in (27). Taking into account this fact and combining (23) and (26) we find that

$$\fint_\Omega |f|\log^\beta\left(e + \frac{|f|}{\|f\|_1}\right)dx \leqq c(p,\beta)\left(\fint_\Omega |f|^p\,dx\right)^{\frac{1}{p}} \tag{28}$$

for every $f \in L\log^\beta L(\Omega;\mathbb{R}^n)$ and β as in (27); since the integrals are averaged, the previous constant does not depend on $|\Omega|$. More precisely, from the elementary inequality

$$t\log^\beta t \leqq \left[\frac{\beta}{e(p-1)}\right]^\beta t^p \quad \forall t \geqq 1,\ p > 1 \tag{29}$$

we infer (see [29] for details) the following asymptotic behaviour as $p \searrow 1$:

$$c(p,\beta) \approx \left(\frac{1}{p-1}\right)^\beta. \tag{30}$$

In particular,

$$(e+t)\log^\beta(e+t) \leqq c(\gamma_1,\gamma_2)\sigma^{-\beta}(e+t)^{1+\sigma/4} \quad \forall t \geqq 0 \tag{31}$$

for every β satisfying (27) and every $0 < \sigma < 1$. Finally, let us record another elementary inequality that will be useful later on. The concavity of the logarithm gives

$$\log(e+ab) \leqq \log(e+a) + \log(e+b)$$

whenever a and b are positive real numbers. Therefore

$$\log^\beta(e+ab) \leqq 2^{\frac{\gamma_1}{\gamma_1-1}-1}\left(\log^\beta(e+a) + \log^\beta(e+b)\right), \tag{32}$$

whenever β satisfies the right hand side inequality in (27).

Gehring's lemma restated. We shall later need the following version of Gehring's lemma; the dependence of the constants we state below can be inferred from the various proofs reported in the literature, in particular adapting that from [6], Section 4, where the dependence on the constants is carefully exploited (see also [21]).

Theorem 4. *Let $Q_{4R_0} \subset \mathbb{R}^n$ be a cube and $s, q > 1$; let $f \in L^s(Q_{4R_0}; \mathbb{R}^n)$, $\phi \in L^{sq}(Q_{4R_0}; \mathbb{R}^n)$ be two functions such that*

$$\left(\fint_{Q_{R/2}} |f|^s \, dx\right)^{1/s} \leqq K \fint_{Q_R} |f| \, dx + H\left(\fint_{Q_R} |\phi|^s \, dx\right)^{1/s}$$

for every (not necessarily concentric) cube $Q_R \subseteqq Q_{4R_0}$, where $K, H > 1$. Then the following holds:

$$\left(\fint_{Q_{R/2}} |f|^{s(1+\sigma)} \, dx\right)^{\frac{1}{1+\sigma}} \leqq \frac{c(n,s)}{(s-1)^{\frac{1}{1+\sigma}}} \fint_{Q_R} |f|^s \, dx + \frac{c(n,s)}{(s-1)^{\frac{1}{1+\sigma}}} \left(\frac{H}{K}\right)^s \left(\fint_{Q_R} |\phi|^{s(1+\sigma)} \, dx\right)^{\frac{1}{1+\sigma}}$$

for all $Q_R \subseteqq Q_{4R_0}$, where $\sigma > 0$ is any number such that

$$\sigma \leqq \min\left\{\frac{c(n,s)(s-1)}{K^{sq}}, q-1\right\}, \tag{33}$$

and $c(n,s)$ is a positive constant.

We remark that the previous result falls far from picking the best possible constant in (33)—indeed the presence of $c(n,s)$ makes the constants not explicit; anyway the estimate above is sufficient for our later purposes.

We conclude the section with the following elementary lemma, whose proof can be promptly adapted from Lemma 2.2 in [11].

Lemma 1. *Let $p \in [\gamma_1, \gamma_2]$ and $\mu \in (0,1]$; there exists a constant $c \equiv c(k, \gamma_1, \gamma_2)$ such that if $v, w \in \mathbb{R}^k$ then:*

$$(\mu^2 + |v|^2)^{\frac{p}{2}} \leqq c(\mu^2 + |w|^2)^{\frac{p}{2}} + c(\mu^2 + |v|^2 + |w|^2)^{\frac{p-2}{2}} |v - w|^2.$$

4. Proof of the results

General setting, I. Here we begin the proof by fixing some objects and notations that will apply to the end of the paper. We consider a "large" cube $Q_{4R_0} \subset\subset \Omega$; during the developement of the section we shall make several restrictions on the size of R_0. Using (6) for the second inequality, we shall initially take R_0 small enough in order to have

$$\begin{cases} \omega(8nR_0) \leqq \sqrt{\dfrac{n+1}{n}} - 1, \\ 0 < \omega(R) \log\left(\dfrac{1}{R}\right) \leqq L \quad \forall R \leqq 8nR_0. \end{cases} \tag{34}$$

We start with a preliminary version of Theorem 1, which rests on an application of Gehring's lemma in the spirit of [3], [36]; we need the following exact statement, emphasizing the precise dependence of the constants.

Theorem 5. *Let $u \in W^{1,p(x)}(\Omega)$ be a weak solution to* (12) *under the assumptions* (4), (6), (10), (11) *and let $F \in L^{p(x)q}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ with $q > 1$. There exist constants $c \equiv c(n,\gamma_1,\gamma_2,\nu,L)$ and $c_g \equiv c_g(n,\gamma_1,\gamma_2,\nu,L)$ such that the following is true: assume R_0 satisfies* (34), *let $Q_{4R_0} \subset\subset \Omega$, set*

$$K_0 := \int_{Q_{4R_0}} |Du|^{p(x)}\,dx + 1 \tag{35}$$

and let $\sigma > 0$ be any number such that

$$\sigma \leqq \min\left\{ \frac{c_g}{K_0^{\frac{2q\omega(8nR_0)}{\gamma_1}}}, q-1, 1 \right\} =: \sigma_0. \tag{36}$$

Then for every $Q_R \subseteqq Q_{4R_0}$ it holds

$$\left(\fint_{Q_{R/2}} |Du|^{p(x)(1+\sigma)}\,dx \right)^{\frac{1}{1+\sigma}} \leqq c \fint_{Q_R} |Du|^{p(x)}\,dx + c\left(\fint_{Q_R} |F|^{p(x)(1+\sigma)}\,dx + 1 \right)^{\frac{1}{1+\sigma}}. \tag{37}$$

Proof. If $Q_R \subseteqq Q_{4R_0}$ we set

$$p_1 := \inf_{Q_R} p(x), \quad p_2 := \sup_{Q_R} p(x);$$

then $p_2 - p_1 \leqq \omega(2R\sqrt{n}) \leqq \omega(2nR)$, and by the first inequality in (34) we have

$$\frac{p_2}{p_1} \leqq \sqrt{\frac{n+1}{n}} =: s. \tag{38}$$

Now take a cut-off function $\eta \in C_0^\infty(Q_R)$ such that $0 \leqq \eta \leqq 1$, $\eta \equiv 1$ on $Q_{R/2}$ and $D\eta \leqq 4/R$. We test (13) with $\varphi \equiv \eta^{p_2}\big(u - (u)_R\big)$ and we estimate the various terms using (10), (11) and Young's inequality:

$$\nu \int_{Q_R} \eta^{p_2} |Du|^{p(x)}\,dx \leqq c \int_{Q_R} \eta^{p_2} \langle a(x,Du), Du\rangle + 1\,dx,$$

$$\int_{Q_R} \eta^{p_2-1} \big| \big\langle a(x,Du), D\eta \otimes \big(u-(u)_R\big)\big\rangle \big|\,dx$$

$$\leqq \zeta \int_{Q_R} \eta^{p_2} |Du|^{p(x)}\,dx + c_\zeta \int_{Q_R} \frac{|u-(u)_R|^{p_2}}{R^{p_2}} + 1\,dx,$$

$$\int_{Q_R} \eta^{p_2} |\langle |F|^{p(x)-2}F, Du\rangle|\,dx \leqq \zeta \int_{Q_R} \eta^{p_2}|Du|^{p(x)}\,dx + c_\zeta \int_{Q_R} |F|^{p(x)}\,dx,$$

$$\int_{Q_R} \eta^{p_2-1} \left|\left\langle |F|^{p(x)-2}F, D\eta \otimes \left(u-(u)_R\right)\right\rangle\right| dx$$

$$\leqq c \int_{Q_R} \frac{|u-(u)_R|^{p_2}}{R^{p_2}} + 1\,dx + c\int_{Q_R} |F|^{p(x)}\,dx$$

where $\zeta \in (0,1)$ and $c \equiv c(n,\gamma_1,\gamma_2,v,L)$. Observe that we used the definition of p_2 to deduce

$$\tilde{p} := \frac{p(x)(p_2-1)}{p(x)-1} \geqq p_2 \quad \forall x \in Q_R$$

and to estimate $\eta^{\tilde{p}} \leqq \eta^{p_2}$ in the second inequality. Choosing $\zeta \equiv \zeta(n,\gamma_1,\gamma_2,v,L)$ small enough and connecting the previous estimates we obtain the following Caccioppoli type inequality:

$$\int_{Q_{R/2}} |Du|^{p(x)}\,dx \leqq c \int_{Q_R} \frac{|u-(u)_R|^{p_2}}{R^{p_2}} + 1\,dx + c\int_{Q_R} |F|^{p(x)}\,dx.$$

Averaging and eventually applying Poincaré's inequality by virtue of (38) yields

$$\begin{aligned}
\fint_{Q_{R/2}} |Du|^{p(x)}\,dx &\leqq c\Big(\fint_{Q_R} |Du|^{\frac{p_1}{s}}\,dx\Big)^{\frac{sp_2}{p_1}} + c\fint_{Q_R} |F|^{p(x)} + 1\,dx \\
&\leqq cR^{-2n\omega(2nR)}\Big(\int_{Q_R} |Du|^{\frac{p(x)}{s}}\,dx\Big)^{\frac{s\omega(2nR)}{p_1}} \\
&\quad \times \Big(\fint_{Q_R} |Du|^{\frac{p(x)}{s}}\,dx\Big)^{s} + c\fint_{Q_R} |F|^{p(x)} + 1\,dx \\
&\leqq cK_0^{\frac{2\omega(8nR_0)}{\gamma_1}}\Big(\fint_{Q_R} |Du|^{\frac{p(x)}{s}}\,dx\Big)^{s} + c\fint_{Q_R} |F|^{p(x)} + 1\,dx,
\end{aligned}$$

where $c \equiv c(n,\gamma_1,\gamma_2,v,L)$ and K_0 is as in (35), and in particular $K_0 \geqq 1$ so we could increase its exponent; we also used the fact that $R^{-\omega(2nR)} \leqq c(n,L)$ as $0 < R \leqq 8nR_0$, by (34). The assertion now follows via Theorem 5 applied with the choice $f \equiv |Du|^{p(x)/s}$ and $\phi \equiv (|F|^{p(x)}+1)^{1/s}$, keeping into account (33). □

Remark 3. *A milder assumption.* We explicitly remark that we applied Gehring's lemma only with the exponent $s \equiv s(n)$ described in (38); in particular the constant c_g above does not in any way depend on R_0, K_0, σ. Moreover, to avoid adding another constant to our already overburdened list, we stated Theorem 5 under the assumption (6); indeed, the result holds as soon as we assume the weaker (18) instead, but then the general-purpose letter L in the second inequality of (34) should be replaced by, say, $M+1$, and the constant c (but not c_g) would depend also on M. □

General setting, II. We remark that since $K_0 \geqq 1$ we have for every $K \geqq K_0$

(39) $$\sigma_0 \geqq \min\{1, q-1, c_g\} K^{-\frac{2q\omega(8nR_0)}{\gamma_1}}.$$

Now let us set

$$K_M := \int_\Omega |Du|^{p(x)} + |F|^{p(x)q} + 2\,dx + 1$$

(this will be larger than all the different versions of K we have met or will meet) and

$$\sigma_m := \min\left\{\frac{c_g}{K_M^{\frac{2q(\gamma_2-\gamma_1)}{\gamma_1}}}, \frac{q-1}{2}, 1\right\} > 0, \quad \sigma_M := c_g + q.$$

Clearly, with $K \leqq K_M$, we will always have

$$\sigma_m \leqq \sigma_0 \leqq \sigma_M. \tag{40}$$

Now we are going to bound the maximal size of a quantity, $\sigma > 0$, that we shall later use as a higher integrability exponent. We shall pick σ of the form

$$\sigma := \tilde{\sigma}\sigma_0, \quad 0 < \tilde{\sigma} < \min\{\gamma_1 - 1, 1/2\}, \tag{41}$$

where σ_0 is the one appearing in (36). In particular by (39) for all β satisfying (27) and all $K \geqq K_0$

$$\sigma^{-\beta} \leqq c\tilde{\sigma}^{-\beta} K^{\beta\frac{2q\omega(8nR_0)}{\gamma_1}} \leqq c(n, \gamma_1, \gamma_2, \nu, L, q)\tilde{\sigma}^{-\beta} K^{\frac{2q\omega(8nR_0)}{\gamma_1 - 1}}. \tag{42}$$

We also remark that by (36) and (41)

$$\sigma < \frac{q-1}{2}. \tag{43}$$

Before proceeding we describe the plot of what will follow, introducing some characters: we feel that this is necessary at this time, because we will proceed with an estimate containing several quantities labeled as "to be determined later," and we will make restrictions on some of them based on the size of some others, and only at the very end the values will be actually determined in order to make everything work. The suspect might arise that in all this reciprocal influence some bug could be creeping, so we show our cards in advance: the proof should be read backwards, but of course it would be totally unreadable should the estimates be derived from the end back. We will take the number ε in the statement of Theorem 2, then, see (98), we will determine the value of $\tilde{\sigma}$ depending on ε (and the data of the problem, such as n, γ_1 and so on), small enough as to satisfy a condition depending on γ_1, see (98). Therefore the quantities σ_m, σ_M and especially $\tilde{\sigma}$ should not be regarded as unknown or to be determined. In the next lemma we will determine, see (77), a quantity A depending only on the data of the problem and later, see (83), we will meet a quantity δ_1 which only depends on the data and K_M; this δ_1 will in turn determine a radius R_1, see (79), which we use to further bound R_0; from R_0, thus bounded, we will deduce the values of K and of σ_0, which will complete the determination of σ; finally from all the above we will deduce, see (84), the value of a quantity that will be called δ, and which will provide us a value for $\tilde{\varepsilon}$, see (81). We are now ready to proceed with the proof.

With the size of σ initially bounded by (41), let us come back to the "large" cube Q_{4R_0}, making further restrictions on the size of R_0, in addition to those already considered in (34). We shall require that

$$\max\left\{2q\omega(8nR_0), \frac{2q\omega(8nR_0)}{\gamma_1 - 1}\right\} \leqq \frac{\tilde{\sigma}\sigma_m}{4}. \tag{44}$$

Note that such restriction leaves R_0 only depending on n, γ_1, γ_2, ν, L, q, the norms $\| |Du(\cdot)|^{p(\cdot)}\|_{L^1(\Omega)}$, $\| |F(\cdot)|^{p(\cdot)}\|_{L^q(\Omega)}$ and the smallness parameter $\tilde{\sigma}$. From (41), (44) and the definition of σ_m it immediately follows that

$$\omega(8nR_0) \leqq \max\left\{2q\omega(8nR_0), \frac{2q\omega(8nR_0)}{\gamma_1 - 1}\right\} \leqq \frac{\tilde{\sigma}\sigma_m}{4} \leqq \frac{\tilde{\sigma}\sigma_0}{4} = \frac{\sigma}{4}. \tag{45}$$

We finally remark that since $\sigma < (q-1)/2$ then $F(x) \in L^{p(x)(1+\sigma)}(Q_{4R_0}; \mathbb{R}^n)$. As a consequence of the previous choices we can apply Theorem 5 in order to deduce that

$$\int_{Q_{2R_0}} |Du|^{p(x)(1+\sigma)}\, dx < \infty, \tag{46}$$

a higher integrability property that we shall use several times in the following. Moreover, we shall always use Theorem 5 under the previous restrictions on R_0 and σ; further restrictions will be done on the size of R_0, starting right now: the complexity of the statement reflects the interplay of constants which we unveiled above.

Lemma 2. *Let $u \in W^{1,p(x)}(\Omega)$ be a weak solution to* (12) *under the assumptions* (4), (6), (8), (9), *and let $\lambda \geqq 1$ and $0 < \tilde{\sigma} < 1$ as in* (41). *There exists a constant $A \equiv A(n, \gamma_1, \gamma_2, \nu, L) \geqq 2$, independent of λ, $\tilde{\sigma}$, u, a, F, such that for every $\delta_1 > 0$ there exists*

$$R_1 \equiv R_1(n, \gamma_1, \gamma_2, \nu, L, q, \tilde{\sigma}, \delta_1) > 0$$

such that: if $R_0 \leqq R_1$ satisfies (34), (44) *and K_0, σ_0 are as in* (35), (36), *setting $\sigma = \tilde{\sigma}\sigma_0$ and*

$$K := \int_{Q_{4R_0}} |Du|^{p(x)} + |F|^{p(x)(1+\sigma)}\, dx + 1 \tag{47}$$

then for every $\delta \geqq \delta_1$ there exists $\tilde{\varepsilon} > 0$, independent of λ, such that the following holds:

If $Q \in \mathscr{D}(Q_{R_0})$ satisfies

$$\begin{aligned} \operatorname{meas}\big(Q \cap \{x \in Q_{R_0} : {}& M^*(|Du(\cdot)|^{p(\cdot)})(x) > AK^\sigma\lambda, \\ & M^*_{1+\sigma}(|F(\cdot)|^{p(\cdot)} + 1)(x) < \tilde{\varepsilon}\lambda\}\big) > \delta|Q| \end{aligned} \tag{48}$$

then its predecessor $\tilde{Q}$ satisfies

$$\tilde{Q} \subseteq \{x \in Q_{R_0} : M^*(|Du(\cdot)|^{p(\cdot)})(x) > \lambda\}, \tag{49}$$

where

$$M^* \equiv M^*_{Q_{4R_0}}, \quad M^*_{1+\sigma} \equiv M^*_{1+\sigma, Q_{4R_0}}$$

denote the restricted Maximal Function Operators relative to Q_{4R_0}.

Proof. *Step* 1: *beginning.* We warn that we increased in places the values of some constants, e.g. replacing $\sqrt{n}$ by n, for clearness of reading (and as long as it does not invalidate the estimates): we could have been more accurate, but the result (Theorems 1 and 2) would still be the same, only the exponents we would have gotten have now a much better look, see Remark 6. We also repeatedly use the fact that for $a, b \geqq 0$

$$\frac{1}{c}(a^\alpha + b^\alpha) \leqq (a+b)^\alpha \leqq c(a^\alpha + b^\alpha)$$

with $c = 1 + 2^{\alpha-1}$.

The proof goes by contradiction; the constants A, $\tilde{\varepsilon}$ and R_1 will be chosen towards the end, see (77), (79), (81) and Remark 4 below. The only restrictions on R_0 at this stage are (34) and (44). Suppose (49) is not satisfied although (48) holds; in this case there exists $x_0 \in \tilde{Q}$ such that

$$\fint_C |Du|^{p(x)}\,dx \leqq \lambda \tag{50}$$

for all cubes $C \subseteqq Q_{4R_0}$ such that $x_0 \in C$. Let us set $S := 2\tilde{Q}$; observe that Q was obtained from Q_{R_0} by at least one subdivision, thus $\tilde{Q} \subseteqq Q_{R_0}$ and $S \subseteqq Q_{2R_0}$: as a consequence

$$s := \operatorname{diam}(2S) \leqq 8nR_0,$$

a fact that we shall use in the following. We observe that (50) gives

$$\fint_{2S} |Du|^{p(x)}\,dx \leqq \lambda \tag{51}$$

because $2S \subseteqq Q_{4R_0}$ and $x_0 \in 2S$. Moreover, since (48) is in force we have that

$$\left|\left\{x \in Q : M^*_{1+\sigma}\big(|F(\cdot)|^{p(\cdot)} + 1\big)(x) < \tilde{\varepsilon}\lambda\right\}\right| > 0 \tag{52}$$

and therefore the last set is not empty, thus by the definition of $M^*_{1+\sigma}$ it follows

$$\left(\fint_S (|F|^{p(x)} + 1)^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}} < \tilde{\varepsilon}\lambda, \quad \left(\fint_{2S} (|F|^{p(x)} + 1)^{1+\sigma}\,dx\right)^{\frac{1}{1+\sigma}} < \tilde{\varepsilon}\lambda. \tag{53}$$

Let us derive some useful preparatory estimates; let

$$p_1 := \min_{\overline{2S}} p(x), \quad p_2 := p(x_M) = \max_{\overline{2S}} p(x), \quad x_M \in \overline{2S}; \tag{54}$$

observe that the numbers p_1 and p_2 depend on the selected cube Q and vary when Q varies in $\mathscr{D}(Q_{R_0})$. Since $2S \equiv 4\tilde{Q} \subseteqq Q_{4R_0}$ we get

$$
\begin{aligned}
p_2 &= (p_2 - p_1) + p_1 \\
&\leqq \omega(s) + p_1 \\
&\leqq p_1\big(1 + \omega(s)\big) \\
&\leqq p(x)\big(1 + \omega(s)\big) \qquad (55) \\
&\leqq p(x)\big(1 + \omega(s) + \sigma/4\big) \\
&\leqq p(x)(1 + \sigma) \quad \forall x \in 2S, \qquad (56)
\end{aligned}
$$

where we used (45) in the last estimate. Also, since (41) implies $\sigma \leqq p_1 - 1$,

$$
\begin{aligned}
p_2(1 + \sigma/4) &\leqq \big(p_1 + \omega(s)\big)(1 + \sigma/4) \\
&\leqq p_1\big(1 + \sigma/4 + \omega(s)\big) \qquad (57) \\
&\leqq p(x)\big(1 + \sigma/4 + \omega(s)\big) \\
&\leqq p(x)(1 + \sigma). \qquad (58)
\end{aligned}
$$

Now, since $\omega(s) \leqq \sigma/4$ by (45), we can use Theorem 5 and formula (37) as follows:

$$
\begin{aligned}
(59)\qquad \fint_S |Du|^{p_2}\,dx &\leqq \fint_S |Du|^{p_2} + 1\,dx \\
&\overset{(55)}{\leqq} 2 \fint_S |Du|^{p(x)(1+\omega(s))} + 1\,dx \\
&\overset{(37),(45)}{\leqq} c\Big(\fint_{2S} |Du|^{p(x)} + 1\,dx\Big)^{1+\omega(s)} + c\fint_{2S} \big(|F|^{p(x)} + 1\big)^{1+\omega(s)}\,dx \\
&\leqq c\Big(\int_{2S} |Du|^{p(x)} + 1\,dx\Big)^{\omega(s)} s^{-n\omega(s)} \times \fint_{2S} |Du|^{p(x)} + 1\,dx \\
&\quad + c\Big(\int_{2S} |F|^{p(x)(1+\omega(s))} + 1\,dx\Big)^{\frac{\omega(s)}{1+\omega(s)}} s^{\frac{-n\omega(s)}{1+\omega(s)}} \\
&\quad \times \Big(\fint_{2S} |F|^{p(x)(1+\omega(s))} + 1\,dx\Big)^{\frac{1}{1+\omega(s)}} \\
&\overset{(45)}{\leqq} cK^{\frac{\sigma}{4}} \fint_{2S} |Du|^{p(x)} + 1\,dx + cK^{\frac{\sigma}{4}}\Big(\fint_{2S} |F|^{p(x)(1+\sigma)} + 1\,dx\Big)^{\frac{1}{1+\sigma}} \\
&\overset{(51),(53)}{\leqq} c(n, \gamma_1, \gamma_2, \nu, L)K^{\frac{\sigma}{4}}\lambda.
\end{aligned}
$$

We crucially used the fact that $s^{-n\omega(s)}$ stays bounded as $0 < s < 8nR_0$, by (34); in the last estimates we used Hölder's inequality, since $\omega(s) \leqq \sigma/4$, and the facts that $\lambda \geqq 1$ and $\tilde{\varepsilon} < 1$. A rewriting of the previous estimates with a different aim gives

$$
(60)\qquad \int_S |Du|^{p_2}\,dx \leqq c\Big(\int_{2S} |Du|^{p(x)} + 1\,dx\Big)^{\omega(s)} s^{-n\omega(s)} \int_{2S} |Du|^{p(x)} + 1\,dx
$$
$$
+ c\int_{2S} |F|^{p(x)(1+\omega(s))} + 1\,dx
$$
$$
\leqq cK^{\frac{\sigma}{4}} \int_{2S} |Du|^{p(x)} + |F|^{p(x)(1+\sigma)} + 1\,dx
$$
$$
\leqq c(n,\gamma_1,\gamma_2,\nu,L)K^{1+\frac{\sigma}{4}}.
$$

Step 2: *a comparison function.* By (59) it follows that $u \in W^{1,p_2}(S)$, therefore we are able to define $w \in \big(u + W_0^{1,p_2}(S)\big) \cap W^{1,p_2}(S)$ as the unique solution to the following Dirichlet problem:

$$
(61)\qquad \begin{cases} \int_S a(x_M, Dw)D\varphi\,dx = 0 & \forall\varphi \in W_0^{1,p_2}(S),\\ w = u & \text{on } \partial S, \end{cases}
$$

where the point $x_M \in \overline{2S}$ was found according to (54); it is harmless that it may happen that $x_M \notin S$. The vector field $z \mapsto a(x_M, z)$ satisfies the following growth and coercivity conditions (with respect to the z variable)

$$
(62)\qquad c^*(\nu)(\mu^2 + |z_1|^2 + |z_2|^2)^{\frac{p_2-2}{2}}|z_2 - z_1|^2 \leqq \langle a(x_M, z_2) - a(x_M, z_1), z_2 - z_1\rangle
$$

and

$$
(63)\qquad |a(x_M, z)| \leqq L(1 + |z|^2)^{\frac{p_2-1}{2}}, \quad \nu|z|^{p_2} \leqq \langle a(x_M, z), z\rangle + c(L)
$$

for every $z, z_1, z_2 \in \mathbb{R}^{nN}$ and $c^* \equiv c^*(n,\gamma_1,\gamma_2,\nu) > 0$. The first inequality is a standard consequence of (8) and of Lemmas 2.1 and 2.2 from [1], which work for any $p \geqq 1$; a bit of care is required here, since $z \mapsto a(x_M, z)$ is only $C^1(\mathbb{R}^n\backslash\{0\})$. The inequalities in (63) trivially follow from (10) and (11), respectively. Since $u \in W^{1,p_2}(S)$, by (62), (63) the existence of w follows from the standard theory of Leray-Lions operators; uniqueness follows from strong monotonicity, (62). As a consequence of the standard regularity theory for degenerate elliptic equations of the type in (61), recalling that $S = 2\tilde{Q}$, the following estimate holds true:

$$
(64)\qquad \sup_{\frac{3}{2}\tilde{Q}} (\mu^2 + |Dw|^2)^{\frac{p_2}{2}} \leqq c(n,\gamma_1,\gamma_2,\nu,L) \fint_S (\mu^2 + |Dw|^2)^{\frac{p_2}{2}}\,dx.
$$

The validity of the previous estimate, and in particular the fact that the constant c can be chosen independent of p_2, specifying its dependence only on the bounds γ_1, γ_2, can be deduced e.g. looking at [25]; although cubes are replaced by balls in [25], the previous inequality can be easily proved by using cubes instead of balls everywhere in the proofs. Also observe that such estimate is usually obtained for the case when the supremum at the left hand side is computed over a smaller subset, namely $\frac{1}{2}S \equiv \tilde{Q}$; by an easy covering argument one sees that the supremum in estimate (64) may be computed over any subset of the type γS, for any $\gamma < 1$, the constant c appearing on the right hand side eventually depending on γ and blowing up when $\gamma \to 1$.

Let us test (61) with $\varphi := u - w$; by (62) and (63) we get

$$\begin{aligned} \nu \int_S |Dw|^{p_2}\,dx &\leqq c \int_S \langle a(x_M, Dw), Dw\rangle + 1\,dx \\ &= c \int_S \langle a(x_M, Dw), Du\rangle + 1\,dx \\ &\leqq c \int_S (1 + |Dw|)^{p_2 - 1}|Du| + 1\,dx \end{aligned}$$

and observing that $\gamma_1 \leqq p_2 \leqq \gamma_2$ and applying Young's inequality we conclude with

$$(65) \qquad \fint_S |Dw|^{p_2}\,dx \leqq c(n, \gamma_1, \gamma_2, \nu, L) \fint_S |Du|^{p_2} + 1\,dx.$$

Combining this last estimate with (64) and then with (59), and using the fact that $\lambda \geqq 1$, we also infer

$$(66) \qquad \begin{aligned} \sup_{\frac{3}{2}\tilde{Q}} (\mu^2 + |Dw|^2)^{\frac{p_2}{2}} &\leqq c(n, \gamma_1, \gamma_2, \nu, L) \fint_S |Du|^{p_2} + 1\,dx \\ &\leqq c_1 K^{\sigma/4}\lambda \leqq c_1 K^{\sigma}\lambda, \end{aligned}$$

where $c_1 \equiv c_1(n, \gamma_1, \gamma_2, \nu, L)$; this constant c_1 will play a central role in the determination of the constant A, see (77).

Step 3: *a comparison estimate.* Our next aim is to derive an estimate for the quantity

$$I := \int_S (\mu^2 + |Du|^2 + |Dw|^2)^{\frac{p_2 - 2}{2}} |Du - Dw|^2\,dx.$$

Using (62) and the fact that both u and w are weak solutions, of (12) and (61) respectively, while $u = w$ on ∂S, we get

$$(67) \qquad \begin{aligned} c^* I &\leqq \int_S \langle a(x_M, Du) - a(x_M, Dw), Du - Dw\rangle\,dx \\ &= \int_S \langle a(x_M, Du), Du - Dw\rangle\,dx \\ &= \int_S \langle a(x_M, Du) - a(x, Du), Du - Dw\rangle\,dx \\ &\quad + \int_S \langle |F|^{p(x)-2}F, Du - Dw\rangle\,dx =: II + III, \end{aligned}$$

where c^* is the constant appearing in (62). We will estimate the quantities II and III; since we use (9), where we find the logarithm of an elaborate quantity which may be less than 1, we employ in this estimate (and in a single line further on) the notation $|\log|^{\alpha} x$ instead of $|\log x|^{\alpha}$. Using (9) and Hölder's inequality we have, still with $s = \mathrm{diam}(2S)$,

$$(68)\qquad II \leqq c\omega(s)\int_S (\mu + |Du|)^{p_2-1}|\log|(\mu + |Du|)|Du - Dw|\,dx$$
$$\leqq c\omega(s)\Big(\int_S (\mu + |Du|)^{p_2}|\log|^{\frac{p_2}{p_2-1}}(\mu + |Du|)\,dx\Big)^{\frac{p_2-1}{p_2}}$$
$$\times \Big(\int_S |Du - Dw|^{p_2}\,dx\Big)^{\frac{1}{p_2}}$$
$$\overset{(65)}{\leqq} c\omega(s)\Big(\int_S (\mu + |Du|)^{p_2}|\log|^{\frac{p_2}{p_2-1}}(\mu + |Du|)\,dx\Big)^{\frac{p_2-1}{p_2}}$$
$$\times \Big(\int_S |Du|^{p_2} + 1\,dx\Big)^{\frac{1}{p_2}}$$

where $c \equiv c(n, \gamma_1, \gamma_2, v, L)$. We remark that by (46) and (58) it also follows that

$$(\mu + |Du|)^{p_2}|\log|^{\frac{p_2}{p_2-1}}(\mu + |Du|) \in L^1(S)$$

and therefore the last quantity in (68) is finite; this fact will be exploited in a few lines. Before continuing with estimate (68) let us point out a preliminary inequality; we have

$$(69)\qquad \Big(\fint_{2S} |F|^{p(x)(1+\sigma/4+\omega(s))}\,dx\Big)^{\frac{1}{1+\sigma/4}}$$
$$\leqq c\Big(\int_{2S} |F|^{p(x)(1+\sigma/4+\omega(s))}\,dx\Big)^{\frac{\omega(s)}{(1+\sigma/4)(1+\sigma/4+\omega(s))}}$$
$$\times\, s^{\frac{-n\omega(s)}{(1+\sigma/4)(1+\sigma/4+\omega(s))}}\Big(\fint_{2S} |F|^{p(x)(1+\sigma/4+\omega(s))}\,dx\Big)^{\frac{1}{1+\sigma/4+\omega(s)}}$$
$$\leqq cK^{\frac{\sigma}{4}}\Big(\fint_{2S} |F|^{p(x)(1+\sigma)}\,dx\Big)^{\frac{1}{1+\sigma}}$$

where $c \equiv c(n, L)$. Again, we used that $s^{-n\omega(s)}$ stays bounded as $0 < s < 8nR_0$ by (34), and we used that $\omega(s) \leqq \sigma/4$ to apply Hölder's inequality in the last estimate. Now we set

$$(70)\qquad \beta := \frac{p_2}{p_2 - 1} \in \left[\frac{\gamma_2}{\gamma_2 - 1}, \frac{\gamma_1}{\gamma_1 - 1}\right].$$

Observe that β satisfies (27) and therefore (32) is available. In the same range for β and with $\gamma_1 \leqq p_2 \leqq \gamma_2$ we also have

$$t^{p_2}|\log|^{\beta} t \leqq c(\gamma_1, \gamma_2), \quad 0 < t \leqq e.$$

We recall that by our definition (25)

$$\|\,|Du|^{p_2}\|_1 := \fint_S |Du|^{p_2}\,dx.$$

Still with $s = \operatorname{diam}(2S)$, we now estimate the term

$$\int_S (\mu + |Du|)^{p_2} \log^\beta(\mu + |Du|)\, dx$$

appearing in (68); we may assume that $|Du(x)| \geqq e$ on a set of positive measure in S, otherwise the term is estimated simply by $c|S| \leqq cs^n$. Then, we have:

$$\begin{aligned}
&\int_S (\mu + |Du|)^{p_2} \log^\beta(\mu + |Du|)\, dx \\
&\leqq \int_{\{x:|Du| \geqq e\}} (\mu + |Du|)^{p_2} \log^\beta(\mu + |Du|^{p_2})\, dx + cs^n \\
&\leqq cs^n \fint_S |Du|^{p_2} \log^\beta(e + |Du|^{p_2})\, dx + cs^n \\
&\overset{(32)}{\leqq} cs^n \fint_S |Du|^{p_2} \log^\beta\left(e + \frac{|Du|^{p_2}}{\|\,|Du|^{p_2}\|_1}\right) dx \\
&\quad + cs^n \fint_S |Du|^{p_2} \log^\beta(e + \|\,|Du|^{p_2}\|_1)\, dx + cs^n \\
&\overset{(28),(30)}{\leqq} c\sigma^{-\beta} s^n \left(\fint_S |Du|^{p_2(1+\sigma/4)}\, dx\right)^{\frac{1}{1+\sigma/4}} \\
&\quad + c \log^\beta\left(s^{-n} e + s^{-n} \int_S |Du|^{p_2}\, dx\right) \int_S |Du|^{p_2}\, dx + cs^n \\
&\overset{(57)}{\leqq} c\sigma^{-\beta} s^n \left(1 + \fint_S |Du|^{p(x)(1+\sigma/4+\omega(s))}\, dx\right)^{\frac{1}{1+\sigma/4}} \\
&\quad + c \log^\beta\left(\frac{1}{s}\right) \int_S |Du|^{p_2}\, dx \\
&\quad + c\left(e + \int_S |Du|^{p_2}\, dx\right) \log^\beta\left(e + \int_S |Du|^{p_2}\, dx\right) + cs^n \\
&\overset{(31),(37),(42),(56)}{\leqq} c(q)\tilde{\sigma}^{-\beta} K^{\frac{2q\omega(8nR_0)}{\gamma_1 - 1}} s^n \left(\fint_{2S} |Du|^{p(x)}\, dx\right)^{\frac{(1+\sigma/4+\omega(s))}{1+\sigma/4}} \\
&\quad + c(q)\tilde{\sigma}^{-\beta} K^{\frac{2q\omega(8nR_0)}{\gamma_1 - 1}} s^n \left(\fint_{2S} |F|^{p(x)(1+\sigma/4+\omega(s))}\, dx\right)^{\frac{1}{1+\sigma/4}} \\
&\quad + c \log^\beta\left(\frac{1}{s}\right) s^n \fint_S |Du|^{p_2}\, dx \\
&\quad + c(q)\tilde{\sigma}^{-\beta} K^{\frac{2q\omega(8nR_0)}{\gamma_1 - 1}} s^n \left(1 + \int_S |Du|^{p_2}\, dx\right)^{\frac{\sigma}{4}} \fint_S |Du|^{p_2}\, dx \\
&\quad + c(q)\tilde{\sigma}^{-\beta} K^{\frac{2q\omega(8nR_0)}{\gamma_1 - 1}} s^n \\
&\overset{(45),(69)}{\leqq} c\tilde{\sigma}^{-\beta} K^{\frac{\sigma}{4}} s^{-n\omega(s)} \left(\int_{2S} |Du|^{p(x)}\, dx\right)^{\frac{\omega(s)}{1+\sigma/4}} \times \int_{2S} |Du|^{p(x)}\, dx \\
&\quad + c\tilde{\sigma}^{-\beta} K^{\frac{\sigma}{2}} s^n \left(\fint_{2S} |F|^{p(x)(1+\sigma)}\, dx\right)^{\frac{1}{1+\sigma}}
\end{aligned}$$

$$
\begin{aligned}
&+ c\log^{\beta}\left(\frac{1}{s}\right)s^n \fint_S |Du|^{p_2}\,dx \\
&+ c\tilde{\sigma}^{-\beta}K^{\frac{\sigma}{4}}s^n\left(1+\int_S |Du|^{p_2}\,dx\right)^{\frac{\sigma}{4}}\fint_S |Du|^{p_2}\,dx \\
&+ c\tilde{\sigma}^{-\beta}K^{\frac{\sigma}{4}}s^n \\
&\overset{\text{(see below)}}{\leqq} c\tilde{\sigma}^{-\beta}K^{\frac{\sigma}{2}}s^n\lambda \\
&+ c\log^{\beta}\left(\frac{1}{s}\right)s^nK^{\frac{\sigma}{4}}\lambda + c\tilde{\sigma}^{-\beta}K^{\frac{\sigma}{2}+\frac{\sigma}{4}\left(1+\frac{\sigma}{4}\right)}s^n\lambda + c\tilde{\sigma}^{-\beta}K^{\frac{\sigma}{4}}s^n \\
&\leqq c(n,\gamma_1,\gamma_2,v,L)\log^{\beta}\left(\frac{1}{s}\right)K^{\sigma}s^n\lambda \\
&+ c(n,\gamma_1,\gamma_2,v,L,q)\tilde{\sigma}^{-\beta}K^{\sigma}s^n\lambda.
\end{aligned}
$$

In the second-last and last estimates we used (45), (51), (53), (59) and (60); as before, we used that $s^{-n\omega(s)}$ stays bounded as $0 < s < 8nR_0$ by (34), and also (several times) that $\lambda, K \geqq 1$. Before proceeding, let us point out that the use of the quantity $[f]_{L\log^{\beta}L(S)}$ in (24), and of the inequality (24), is essential to get the right dependence of the constant in the last estimate. Combining the previous inequalities with (59) and (68) we get

$$
\begin{aligned}
II &\leqq c(n,\gamma_1,\gamma_2,v,L)K^{\sigma}\omega(s)\log\left(\frac{1}{s}\right)s^n\lambda \\
&\quad + c(n,\gamma_1,\gamma_2,v,L,q)\tilde{\sigma}^{-1}K^{\sigma}\omega(s)s^n\lambda.
\end{aligned}
\tag{71}
$$

We now estimate III; let us note that since $p_2 \geqq p(x)$ in S, then

$$
\frac{p_2\big(p(x)-1\big)}{p_2-1} \leqq p(x) \quad \forall x \in S.
\tag{72}
$$

Therefore, again by Hölder's inequality we get

$$
\begin{aligned}
III &\leqq \int_S |F|^{p(x)-1}|Du-Dw|\,dx \\
&\leqq cs^n\left(\fint_S |F|^{\frac{p_2(p(x)-1)}{p_2-1}}\,dx\right)^{\frac{p_2-1}{p_2}}\left(\fint_S |Du-Dw|^{p_2}\,dx\right)^{\frac{1}{p_2}} \\
&\overset{(65),(72)}{\leqq} cs^n\left(\fint_S |F|^{p(x)}+1\,dx\right)^{\frac{p_2-1}{p_2}}\left(\fint_S |Du|^{p_2}+1\,dx\right)^{\frac{1}{p_2}} \\
&\leqq cs^n\left(\fint_S (|F|^{p(x)}+1)^{1+\sigma}\,dx\right)^{\frac{p_2-1}{p_2(1+\sigma)}}\left(\fint_S |Du|^{p_2}+1\,dx\right)^{\frac{1}{p_2}} \\
&\overset{(53),(59)}{\leqq} c(n,\gamma_1,\gamma_2,v,L)^n K^{\frac{\sigma}{4p_2}}\tilde{\varepsilon}^{\frac{p_2-1}{p_2}}s^n\lambda.
\end{aligned}
\tag{73}
$$

Combining in sequel (67), (71) and (73) and passing to averages yields

$$
\begin{aligned}
(74)\qquad & \fint_S (\mu^2+|Du|^2+|Dw|^2)^{\frac{p_2-2}{2}}|Du-Dw|^2\,dx \\
&\leqq c_2K^\sigma\omega(s)\log\left(\frac{1}{s}\right)\lambda + c_{22}K^\sigma\tilde{\sigma}^{-1}\omega(s)\lambda + c_2K^{\frac{\sigma}{4p_2}}\tilde{\varepsilon}^{\frac{p_2-1}{p_2}}\lambda \\
&\leqq c_2K^\sigma\omega(s)\log\left(\frac{1}{s}\right)\lambda + c_{22}K^\sigma\tilde{\sigma}^{-1}\omega(s)\lambda + c_2K^\sigma\tilde{\varepsilon}^{\frac{\gamma_1-1}{\gamma_1}}\lambda
\end{aligned}
$$

where $c_2 \equiv c_2(n,\gamma_1,\gamma_2,v,L)$, $c_{22} \equiv c_{22}(n,\gamma_1,\gamma_2,v,L,q)$, and $s = \operatorname{diam}(2S)$. This is the estimate we needed (here is one of the points where we increase something for the sake of readability: as one sees from (73), the K^σ close to $\tilde{\varepsilon}$ in (74) is indeed a $K^{\sigma/4\gamma_1}$).

Step 4: *estimate at the higher level.* We shall use the Restricted Maximal Operator with respect to $\frac{3}{2}\tilde{Q}$, which for cleanliness we shall denote by

$$M^{**} \equiv M^*_{\frac{3}{2}\tilde{Q}}.$$

Instead, we shall denote by $M^* \equiv M^*_{Q_{4R_0}}$ the maximal operator appearing in the statement of the lemma we are proving. Lemma 1 implies that for all $x \in S = 2\tilde{Q}$

$$
\begin{aligned}
(75)\qquad (\mu^2+|Du|^2)^{\frac{p_2}{2}} &\leqq c_3(\mu^2+|Dw|^2)^{\frac{p_2}{2}} \\
&\quad + c_3(\mu^2+|Du|^2+|Dw|^2)^{\frac{p_2-2}{2}}|Du-Dw|^2 \\
&=: c_3G_1(x) + c_3G_2(x),
\end{aligned}
$$

with $c_3 \equiv c_3(n,\gamma_1,\gamma_2)$. From (66) we immediately deduce

$$(76)\qquad M^{**}(G_1)(x) \leqq c_1K^\sigma\lambda \quad \forall x \in \frac{3}{2}\tilde{Q};$$

we remark that $\frac{3}{2}\tilde{Q}$ is a neighborhood of $\bar{Q}$.

Accordingly, let us take

$$(77)\qquad C := 5^{n+3}c_1c_3, \quad A = 2C$$

and without loss of generality assume $c_1, c_3 \geqq 1$; observe that now $A \geqq 2$ is determined, with the dependence upon the constants specified in the statement; in particular, it is independent of $\lambda, K \geqq 1$. With such a choice we estimate as follows:

$$
\begin{aligned}
(78)\qquad & |\{x \in Q : M^{**}(|Du|^{p_2})(x) > CK^{\sigma}\lambda\}| \\
&\overset{(75)}{\leqq} \left|\left\{x \in Q : M^{**}(G_1)(x) + M^{**}(G_2)(x) > \frac{CK^{\sigma}}{c_3}\lambda\right\}\right| \\
&\leqq \left|\left\{x \in Q : M^{**}(G_1)(x) > \frac{CK^{\sigma}}{2c_3}\lambda\right\}\right| \\
&\quad + \left|\left\{x \in Q : M^{**}(G_2)(x) > \frac{CK^{\sigma}}{2c_3}\lambda\right\}\right| \\
&\overset{(76),(77)}{=} \left|\left\{x \in Q : M^{**}(G_2)(x) > \frac{CK^{\sigma}}{2c_3}\lambda\right\}\right| \\
&\overset{(20)}{\leqq} \frac{c_W}{8^n c_1 K^{\sigma}\lambda}\int_S (\mu^2 + |Du|^2 + |Dw|^2)^{\frac{p_2-2}{2}} |Du - Dw|^2 \, dx \\
&\overset{(74)}{\leqq} c_4\omega(s)\log\left(\frac{1}{s}\right)|Q| + c_{44}\tilde{\sigma}^{-1}\omega(s)|Q| + c_4\tilde{\varepsilon}^{\frac{\gamma_1-1}{\gamma_1}}|Q|
\end{aligned}
$$

where, obviously $c_4 := c_W c_2$, $c_{44} := c_{22} c_W$; c_2 and c_{22} are the constants appearing in (74) and therefore $c_4 \equiv c_4(n, \gamma_1, \gamma_2, \nu, L)$, $c_{44} \equiv c_{44}(n, \gamma_1, \gamma_2, \nu, L, q)$.

Now let δ_1 be chosen as in the statement: by (6) we may determine the radius $R_1 \equiv R_1(n, \gamma_1, \gamma_2, \nu, L, q, \tilde{\sigma}, \delta_1) > 0$ small enough in order to have

$$
(79)\qquad c_4\omega(s)\log\left(\frac{1}{s}\right) \leqq \frac{\delta_1}{8}, \quad c_{44}\omega(s) \leqq \frac{\delta_1\tilde{\sigma}}{8}, \quad \text{if } s \leqq 8nR_1;
$$

clearly, if $R_0 \leqq R_1$ satisfies (34), (44) we have

$$
R_0 \equiv R_0\big(n, \gamma_1, \gamma_2, \nu, L, q, \| \, |Du(\cdot)|^{p(\cdot)} \|_{L^1(\Omega)}, \| \, |F(\cdot)|^{p(\cdot)} \|_{L^q(\Omega)}, \tilde{\sigma}, \delta_1\big).
$$

For $R_0 \leqq R_1$ and for every $\delta \geqq \delta_1$ we obviously have

$$
(80)\qquad c_4\omega(s)\log\left(\frac{1}{s}\right) \leqq \frac{\delta}{8}, \quad c_{44}\omega(s) \leqq \frac{\delta\tilde{\sigma}}{8}, \quad \text{if } s \leqq 8nR_0;
$$

we may now choose $\tilde{\varepsilon} \equiv \tilde{\varepsilon}(n, \gamma_1, \gamma_2, \nu, L, \delta) \in (0, 1)$ by

$$
(81)\qquad c_4\tilde{\varepsilon}^{\frac{\gamma_1-1}{\gamma_1}} = \frac{\delta}{8}.
$$

With the previous choices we obtain

$$
(82)\qquad |\{x \in Q_{R_0} : M^{**}(|Du|^{p_2}) > CK^{\sigma}\lambda\}| \leqq \frac{\delta}{2}|Q|;
$$

observe that the choice of both $\tilde{\varepsilon}$ and R_0 is done here giving the dependence upon the constants described in the statement; actually, the dependence of R_0 on the norms

$\| |Du(\cdot)|^{p(\cdot)} \|_{L^1(\Omega)}$, $\| |F(\cdot)|^{p(\cdot)} \|_{L^q(\Omega)}$ only comes from the restrictions made in (44). Also observe that the first inequality in (79) is *the only point* where (6) is needed; otherwise we could have simply assumed the limit in (6) to be finite, as in (18), compare Remarks 2 and 3. Now, recalling (54), observe that since $p_2 \geqq p(x)$ whenever $x \in 2\tilde{Q}$ we have that

$$\fint_Q |Du|^{p(x)}\,dx \leqq \fint_Q |Du|^{p_2}\,dx + 1$$

as soon as $Q \subset \frac{3}{2}\tilde{Q}$ is a cube; as a consequence, if $x \in Q$ then

$$M^{**}\big(|Du(\cdot)|^{p(\cdot)}\big)(x) \leqq M^{**}(|Du|^{p_2} + 1)(x).$$

But λ, K and C are larger than 1, see (77), so $CK^\sigma\lambda \geqq 1$ and we have

$$\begin{aligned} &M^{**}\big(|Du(\cdot)|^{p(\cdot)}\big)(x) > AK^\sigma\lambda \\ &\qquad \Rightarrow M^{**}(|Du|^{p_2})(x) + CK^\sigma\lambda \geqq M^{**}(|Du|^{p_2} + 1)(x) > AK^\sigma\lambda = 2CK^\sigma\lambda, \end{aligned}$$

so from (82) we deduce that

$$\big|\big\{x \in Q : M^{**}\big(|Du(\cdot)|^{p(\cdot)}\big)(x) > AK^\sigma\lambda\big\}\big| \leqq \frac{\delta}{2}|Q|.$$

Now let ℓ be the side length of Q, take any point $x \in Q$ and remark that both x and x_0, the point to which (50) refers, belong to $\tilde{Q}$, a cube with side length 2ℓ. If $C' \subseteqq Q_{4R_0}$ is a cube containing x and with side length ℓ' larger than $\ell/2$, then since $C' \cap \tilde{Q} \neq \emptyset$ there is a cube $C'' \subseteqq Q_{4R_0}$ containing both C' and $\tilde{Q}$ and whose side length ℓ'' satisfies

$$\ell'' \leqq 2\ell + \ell' \leqq 5\ell',$$

so

$$\fint_{C'} |Du|^{p(x)}\,dx \leqq \frac{1}{|C'|}\int_{C''} |Du|^{p(x)}\,dx \overset{(50)}{\leqq} \frac{|C''|}{|C'|}\lambda \leqq 5^n\lambda.$$

On the other hand, if instead $\ell' \leqq \ell/2$ then $C' \subset \frac{3}{2}\tilde{Q}$ and

$$\fint_{C'} |Du|^{p(x)}\,dx \leqq M^{**}\big(|Du(\cdot)|^{p(\cdot)}\big).$$

Therefore

$$M^{*}\big(|Du(\cdot)|^{p(\cdot)}\big)(x) \leqq \max\big\{M^{**}\big(|Du(\cdot)|^{p(\cdot)}\big)(x), 5^n\lambda\big\} \quad \forall x \in Q.$$

Since $CK^\sigma \geqq 5^{n+1}$ by (77), it also follows that

$$\big|\big\{x \in Q : M^{*}\big(|Du(\cdot)|^{p(\cdot)}\big)(x) > AK^\sigma\lambda\big\}\big| \leqq \frac{\delta}{2}|Q|,$$

which contradicts (49). The proof of the lemma is complete. □

Remark 4. *The role of* (49). The possibility to perform the estimates is given by the failure of (49) and by (52), that is $\{x \in Q_{R_0} : M^*_{1+\sigma}(|F(\cdot)|^{p(\cdot)} + 1) < \tilde{\varepsilon}\lambda\} \neq \emptyset$ for some $\tilde{\varepsilon} \in (0,1)$: neither the measure of the set nor the value of $\tilde{\varepsilon}$ have been used in the estimate. □

Proof of Theorems 1 *and* 2. In Lemma 2 we take

$$\delta_1 := \frac{1}{2A^q K_M^{\sigma_M q}} \tag{83}$$

and we determine R_1 according to (79), then we take the greatest number $R_0 \leqq R_1$ which satisfies (34), (44); now that R_0 is set we may define K_0, σ_0 as in (35), (36), we set $\sigma = \tilde{\sigma}\sigma_0$ and we take K as in (47). Now comes the crucial moment: we link the number δ, which we still have to choose, to K, by setting

$$\delta := \frac{1}{2A^q K^{\sigma q}} \tag{84}$$

(remark that such a value is admissible in Lemma 2, as clearly $\delta \geqq \delta_1$), which also forces the value of $\tilde{\varepsilon}$, see (81). We remark that, beside (34), (44), the choice of R_0 and $\tilde{\varepsilon}$ done in (79), (81) means that

$$\omega(R_0)\log\left(\frac{1}{R_0}\right) \leqq \frac{1}{16c_4A^q}\left(\int_\Omega |Du|^{p(x)} + |F|^{p(x)q}\,dx + 1\right)^{-q}, \tag{85}$$

$$\omega(R_0) \leqq \frac{\tilde{\sigma}}{16c_{44}A^q}\left(\int_\Omega |Du|^{p(x)} + |F|^{p(x)q}\,dx + 1\right)^{-q}, \tag{86}$$

$$\tilde{\varepsilon} = \left(\frac{1}{16c_4A^q}\right)^{\frac{\gamma_1}{\gamma_1-1}}\left(\int_{Q_{4R_0}} |Du|^{p(x)} + |F|^{p(x)(1+\sigma)}\,dx + 1\right)^{-\frac{\gamma_1\sigma q}{\gamma_1-1}}. \tag{87}$$

Therefore R_0 and $\tilde{\varepsilon}$ are now fixed; this is the final choice of these three quantities, that, as far as R_0 is concerned, together with the restrictions in (34) and (44) widely discussed above, yields the dependence on the constants announced in the statement of Theorem 2. In the following we shall denote

$$\mu_1(t) := \left|\{x \in Q_{R_0} : M^*(|Du(\cdot)|^{p(\cdot)})(x) > t\}\right|,$$
$$\mu_2(t) := \left|\{x \in Q_{R_0} : M^*_{1+\sigma}(|F(\cdot)|^{p(\cdot)} + 1)(x) > t\}\right|,$$

where all maximal operators are restricted to Q_{4R_0}, and we define

$$\begin{aligned}\lambda_0 &:= \frac{5^{n+2}c_W}{\delta} \fint_{Q_{4R_0}} |Du|^{p(x)}\,dx + 1 \\ &\leqq 5^{n+3}c_W A^q K^{\sigma q} \fint_{Q_{4R_0}} |Du|^{p(x)}\,dx + 1,\end{aligned} \tag{88}$$

$c_W \equiv c_W(n)$ being the constant which appears in (20). With such a choice (84) gives

$$\mu_1(\lambda_0) \leqq \frac{c_W}{\lambda_0} \int_{Q_{4R_0}} |Du|^{p(x)}\,dx < \frac{\delta}{2}|Q_{R_0}|. \tag{89}$$

We set

$$\tilde{A} := AK^{\sigma} \geqq 2. \tag{90}$$

From (89) it follows that

$$\mu_1(\tilde{A}^h\lambda_0) < \frac{\delta}{2}|Q_{R_0}| \quad \forall h \in \mathbb{N}. \tag{91}$$

We will check, by induction, that actually

$$\mu_1(\tilde{A}^{h+1}\lambda_0) \leqq \delta^{h+1}\mu_1(\lambda_0) + \sum_{i=0}^{h} \delta^{h-i}\mu_2(\tilde{A}^i\tilde{\varepsilon}\lambda_0) \tag{92}$$

for every integer $h \geqq 0$. The case $h = 0$ reduces to

$$\mu_1(\tilde{A}\lambda_0) \leqq \delta\mu_1(\lambda_0) + \mu_2(\tilde{\varepsilon}\lambda_0)$$

which is a consequence of Proposition 1 (applied to the sets X, Y of points where the conditions in (48) and (49) hold respectively) and Lemma 2 with the choice $\lambda \equiv \lambda_0 \geqq 1$ and δ as in (84), keeping into account that (89) holds. Moreover, assuming (92) for a certain $n \geqq 0$, we apply again Proposition 1 and Lemma 2 with the choice $\lambda \equiv \tilde{A}^{h+1}\lambda_0$ and δ as in (84), keeping in mind (91), in order to have

$$\begin{aligned}
\mu_1(\tilde{A}^{h+2}\lambda_0) &\leqq \delta\mu_1(\tilde{A}^{h+1}\lambda_0) + \mu_2(\tilde{A}^{h+1}\tilde{\varepsilon}\lambda_0) \\
&\overset{(92)}{\leqq} \delta\left[\delta^{h+1}\mu_1(\lambda_0) + \sum_{i=0}^{h} \delta^{h-i}\mu_2(\tilde{A}^i\tilde{\varepsilon}\lambda_0)\right] + \mu_2(\tilde{A}^{h+1}\tilde{\varepsilon}\lambda_0) \\
&= \delta^{h+2}\mu_1(\lambda_0) + \sum_{i=0}^{h+1} \delta^{h+1-i}\mu_2(\tilde{A}^i\tilde{\varepsilon}\lambda_0),
\end{aligned}$$

and (92) is completely proved for every $h \in \mathbb{N}$. Now, from (92) it follows that for every $M \in \mathbb{N}$

$$\begin{aligned}
\sum_{h=0}^{M} \tilde{A}^{q(h+1)}\mu_1(\tilde{A}^{h+1}\lambda_0) \leqq{}& \left(\sum_{h=0}^{M} (\delta\tilde{A}^q)^{h+1}\right)\mu_1(\lambda_0) \\
&+ \sum_{h=0}^{M}\sum_{i=0}^{h} \tilde{A}^{q(h+1)}\delta^{h-i}\mu_2(\tilde{A}^i\tilde{\varepsilon}\lambda_0).
\end{aligned} \tag{93}$$

By (84)

$$\sum_{h=0}^{M} (\delta\tilde{A}^q)^{h+1}\mu_1(\lambda_0) \leqq \mu_1(\lambda_0) \quad \forall M \in \mathbb{N}.$$

Concerning the last sum at the right-hand side of (93) we have, again by (84)

$$\sum_{h=0}^{M}\sum_{i=0}^{h}\tilde{A}^{q(h+1)}\delta^{h-i}\mu_2(\tilde{A}^i\tilde{\varepsilon}\lambda_0) \overset{h=k-i}{=} \tilde{A}^q\sum_{i=0}^{M}\tilde{A}^{qi}\mu_2(\tilde{A}^i\tilde{\varepsilon}\lambda_0)\sum_{h=0}^{M-i}(\delta\tilde{A}^q)^h$$

$$\overset{i\to k}{\leqq} 2\tilde{A}^q\sum_{k=0}^{M}\tilde{A}^{qk}\mu_2(\tilde{A}^k\tilde{\varepsilon}\lambda_0).$$

Using the last two estimates in (93) and letting $M\to\infty$ we get

$$\text{(94)}\qquad \sum_{k=1}^{\infty}\tilde{A}^{qk}\mu_1(\tilde{A}^k\lambda_0) = \sum_{k=0}^{\infty}\tilde{A}^{q(k+1)}\mu_1(\tilde{A}^{k+1}\lambda_0)$$

$$\leqq \mu_1(\lambda_0) + 2\tilde{A}^q\sum_{k=0}^{\infty}\tilde{A}^{qk}\mu_2(\tilde{A}^k\tilde{\varepsilon}\lambda_0).$$

Now we will once more do a straight, readable estimate, although it is justified only if read backwards: when the series in (96) will be shown to converge, we will have proved that the power q of the maximal function is integrable, which implies that also the first integral we are about to write is finite. We observe that

$$\text{(95)}\qquad \int_{Q_{R_0}}|Du|^{p(x)q}\,dx \leqq \int_{Q_{R_0}}\left|M^*\left(|Du(\cdot)|^{p(\cdot)}\right)(x)\right|^q dx$$

$$= \int_0^{\infty} q\lambda^{q-1}\mu_1(\lambda)\,d\lambda = \int_0^{\lambda_0}[\ldots]\,d\lambda + \int_{\lambda_0}^{\infty}[\ldots]\,d\lambda$$

and

$$\int_0^{\lambda_0} q\lambda^{q-1}\mu_1(\lambda)\,d\lambda \leqq \lambda_0^q|Q_{R_0}| \overset{(88)}{\leqq} cK^{\sigma q^2}\Big(\fint_{Q_{4R_0}}|Du|^{p(x)}\,dx + 1\Big)^q|Q_{R_0}|,$$

where $c\equiv c(n,\gamma_1,\gamma_2,\nu,L,q)$ since $A\equiv A(n,\gamma_1,\gamma_2,\nu,L)$. In a similar way we have

$$\int_{\lambda_0}^{\infty} q\lambda^{q-1}\mu_1(\lambda)\,d\lambda = \sum_{n=0}^{\infty}\int_{\tilde{A}^n\lambda_0}^{\tilde{A}^{(n+1)}\lambda_0} q\lambda^{q-1}\mu_1(\lambda)\,d\lambda \leqq (\tilde{A}\lambda_0)^q\sum_{n=0}^{\infty}\tilde{A}^{nq}\mu_1(\tilde{A}^n\lambda_0).$$

Again,

$$(\tilde{A}\lambda_0)^q\mu_1(\lambda_0) = \tilde{A}^q\lambda_0^q\mu_1(\lambda_0) \overset{(89)}{\leqq} cK^{\sigma q}\lambda_0^{q-1}\int_{Q_{4R_0}}|Du|^{p(x)}\,dx$$

$$\overset{(88)}{\leqq} \frac{cK^{\sigma q}|Q_{R_0}|}{\delta^{q-1}}\Big(\fint_{Q_{4R_0}}|Du|^{p(x)} + 1\,dx\Big)^q$$

$$\overset{(84)}{=} c(n,\gamma_1,\gamma_2,\nu,L,q)K^{\sigma q^2}\Big(\fint_{Q_{4R_0}}|Du|^{p(x)} + 1\,dx\Big)^q|Q_{R_0}|.$$

Joining the last three estimates to (95) yields

$$(96)\qquad \int_{Q_{R_0}} |Du|^{p(x)q}\,dx \leqq cK^{\sigma q^2}\Big(\fint_{Q_{4R_0}} |Du|^{p(x)} + 1\,dx\Big)^q |Q_{R_0}| + (\tilde{A}\lambda_0)^q\mu_1(\lambda_0) + (\tilde{A}\lambda_0)^q\sum_{k=1}^{\infty}\tilde{A}^{kq}\mu_1(\tilde{A}^k\lambda_0)$$

$$\overset{(94)}{\leqq} cK^{\sigma q^2}\Big(\fint_{Q_{4R_0}} |Du|^{p(x)} + 1\,dx\Big)^q |Q_{R_0}| + 2(\tilde{A}\lambda_0)^q\mu_1(\lambda_0) + 2(\tilde{A}\lambda_0)^q\tilde{A}^q\sum_{k=0}^{\infty}\tilde{A}^{qk}\mu_2(\tilde{A}^k\tilde{\varepsilon}\lambda_0)$$

$$\leqq cK^{\sigma q^2}\Big(\fint_{Q_{4R_0}} |Du|^{p(x)} + 1\,dx\Big)^q |Q_{R_0}| + cK^{2\sigma q}\lambda_0^q\sum_{k=0}^{\infty}\tilde{A}^{kq}\mu_2(\tilde{A}^k\tilde{\varepsilon}\lambda_0),$$

where $c \equiv c(n, \gamma_1, \gamma_2, \nu, L, q)$. It remains to estimate the last series.

To this aim, observe that, as before,

$$\int_{Q_{R_0}} \big|M^*_{1+\sigma}\big(|F(\cdot)|^{p(\cdot)} + 1\big)(x)\big|^q\,dx = \int_0^{\infty} q\lambda^{q-1}\mu_2(\lambda)\,d\lambda = \int_0^{\tilde{\varepsilon}\lambda_0}[\dots]\,d\lambda + \int_{\tilde{\varepsilon}\lambda_0}^{\infty}[\dots]\,d\lambda.$$

Then

$$\int_0^{\tilde{\varepsilon}\lambda_0} q\lambda^{q-1}\mu_2(\lambda)\,d\lambda \geqq (\tilde{\varepsilon}\lambda_0)^q\mu_2(\tilde{\varepsilon}\lambda_0),$$

and, using also (90)

$$\int_{\tilde{\varepsilon}\lambda_0}^{\infty} q\lambda^{q-1}\mu_2(\lambda)\,d\lambda = \sum_{k=0}^{\infty}\int_{\tilde{A}^k\tilde{\varepsilon}\lambda_0}^{\tilde{A}^{k+1}\tilde{\varepsilon}\lambda_0} q\lambda^{q-1}\mu_2(\lambda)\,d\lambda$$

$$\geqq \sum_{k=0}^{\infty}\mu_2(\tilde{A}^{k+1}\tilde{\varepsilon}\lambda_0)[(\tilde{A}^{k+1}\tilde{\varepsilon}\lambda_0)^q - (\tilde{A}^k\tilde{\varepsilon}\lambda_0)^q]$$

$$= (\tilde{\varepsilon}\lambda_0)^q\sum_{k=0}^{\infty}\tilde{A}^{(k+1)q}\mu_2(\tilde{A}^{k+1}\tilde{\varepsilon}\lambda_0)[1 - \tilde{A}^{-q}]$$

$$\geqq \frac{1}{2}(\tilde{\varepsilon}\lambda_0)^q\sum_{k=1}^{\infty}\tilde{A}^{kq}\mu_2(\tilde{A}^k\tilde{\varepsilon}\lambda_0).$$

Combining the last estimates with the maximal inequality (22) we finally get

$$
\begin{aligned}
&\frac{1}{2q}(\tilde{\varepsilon}\lambda_0)^q \sum_{k=1}^{\infty} \tilde{A}^{kq}\mu_2(\tilde{A}^k\tilde{\varepsilon}\lambda_0) + \frac{(\tilde{\varepsilon}\lambda_0)^q}{q}\mu_2(\tilde{\varepsilon}\lambda_0) \\
&\quad \leqq \frac{1}{q}(\tilde{\varepsilon}\lambda_0)^q \sum_{k=0}^{\infty} \tilde{A}^{kq}\mu_2(\tilde{A}^k\tilde{\varepsilon}\lambda_0) \\
&\quad \leqq \frac{2}{q}\int_{Q_{R_0}} \left|M^*_{1+\sigma}\big(|F(\cdot)|^{p(\cdot)}+1\big)(x)\right|^q dx \\
&\quad \overset{(20)}{\leqq} \frac{c(n)q}{(1+\sigma)(q-1-\sigma)}\int_{Q_{4R_0}} (|F|^{p(x)}+1)^q\, dx \\
&\quad \overset{(43)}{\leqq} \frac{2^{q+1}c(n)q}{q-1}\int_{Q_{4R_0}} |F|^{p(x)q}+1\, dx.
\end{aligned}
$$

Using this estimate in (96) and passing to averages we have

$$
\begin{aligned}
(97)\qquad &\left(\fint_{Q_{R_0}} |Du|^{p(x)q}\, dx\right)^{\frac{1}{q}} \\
&\quad \leqq cK^{\sigma q}\fint_{Q_{4R_0}} |Du|^{p(x)}+1\, dx + c\frac{K^{2\sigma}}{\tilde{\varepsilon}}\left(\fint_{Q_{4R_0}} |F|^{p(x)q}\, dx\right)^{\frac{1}{q}} \\
&\quad \overset{(81)}{\leqq} cK^{\frac{\sigma 2q\gamma_1}{\gamma_1-1}}\fint_{Q_{4R_0}} |Du|^{p(x)}\, dx + cK^{\frac{\sigma 2q\gamma_1}{\gamma_1-1}}\left(\fint_{Q_{4R_0}} |F|^{p(x)q}\, dx + 1\right)^{\frac{1}{q}}
\end{aligned}
$$

where now $c \equiv c(n, \gamma_1, \gamma_2, \nu, L, q)$. Summarizing what we have done up to now we see that we have proved estimate (14) for every σ as above provided and for every radius $R \leqq R_0$ where R_0 satisfies conditions (34), (44), (85) and (86); with ε fixed, now (14) follows choosing $\tilde{\sigma}$ in (41) small enough to meet

$$
\sigma \leqq \frac{\varepsilon(\gamma_1 - 1)}{2q\gamma_1},
$$

thus we may take, see (40),

$$
(98)\qquad \tilde{\sigma} := \frac{\varepsilon(\gamma_1 - 1)}{2q\gamma_1\sigma_M}.
$$

Observing that $\sigma \leqq \varepsilon$ implies that

$$
K \leqq \int_{Q_{4R}} |Du|^{p(x)} + |F|^{p(x)(1+\varepsilon)}\, dx + 2,
$$

we can replace K by this last expression in (14), obtaining the full statement. The estimate trivially follows for larger values of ε since $K \geqq 1$. Since our reasoning applies to any cube Q_R such that $R \leqq R_0$ and $Q_{4R} \subset\subset \Omega$, the fact that $|Du|^{p(x)} \in L^q_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ follows from (14) via a standard covering argument. We finally remark that the precise way we deduced our estimates allows to conclude that, after we chose $\tilde{\sigma} \in (0,1)$ as in (98), the constant c appearing in (14) does not depend on Du: the dependence on the solution is explicitly computed via the appearance of the quantity K. □

Remark 5. *On the radius* R_0. Our derivation of the estimates is precise enough to allow an estimate on the radius R_0. Suppose for instance that $p(x)$ is Hölder continuous, and therefore $\omega(s) \leqq cs^{\alpha}$ for some $\alpha \in (0,1)$: keeping in mind (85) and (86) we get

$$R_0(\varepsilon) \approx \left(\frac{\varepsilon}{c(n,\gamma_1,\gamma_2,\nu,L)^q c(\gamma)}\right)^{\frac{1}{\gamma}} \left(\int_\Omega |Du|^{p(x)} + |F|^{p(x)}\,dx + 1\right)^{\frac{-\varepsilon q}{\gamma}},$$

for every $\gamma < \alpha$. As one may check by tracing the dependence of the constants in (66) and (76), here $c(n,\gamma_1,\gamma_2,\nu,L,\sigma)$ is the constant in the weak type Harnack inequality valid for solutions to p-Laplacean type equations (or systems) introduced in (64). □

Remark 6. *A more accurate estimate.* One may be more careful with some exponents, as we remarked after (74): then in (78) a $K^{\sigma\left(\frac{1}{4\gamma_1}-1\right)}$ appears before $\tilde{\varepsilon}$, and filing some more exponents we would have ended estimate (97) with the power

$$\frac{\sigma 2q\gamma_1}{\gamma_1 - 1}$$

replaced by

$$2\sigma + \frac{\sigma\gamma_1}{\gamma_1 - 1}\left(q - 1 + \frac{1}{4\gamma_1}\right)$$

and we would need to reduce $\tilde{\sigma}$ accordingly to an uglier value. □

Proof of Theorem 3. This is essentially the same as the previous one. We summarize it via the following observations: to begin with, Theorem 5 applies in the case of the $p(x)$-Laplacean system, since it only depends on the monotonicity and growth assumptions imposed on the vector field $a(x,\cdot)$. More precisely, if we define the vector field $a : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ as

$$a_i^{\alpha}(x,z) := |z|^{p(x)-2} z_i^{\alpha}, \quad i \in \{1,\dots,n\},\ \alpha \in \{1,\dots,N\} \tag{99}$$

then $a(x,z)$ satisfies, with a suitable choice of ν and L the assumptions (8), (9) (once recast in a way that fits the vectorial case and the degenerate structure, that is $\lambda, z \in \mathbb{R}^{nN}$, $\mu = 0$, and so on). These are the only ones used in the proof of Theorem 5. Observe that the vector field in (99) is not of class C^1 when $p(x) < 2$; this fact does not affect the proof of Theorem 5; anyway, this lack of regularity, due to the singularity of the sub-quadratic case, can be easily dealt with via a by now classical approximation argument [1]. When passing to the proof of Lemma 2, everything goes as before but at the point where the a priori estimate for the solution of the (frozen) auxiliary problem (61) comes into the play, (66). Due to the particular structure of the $p(x_M)$-Laplacean system

$$-\operatorname{div}(|Du|^{p(x_M)-2} Du) = 0$$

estimate (66) is still valid in this case [33], [1], [9]. At this point the rest of the proof proceeds exactly in the same way, giving the announced dependence of the constants. □

References

[1] *E. Acerbi* and *N. Fusco*, Regularity for minimizers of nonquadratic functionals: the case $1 < p < 2$, J. Math. Anal. Appl. **140** (1989), 115–135.

[2] *E. Acerbi* and *G. Mingione*, Regularity results for a class of functionals with non-standard growth, Arch. Rat. Mech. Anal. **156** (2001), 121–140.

[3] *E. Acerbi* and *G. Mingione*, Regularity results for stationary electro-rheological fluids, Arch. Rat. Mech. Anal. **164** (2002), 213–259.

[4] *E. Acerbi*, *G. Mingione* and *G. A. Seregin*, Regularity results for parabolic systems related to a class of Non Newtonian fluids, Ann. Inst. H. Poincaré Anal. Non Lin. **21** (2004), 25–60.

[5] *K. Astala*, *T. Iwaniec*, *P. Koskela* and *G. Martin*, Mappings of BMO-bounded distortion, Math. Ann. **317** (2000), 703–726.

[6] *B. Bojarski* and *T. Iwaniec*, Analytical foundations of the theory of quasiconformal mappings in $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. (A I Math.) **8** (1983), 257–324.

[7] *L. Caffarelli*, Interior a priori estimates for solutions of fully nonlinear equations, Ann. Math. **130** (1989), 189–213.

[8] *L. Caffarelli* and *I. Peral*, On $W^{1,p}$ estimates for elliptic equations in divergence form, Comm. Pure Appl. Math. **51** (1998), 1–21.

[9] *A. Coscia* and *G. Mingione*, Hölder continuity of the gradient of $p(x)$-harmonic mappings, C. R. Acad. Sci. Paris (I Math.) **328** (1999), 363–368.

[10] *D. Cruz-Uribe*, *A. Fiorenza* and *C. J. Neugebauer*, The maximal function on variable L^p spaces, Ann. Acad. Sci. Fenn. (A I Math.) **28** (2003), 223–238.

[11] *G. Cupini*, *N. Fusco* and *R. Petti*, Hölder continuity of local minimizers, J. Math. Anal. Appl. **235** (1999), 578–597.

[12] *L. Diening* and *M. Růžička*, Calderón-Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics, J. reine angew. Math. **563** (2003), 197–220.

[13] *L. Diening* and *M. Růžička*, Integral operators on the halfspace in generalized Lebesgue spaces $L^{p(\cdot)}$, I and II, J. Math. Anal. Appl. **298** (2004) no 2, 559–571 and 572–588.

[14] *D. Edmunds* and *J. Rakosnik*, Sobolev embeddings with variable exponent, Stud. Math. **143** (2000), 267–293.

[15] *L. Esposito*, *F. Leonetti* and *G. Mingione*, Higher integrability for minimizers of integral functionals with (p,q) growth, J. Diff. Equ. **157** (1999), 414–438.

[16] *E. Giusti*, Direct methods in the calculus of variations, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[17] *P. Harjulehto*, *P. Hästö* and *O. Martio*, Fuglede's theorem in variable exponent Sobolev space, Collect. Math. **55** (2004) no 3, 315–324.

[18] *P. Hästö*, Counter-examples of regularity in variable exponent Sobolev spaces, Contemp. Math., to appear.

[19] *T. Iwaniec*, Projections onto gradient fields and L^p-estimates for degenerated elliptic operators, Stud. Math. **75** (1983), 293–312.

[20] *T. Iwaniec*, p-harmonic tensors and quasiregular mappings, Ann. Math. (2) **136** (1992), 589–624.

[21] *T. Iwaniec*, The Gehring lemma, Quasiconformal mappings and analysis (Ann Arbor, MI, 1995), Springer, New York (1998), 181–204.

[22] *T. Iwaniec* and *A. Verde*, On the operator $\mathscr{L}(f) = f\log|f|$, J. Funct. Anal. **169** (1999), 391–420.

[23] *J. Kinnunen* and *S. Zhou*, A local estimate for nonlinear equations with discontinuous coefficients, Comm. Part. Diff. Equ. **24** (1999), 2043–2068.

[24] *S. Levine*, *J. Stanich* and *Y. Chen*, Image restoration via non standard diffusion, to appear.

[25] *J. J. Manfredi*, Regularity for minima of functionals with p-growth, J. Diff. Equ. **76** (1988), 203–212.

[26] *P. Marcellini*, Regularity and existence of solutions of elliptic equations with p,q-growth conditions, J. Diff. Equ. **90** (1991), 1–30.

[27] *L. Pick* and *M. Růžička*, An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded, Expo. Math. **19** (2001), 369–371.

[28] *K. R. Rajagopal* and *M. Růžička*, Mathematical modeling of electrorheological materials, Contin. Mech. Thermodyn. **13** (2001), 59–78.

[29] *M. M. Rao* and *Z. D. Ren*, Theory of Orlicz spaces, Monogr. Textb. Pure Appl. Math. **146**, Marcel Dekker, Inc., New York 1991.

[30] *M. Růžička*, Electrorheological fluids: modeling and mathematical theory, Springer Lect. Notes Math. **1748** (2000).
[31] *M. Růžička*, Flow of shear dependent electrorheological fluids, C. R. Acad. Sci. Paris (I Math.) **329** (1999), 393–398.
[32] *E. M. Stein*, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.
[33] *K. Uhlenbeck*, Regularity for a class of non-linear elliptic systems, Acta Math. **138** (1977), 219–240.
[34] *N. Ural'tseva*, Quasilinear degenerate systems (russian), Leningrad Odtel. Mat. Inst. Steklov (LOMI) **7** (1968), 184–222.
[35] *V. V. Zhikov*, Averaging of functionals of the calculus of variations and elasticity theory (russian), Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), 675–710, 877.
[36] *V. V. Zhikov*, On Lavrentiev's phenomenon, Russian J. Math. Phys. **3** (1995), 249–269.

Dipartimento di Matematica dell'Università di Parma, via M. d'Azeglio 85/a, 43100 Parma
e-mail: emilio.acerbi@unipr.it
e-mail: giuseppe.mingione@unipr.it

Eingegangen 6. Mai 2004