GRADIENT ESTIMATES FOR A CLASS OF PARABOLIC SYSTEMS

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Abstract

We establish local Calderón-Zygmund-type estimates for a class of parabolic problems whose model is the nonhomogeneous, degenerate/singular parabolic p-Laplacian system

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F),$$

proving that

$$F \in L^q_{\mathrm{loc}} \Longrightarrow Du \in L^q_{\mathrm{loc}}, \quad \forall\, q \geq p.$$

We also treat systems with discontinuous coefficients of vanishing mean oscillation (VMO) type.

1. Introduction

The aim of this article is to present Calderón-Zygmund-type estimates for weak solutions to a class of degenerate/singular parabolic systems and equations, a prominent model example of which is the nonhomogeneous, parabolic p-Laplacian system

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F), \quad p > \frac{2n}{n+2}, \tag{1}$$

considered in the cylindrical domain $C := \Omega \times [0, T)$. Here, $\Omega \subset \mathbb{R}^n$ is a bounded open set and $u \in C^0((0,T); L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$, $N \geq 1$, while $F \in L^p(C, \mathbb{R}^{nN})$. Such a system is degenerate when $p > 2$ and singular when $p < 2$; the lower bound on the exponent p assumed in (1) is standard in the theory of the parabolic p-Laplacian operator and unavoidable for the type of regularity which we consider here.

For system (1), we prove that

$$F \in L^q_{\mathrm{loc}}(C, \mathbb{R}^{nN}) \Longrightarrow Du \in L^q_{\mathrm{loc}}(C, \mathbb{R}^{nN}) \tag{2}$$

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for any $q \geq p$. In the elliptic, stationary case

$$\operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F), \tag{3}$$

the result in (2) was essentially obtained by T. Iwaniec [14] in the scalar case ($N = 1$) and by DiBenedetto and Manfredi [10] for systems ($N > 1$). The extension to anisotropic elliptic equations with possibly discontinuous, vanishing mean oscillation (VMO) coefficients has been achieved by Kinnunen and Zhou [19], [20], while a class of general nonlinear elliptic equations and systems in divergence form, under nonstandard growth conditions, has been treated by Acerbi and Mingione [1].

There has recently been a great deal of work concerning the integrability properties of weak and very weak solutions to systems similar to (1) (see [16], [17], [18]). In particular, in the interesting article [16], Kinnunen and Lewis proved higher integrability of the spatial gradient for solutions of general nonlinear parabolic systems with p-growth including (1), introducing a localization method to overcome the lack of homogeneity of parabolic systems with p-growth when $p \neq 2$. They came up with a sort of reverse-type Hölder inequality. The new ingredient offered by these authors is a suitable application of DiBenedetto's intrinsic geometry method for degenerate/singular parabolic systems (see [8]) in the setting of Gehring-type estimates. Subsequently, Misawa [22] considered higher integrability of the gradient of solutions to (1), assuming that $F \in L^\infty$, and therefore in L^q for every $q > 1$.

In this article, by means of a new technique, we use the result of Kinnunen and Lewis and partially some methods adapted from [5] and [1] to be able finally to prove the natural integrability result in (2).

A main difficulty of the problem is that no use of classical harmonic analysis tools can be made here: system (1) is nonlinear in the gradient, and therefore the use of singular integrals is ruled out, while, since it is degenerate/singular and scales differently in space and time, no maximal function operator is naturally associated with the problem. We therefore again adopt an intrinsic geometry viewpoint, arguing directly on certain Calderón-Zygmund-type covering arguments and completely avoiding the use of the maximal function operator or of other harmonic analysis principles such as the *good-λ-inequality* one. A peculiar aspect of our work, which allows us to treat the general situation considered here, is that instead of using the $C^{1,\alpha}$-estimates for the homogeneous ($F \equiv 0$) p-Laplacian systems, as done in [10], [14], and [22] for both the elliptic and the parabolic cases, we use only the $C^{0,1}$-estimates (see [8]), which immediately exhibit the right scaling properties when considered on intrinsic cylinders and perfectly fit in this context. This is a natural attempt since we want to prove L^q-estimates for Du, whose limit case is indeed given by the $C^{0,1}$-estimates; anyway, the proof is quite delicate. An approach to gradient estimates for equations in divergence form, making use of $C^{0,1}$-estimates and working via maximal functions, has been introduced in the elliptic, homogeneous case by Caffarelli and Peral in [5]; such an approach works for (homogeneous) parabolic equations only when $p = 2$ (see [24]), again for the reasons explained above. As already mentioned, it is worth pointing out that we cannot use here the so called good-λ-inequality principle; we instead replace it with a new, direct argument that we like to call the *large-M-inequality* principle (see (81)). We like to mention that, apart from the different scaling procedures adopted for the singular and degenerate cases, the proof offered here does not distinguish between the cases $p < 2$ and $p \geq 2$.

Our results cover a more general class of degenerate/singular parabolic systems of the type

$$u_t - \operatorname{div}\big(a(z)|Du|^{p-2}Du\big) = \operatorname{div}(|F|^{p-2}F), \tag{4}$$

whose coefficients $a(z) \equiv a(x,t)$ may be discontinuous in a VMO/bounded mean oscillation (BMO) fashion (see Section 2), for which we still prove (2); furthermore, extensions involving operators different from the p-Laplacian are outlined in Section 5. We also derive natural and neat local estimates for solutions in the form of certain nonhomogeneous reverse-type Hölder inequalities (see (9)). Here, the nonhomogeneity of the estimates precisely reflects that of the system (space/time) via the *scaling deficit exponent d* introduced in (10).

The problem of deriving Calderón-Zygmund-type estimates for elliptic and parabolic equations, eventually with discontinuous coefficients, is a classical one, and it already has a long tradition. In the elliptic and scalar cases, it has usually been faced via harmonic analysis tools such as nonlinear commutators (see [6]), Riezs transform (see [15]), or the maximal function operator (see [19]; see also [12], [23]). Parabolic equations with coefficients of VMO/BMO type have been treated only in the linear case and, in particular, again when $p = 2$, making use of harmonic analysis tools such as nonlinear commutators (see [3]) and, more recently, of the maximal function operator (see [4]); needless to say, such ingredients are not available in the case of the evolutionary p-Laplacian operator.

2. Results

General notation. We establish some notation in addition to what was given in the introduction. By *cylinder* $Q_z(\theta, \varrho) \subset \mathbb{R}^{n+1}$ *centered* at the point $z \equiv (x,t) \in \mathbb{R}^{n+1}$ with $\theta, \varrho > 0$, we always mean a set of the type $Q_z(\theta,\varrho) = B_x(\varrho) \times (t-\theta, t+\theta)$, where, as usual, $B_x(\varrho) := \{y \in \mathbb{R}^N \,:\, |x-y| < \varrho\}$; with abuse of terminology, such cylinders are also called *cubes*. As a partial exception, we write B^1 to denote the unit ball centered at the origin of $\mathbb{R}^N$. When not essential, the center of a cylinder is not specified; that is, $Q(\theta,\varrho) \equiv Q_z(\theta,\varrho)$. In the case of the standard parabolic cylinders, that is, when $\theta = \varrho^2 = R^2$, we simply write $Q_R \equiv Q(R^2, R)$. The parabolic boundary

$\partial_p Q$ of a cylinder $Q_{\tilde{z}}(\theta, \varrho)$ is the union of the lower base $B_{\tilde{x}}(\varrho) \times \{\tilde{t} - \theta\}$ and the side surface $\{|x - \tilde{x}| = \varrho\} \times [\tilde{t} - \theta, \tilde{t} + \theta]$. Adopting a usual convention, c denotes a constant whose value may change in any two occurrences, and only the relevant dependences are specified, as, for example, $c(\gamma, p)$; particular constants are denoted by c_1, $\tilde{c}$, and the like. For the Lebesgue measure of a measurable set A, we employ either of the notations $|A| = \mathrm{meas}(A)$; then we define the mean value on a cylinder $Q \subset \mathbb{R}^{n+1}$ of an integrable function $v \in L^1(Q)$ by

$$(v)_Q \equiv \fint_Q v\, dx := \frac{1}{|Q|} \int_Q v\, dx.$$

When $Q = Q_R$, we also employ the notation $(v)_R \equiv (v)_{Q_R}$.

Strong VMO/BMO functions

Here, we define the class of coefficients $a(z) \equiv a(x, t)$ which we use when treating systems of type (4). In order to preserve the basic parabolicity properties of the systems, and allowing a degeneration caused only by the presence of the factor $|Du|^{p-2}$ in (4), we always assume that the function $a : C \to \mathbb{R}$ satisfies

$$0 < \nu \leq a(z) \leq L < \infty, \quad \forall\, z \in C. \tag{5}$$

Definition 1

We say that a function $a(z)$ satisfies the strong VMO condition if

$$\lim_{R \to 0} \omega(R) = 0, \tag{6}$$

where

$$\omega(R) := \sup_{Q \Subset C} \fint_Q |a(z) - (a)_Q|\, dz \tag{7}$$

and the supremum is taken among all cylinders of the type $Q_z(\theta, \varrho)$ with $\varrho \leq R$ and $\theta \leq R^2$. We say that the function $a(z)$ satisfies the strong BMO condition if

$$[a]_{\mathrm{BMO}} := \sup_{R > 0} \omega(R) < \infty.$$

We have a few comments about this definition. To adapt to the nonlinear parabolic structure we are allowed to pick more cylinders with respect to a usual elliptic-style VMO/BMO definition (see [25]), allowing for the size of the space radius ϱ of the cylinder Q to be unrelated to the time height θ. This class includes, for instance, all continuous coefficients $a(z)$, and it is large enough to include many possibly discontinuous functions. For instance, in (4) we may take $a(x, t) = b(x)c(t)$, where both $b(x)$ and $c(t)$ are usual VMO/BMO functions, in Ω and $[0, T)$, respectively, and satisfying (5). The strong VMO/BMO condition is, in our opinion, the natural one in order to treat situations such as in (4). Indeed, when dealing with partial differential equations (PDEs), especially elliptic and parabolic ones, the notion of VMO/BMO is usually given using a family of cubes or cylinders that are relevant both for the scaling properties and for the geometry of the equation. Since the works of DiBenedetto (see [8] and references therein), it is known that the natural class of cylinders $Q(\theta, \varrho)$ occurring in connection with (1) is the one having the ratio ϱ/θ not related to the coefficient $a(z)$ but depending on the solution u itself, via quantities like, for instance, $|(Du)_Q|^{p-2}$, which are a priori arbitrary. Therefore, when treating such problems, we have to allow for a larger freedom in the choice of the suitable VMO/BMO-like definition. Anyway, the class considered here is already used implicitly in [22].

Main results

When $F \in L^p(C, \mathbb{R}^{nN})$ is a vector field, following [8, pages 17, 215], a weak solution to system (4) is a map

$$u \in C^0\big((0,T); L^2(\Omega, \mathbb{R}^N)\big) \cap L^p\big(0,T; W^{1,p}(\Omega, \mathbb{R}^N)\big)$$

such that for every $0 < t_1 < t_2 < T$,

$$\int_\Omega u\varphi(x,t)\,dx\Big|_{t_1}^{t_2} - \int_\Omega \int_{t_1}^{t_2} u\varphi_t + a(z)\langle |Du|^{p-2}Du, D\varphi\rangle\,dz$$
$$= -\int_\Omega \int_{t_1}^{t_2} \langle |F|^{p-2}F, D\varphi\rangle\,dz$$

for every test function $\varphi \in W^{1,2}_{\mathrm{loc}}(0,T; L^2(\Omega, \mathbb{R}^N)) \cap L^p_{\mathrm{loc}}(0,T; W^{1,p}_0(\Omega, \mathbb{R}^N))$. When dealing with weak solutions, we always adopt the formulation via Steklov averages (see again DiBenedetto's book [8, pages 11, 21]).

THEOREM 1

Let $u \in C^0((0,T); L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution to (4), where

$$p > \frac{2n}{n+2} \tag{8}$$

and the function $a : C \to \mathbb{R}$ satisfies (5) and is strongly VMO. Assume that $|F|^p \in L^q_{\mathrm{loc}}(C)$ for some $q > 1$. Then $|Du|^p \in L^q_{\mathrm{loc}}(C)$. Moreover, there exists a constant $c \equiv c(n, N, p, \nu, L, q, \omega(\cdot)) > 1$ such that if $Q_{2R} \Subset C$, then

$$\left(\fint_{Q_R} |Du|^{pq}\,dz\right)^{1/q} \le c\left[\fint_{Q_{2R}} |Du|^p\,dz + \left(\fint_{Q_{2R}} |F|^{pq}\,dz + 1\right)^{1/q}\right]^d, \tag{9}$$

where

$$1 \le d := \begin{cases} \frac{p}{2} & \text{if } p \ge 2, \\ \frac{2p}{p(n+2)-2n} & \text{if } p < 2. \end{cases} \tag{10}$$

We also have a result concerning coefficients $a(z)$ that are not necessarily VMO but rather have suitably small BMO seminorm.

THEOREM 2
Let $u \in C^0((0,T); L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega,\mathbb{R}^N))$ *be a weak solution to (4), where the function* $a : C \to \mathbb{R}$ *satisfies (5) and* p *is as in (8). Fix* $q > 1$*, and assume that* $|F|^p \in L^q_{\text{loc}}(C)$*. For every* $q > 1$*, there exists a number* $\varepsilon \equiv \varepsilon(n, N, p, \nu, L, q) > 0$ *such that if* $[a]_{BMO} \le \varepsilon$*, then* $|Du|^p \in L^q_{\text{loc}}(C)$*. Moreover, there exists a constant* $c \equiv c(n, N, p, \nu, L, q) > 1$ *such that (9) holds for every* $Q_{2R} \Subset C$*, with* d *as in (10).*

Remark 1
The exponent d outside the square bracket in (9) prevents the estimate from being homogeneous and of reverse-type Hölder. The occurrence of d is absolutely natural and reflects the nonhomogeneity of system (4) due to the fact that the evolutionary part of the system scales differently from the diffusion one: multiplying a solution by a constant does not yield another solution, even when $F \equiv 0$. Of course, $d = 1$ if and only if $p = 2$, and the system is not degenerate/singular; moreover, $d \nearrow \infty$ when $p \searrow 2n/(n+2)$ (for more comments on the dependence of the constant c on the number q, see Remark 3). Finally, we notice that it is possible to apply the method presented here also in the elliptic case (3); this would yield a true, homogeneous reverse-type Hölder inequality, that is, (9) with $d = 1$.

3. Preliminary material

In this section we collect some known results that are crucial in the rest of the article, and we operate a few manipulations on known estimates in order to get them in the exact form that we later need. We start with the higher integrability result of Kinnunen and Lewis [16] which is essential here in treating the case where the coefficient function $a(z)$ is not continuous. The version reported here is adapted to our setting from the more general right-hand-side structure in [16] (for the equivalence, see (88) with $f \equiv h_1$ in the notation of [16, (2.3)]). Also, Kinnunen and Lewis assert that (11) holds for some $\delta_0 > 0$, but the fact that it then holds for all $\delta < \delta_0$ may be deduced from their proof following [16, (4.13)]; indeed, the only condition on δ_0 is that it must be small.

THEOREM 3

Let $u \in C^0((0,T); L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega,\mathbb{R}^N))$ *be a weak solution to (4), where (8) is in force and the function* $a : C \to \mathbb{R}$ *satisfies (5). Assume that* $|F|^p \in L^q_{\mathrm{loc}}(C)$ *for some* $q > 1$. *Then there exists* $\delta_0 \equiv \delta_0(n, N, p, \nu, L)$ *with* $0 < \delta_0 < p(q-1)$ *such that* $|Du| \in L^{p+\delta}_{\mathrm{loc}}(C)$ *for every* $0 < \delta \le \delta_0$. *Moreover, there exists a constant* $c \equiv c(n, p, L, \nu) > 1$ *such that if* $Q_{z_0}(R^p, R) \Subset C$, *then*

$$\left(\fint_{Q_{z_0}(R^p/2^p, R/2)} |Du|^{p+\delta}\, dz\right)^{1/(p+\delta)} \le cR^{\sigma p-1}\left(\fint_{Q_{z_0}(R^p,R)} |Du|^p\, dz\right)^{\sigma} + \frac{c}{R} + c\left(\fint_{Q_{z_0}(R^p,R)} |F|^{p+\delta}\, dz\right)^{1/(p+\delta)}, \quad (11)$$

where

$$\sigma := \frac{2+\delta}{2(p+\delta)}. \quad (12)$$

In the remainder of the article, we eventually take $\delta < \delta_0$ in order to have

$$s := p+\delta \le \min\left\{\frac{(p+pq)}{2}, p+1\right\} < pq, \quad (13)$$

and we notice that this implies

$$\frac{pq}{pq-s} \le \frac{2q}{q-1}. \quad (14)$$

The first two lemmas are a consequence of the fundamental L^∞-gradient estimates of DiBenedetto [8] and DiBenedetto and Friedman [9].

LEMMA 1

Let $v \in C^0((t_1,t_2); L^2(A,\mathbb{R}^N)) \cap L^p(t_1,t_2; W^{1,p}(A,\mathbb{R}^N))$ *be a weak solution to*

$$v_t - \mathrm{div}(\tilde{a}|Dv|^{p-2}Dv) = 0 \quad \text{in } A \times [t_1,t_2), \quad (15)$$

where $A \subset \mathbb{R}^n$ *is an open set,* $t_1 < t_2$, $p \ge 2$, *and* $\nu \le \tilde{a} \le L$. *Assume that*

$$\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Dv|^p\, dz \le c_1\lambda^p \quad (16)$$

for some $\lambda > 0$ *and some cylinder* $Q(\lambda^{2-p}\varrho^2,\varrho) \Subset A \times [t_1,t_2)$, *where* c_1 *is a given positive constant. Then there exists a constant* $c > 0$, *depending only on* n, N, p, ν, L, *and c_1, such that*

$$\sup_{Q((1/2)\lambda^{2-p}\varrho^2,(1/2)\varrho)} |Dv| \le c\lambda. \tag{17}$$

Proof
From [8, Chapter 8, Theorem 5.1] and, in particular, [8, (5.1), page 238], by taking $\sigma = 3/4$, we have the fact that if $Q(\theta, \gamma) \Subset A \times [t_1, t_2)$ is a nondegenerate cylinder, then

$$\sup_{Q(\theta/2,\gamma/2)} |Dv| \le c(n, N, p, \nu, L)\sqrt{\frac{\theta}{\gamma^2}}\Big(\fint_{Q(\theta,\gamma)} |Dv|^p \, dz\Big)^{1/2} + \Big(\frac{\gamma^2}{\theta}\Big)^{1/(p-2)}. \tag{18}$$

Then we take $\theta = \lambda^{2-p}\gamma^2$ and $\gamma = \varrho$, so that $\sqrt{\theta/\gamma^2} = \lambda^{(2-p)/2}$. Using this fact in the previous inequality, and finally using (16), we immediately obtain (17). □

In the case where $p < 2$, the estimate that one is allowed to use is different, so we need another statement (and another proof, albeit very similar).

LEMMA 2
Let $v \in C^0((t_1, t_2); L^2(A, \mathbb{R}^N)) \cap L^p(t_1, t_2; W^{1,p}(A, \mathbb{R}^N))$ be a weak solution to (15), where $A \subset \mathbb{R}^n$ is an open set, $t_1 < t_2$, $\nu \le \tilde{a} \le L$, and $p < 2$ satisfies (8). Assume that

$$\fint_{Q(\varrho^2,\lambda^{(p-2)/2}\varrho)} |Dv|^p \, dz \le c_1\lambda^p \tag{19}$$

for some $\lambda > 0$ and some cylinder $Q(\varrho^2, \lambda^{(p-2)/2}\varrho) \Subset A \times [t_1, t_2)$, where c_1 is a given positive constant. Then there exists a constant $c > 0$, depending only on n, N, p, ν, L, and c_1, such that

$$\sup_{Q((1/2)\varrho^2,(1/2)\lambda^{(p-2)/2}\varrho)} |Dv| \le c\lambda. \tag{20}$$

Proof
This time we use [8, Chapter 8, Theorem 5.2] and, in particular, [8, (5.3), page 239], where we can take $r = p$ since p is assumed to satisfy (8). Again taking $\sigma = 3/4$, we have the fact that if $Q(\theta, \gamma) \Subset A \times [t_1, t_2)$ is a nondegenerate cylinder, then

$$\sup_{Q(\theta/2,\gamma/2)} |Dv| \le c(n, N, p, \nu, L)\Big(\frac{\gamma^2}{\theta}\Big)^{n/[p(n+2)-2n]}\Big(\fint_{Q(\theta,\gamma)} |Dv|^p \, dz\Big)^{2/[p(n+2)-2n]} + \Big(\frac{\theta}{\gamma^2}\Big)^{1/(2-p)}. \tag{21}$$

Then we take $\gamma = \lambda^{(p-2)/2}\varrho$ and $\theta = \varrho^2$, so that $\sqrt{\gamma^2/\theta} = \lambda^{(p-2)/2}$. Using this fact in the previous inequality, and finally using (19), we immediately obtain (20). □

The following twinned lemmas, Lemmas 3 and 4, show how solutions to (4) satisfy real reverse-type Hölder inequalities when considered on cylinders built according to the intrinsic geometry.

LEMMA 3
Let $u \in C^0((0,T); L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega,\mathbb{R}^N))$ be a weak solution to (4), where $p \geq 2$, and the function $a : C \to \mathbb{R}$ satisfies (5). Assume that

$$\left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p \, dz \right)^{1/p} \leq c_1 \lambda \tag{22}$$

and

$$\lambda \leq c_2 \left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p \, dz \right)^{1/p} + c_2 \left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} M^s |F|^s \, dz \right)^{1/s} \tag{23}$$

hold for some $\lambda > 0$ and some cylinder $Q(\lambda^{2-p}\varrho^2, \varrho) \Subset C$, where $s > p$ is defined in (13) via Theorem 3, c_1 and c_2 are two given positive constants, and $M \geq 1$. Then there exists a constant $c_3 \equiv c_3(n, N, p, \nu, L, c_1, c_2)$ such that

$$\left(\fint_{Q((1/2^p)\lambda^{2-p}\varrho^2,(1/2)\varrho)} |Du|^s \, dz \right)^{1/s} \leq c_3 \left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p \, dz \right)^{1/p} + c_3 \left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} (1 + M^s |F|^s) \, dz \right)^{1/s}. \tag{24}$$

Proof
Without loss of generality, we may assume that the cylinder $Q(\lambda^{2-p}\varrho^2, \varrho)$ is centered at the origin. Let us consider the rescaled maps

$$\tilde{u}(x,t) := \frac{u(\varrho x, \lambda^{2-p}\varrho^2 t)}{\varrho\lambda}, \qquad \tilde{F}(x,t) := \frac{F(\varrho x, \lambda^{2-p}\varrho^2 t)}{\lambda},$$

with $(x,t) \in Q_1$, and the rescaled coefficients $\tilde{a}(x,t) := a(\varrho x, \lambda^{2-p}\varrho^2 t)$. It is easy to check that $\tilde{u} \in C^0((0,1); L^2(B_1,\mathbb{R}^N)) \cap L^p(0,1; W^{1,p}(B_1,\mathbb{R}^N))$ is a weak solution to the system

$$\tilde{u}_t - \operatorname{div}\big(\tilde{a}(z)|D\tilde{u}|^{p-2}D\tilde{u}\big) = \operatorname{div}(|\tilde{F}|^{p-2}\tilde{F}) \quad \text{in } Q_1.$$

Therefore we may apply Theorem 3 and, in particular, estimate (11) in order to get the fact that there exists a constant c, depending only on n, N, p, ν, L, such that

$$\Big(\fint_{Q_{(1/2^p,1/2)}} |D\tilde{u}|^s \, dz \Big)^{1/s} \le c \Big(\fint_{Q_1} |D\tilde{u}|^p \, dz \Big)^{\sigma} + c \Big(\fint_{Q_1} |\tilde{F}|^s \, dz \Big)^{1/s} + c, \tag{25}$$

where, according to (12), $\sigma = (2-p+s)/2s$. Scaling back in (25) yields

$$\Big(\fint_{Q((1/2^p)\lambda^{2-p}\varrho^2,(1/2)\varrho)} |Du|^s \, dz \Big)^{1/s} \le c\lambda^{1-\sigma p} \Big(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p \, dz \Big)^{\sigma} + c \Big(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |F|^s \, dz \Big)^{1/s} + c\lambda. \tag{26}$$

Here, $c \equiv c(n, N, p, \nu, L)$. But using (22) and (23), we have

$$\lambda^{1-\sigma p} \Big(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p \, dz \Big)^{\sigma} \le c\lambda \le c \Big(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p \, dz \Big)^{1/p} + c \Big(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} M^s |F|^s \, dz \Big)^{1/s},$$

where $c \equiv c(c_1, c_2)$. Finally, (24) follows, connecting the last inequalities to (26). □

LEMMA 4
Let $u \in C^0((0,T); L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$ *be a weak solution to (4), where* $2n/(n+2) < p \le 2$*, and the function* $a : C \to \mathbb{R}$ *satisfies (5). Assume that*

$$\Big(\fint_{Q(\varrho^2,\lambda^{(p-2)/2}\varrho)} |Du|^p \, dz \Big)^{1/p} \le c_1 \lambda$$

and

$$\lambda \le c_2 \Big(\fint_{Q(\varrho^2,\lambda^{(p-2)/2}\varrho)} |Du|^p \, dz \Big)^{1/p} + c_2 \Big(\fint_{Q(\varrho^2,\lambda^{(p-2)/2}\varrho)} M^s |F|^s \, dz \Big)^{1/s}$$

hold for some $\lambda > 0$ *and some cylinder* $Q(\varrho^2, \lambda^{(p-2)/2}\varrho) \Subset C$*, where* $s > p$ *is defined in (13) via Theorem 3,* c_1 *and* c_2 *are two given positive constants, and* $M \ge 1$*. Then there exists a constant* $c_3 \equiv c_3(n, N, p, \nu, L, c_1, c_2)$ *such that*

$$\Big(\fint_{Q((1/2^p)\varrho^2,(1/2)\lambda^{(p-2)/2}\varrho)} |Du|^s \, dz \Big)^{1/s} \le c_3 \Big(\fint_{Q(\varrho^2,\lambda^{(p-2)/2}\varrho)} |Du|^p \, dz \Big)^{1/p} + c_3 \Big(\fint_{Q(\varrho^2,\lambda^{(p-2)/2}\varrho)} (1 + M^s |F|^s) \, dz \Big)^{1/s}.$$

Proof
Again, we assume that $Q(\varrho^2, \lambda^{(p-2)/2}\varrho)$ is centered at the origin. This time, we consider the rescaled maps

$$\tilde{u}(x,t) := \frac{u(\lambda^{(p-2)/2}\varrho x, \varrho^2 t)}{\varrho\lambda^{p/2}}, \qquad \tilde{F}(x,t) := \frac{F(\lambda^{(p-2)/2}\varrho x, \varrho^2 t)}{\lambda},$$

with $(x,t) \in Q_1$, and the rescaled coefficients $\tilde{a}(x,t) := a(\lambda^{(p-2)/2}\varrho x, \varrho^2 t)$. The remainder of the proof now follows exactly as in Lemma 3. □

We conclude the section with a couple of elementary results: the first can be promptly adapted from [7, Lemma 2.2]; the second can be found in [8, page 13] with slight modifications.

LEMMA 5
Let $p > 1$, and let $\mu \in [0,1]$; there exists a constant $c \equiv c(n, N, p)$ such that if $v, w \in \mathbb{R}^{nN}$, then

$$(\mu^2 + |A|^2)^{p/2} \le c(\mu^2 + |B|^2)^{p/2} + c(\mu^2 + |B|^2 + |A|^2)^{(p-2)/2}|B - A|^2.$$

LEMMA 6
Let $1 < p < \infty$, and let $\mu \in [0,1]$. There exists a constant $c \equiv c(n, N, p)$, independent of μ, such that for any $A, B \in \mathbb{R}^{nN}$,

$$\begin{aligned}(\mu^2 + |B|^2 + |A|^2)^{(p-2)/2}|B - A|^2 \\ \le c\big\langle(\mu^2 + |B|^2)^{(p-2)/2}B - (\mu^2 + |A|^2)^{(p-2)/2}A, B - A\big\rangle.\end{aligned}$$

When $\mu = |A| = |B| = 0$ and $p < 2$, the quantities involved in the previous inequality are meant to be zero.

4. Proofs of Theorems 1 and 2

By an approximation argument, in Step 1 we reduce the proof of Theorem 1 to proving (9) when the solution has locally bounded gradient. Then we devote the remaining steps to this last task; Theorem 2 then follows easily.

Proof of Theorem 1
Step 1: Approximation. We first show how to approximate the solution u of (4), in a neighborhood of a given cylinder, with a sequence u_ε of solutions to similar problems whose gradients are bounded. Let $Q_{2R} \equiv (t_0 - (2R)^2, t_0 + (2R)^2) \times B_{x_0}(2R) \Subset C$ be as in the statement of Theorem 1, and let $Q_{2\bar{R}} \Subset C$ be a cylinder, concentric with Q_{2R}, with $\bar{R} > R$. Let $\phi_1 : \mathbb{R}^n \to \mathbb{R}$ and $\phi_2 : \mathbb{R} \to \mathbb{R}$ be two standard mollifiers with compact support in B^1 and $(-1, 1)$, respectively, and for all $\varepsilon < (1/2)\,\mathrm{dist}(Q_{2\bar{R}}, \partial C)$

and $(x,t)\in Q_{2\bar R}$, define

$$F_\varepsilon(x,t) := \int_{Q_1} F(x+\varepsilon y, t+\varepsilon s)\phi_1(y)\phi_2(s)\,dy\,ds$$

and

$$a_\varepsilon(x,t) := \int_{Q_1} a(x+\varepsilon y, t+\varepsilon s)\phi_1(y)\phi_2(s)\,dy\,ds.$$

Clearly, $F_\varepsilon \in C^\infty(Q_{2\bar R}, \mathbb{R}^{nN})$ and $a_\varepsilon \in C^\infty(Q_{2\bar R})$. Moreover,

$$F_\varepsilon \to F \quad \text{strongly in } L^{pq}(Q_{2\bar R}, \mathbb{R}^{nN}), \tag{27}$$

$$a_\varepsilon \to a \quad \text{strongly in } L^{t}(Q_{2\bar R}, \mathbb{R}^{nN}),\ \forall t<\infty. \tag{28}$$

Finally, the new functions a_ε satisfy (5). Now we define the map

$$\begin{aligned} u_\varepsilon \in C^0\big((t_0-(2\bar R)^2, t_0+(2\bar R)^2); L^2(B_{x_0}(\bar R), \mathbb{R}^N)\big) \\ \cap L^p\big(t_0-(2\bar R)^2, t_0+(2\bar R)^2; W^{1,p}(B_{x_0}(\bar R), \mathbb{R}^N)\big) \end{aligned}$$

as the unique solution to the following Cauchy-Dirichlet problem:

$$\begin{cases} (u_\varepsilon)_t - \operatorname{div}\big(a_\varepsilon(z)|Du_\varepsilon|^{p-2}Du_\varepsilon\big) = \operatorname{div}(|F_\varepsilon|^{p-2}F_\varepsilon) & \text{in } Q_{2\bar R}, \\ u_\varepsilon \equiv u & \text{on } \partial_p Q_{2\bar R}. \end{cases} \tag{29}$$

The existence of such u_ε follows from the theory of monotone operators or via Galerkin approximation (see [21]); for such problems and their exact meaning, see [8, pages 20–21, 296]. Our aim is now to show

$$Du_\varepsilon \to Du \quad \text{strongly in } L^{p}(Q_{2\bar R}, \mathbb{R}^{nN}). \tag{30}$$

Using the fact that both u and u_ε are weak solutions, we have

$$\begin{aligned} &(u_\varepsilon - u)_t - \operatorname{div}\big(a_\varepsilon(z)(|Du_\varepsilon|^{p-2}Du_\varepsilon - |Du|^{p-2}Du)\big) \\ &\quad = \operatorname{div}\big((a_\varepsilon(z)-a(z))|Du|^{p-2}Du\big) + \operatorname{div}(|F_\varepsilon|^{p-2}F_\varepsilon - |F|^{p-2}F). \end{aligned}$$

Now we test the previous identity with the map $u_\varepsilon - u$, which is possible modulo Steklov averages (for the definition, see [8, pages 11, 21]); note that this is an admissible test map since $u \equiv u_\varepsilon$ on $\partial_p Q_{2\bar{R}}$. After a simple computation, we arrive at

$$\begin{aligned} \sup_{\substack{t_0-(2\bar{R})^2 \le t \\ < t_0+(2\bar{R})^2}} \int_{B_{x_0}(2\bar{R})} |u_\varepsilon(x,t) - u(x,t)|^2 \, dx \\ + \int_{Q_{2\bar{R}}} a_\varepsilon(z) \langle |Du_\varepsilon|^{p-2} Du_\varepsilon - |Du|^{p-2} Du, Du_\varepsilon - Du \rangle \, dz \\ \le c \Big| \int_{Q_{2\bar{R}}} \big(a_\varepsilon(z) - a(z)\big) \langle |Du|^{p-2} Du, Du_\varepsilon - Du \rangle \, dz \Big| \\ + c \Big| \int_{Q_{2\bar{R}}} \langle |F_\varepsilon|^{p-2} F_\varepsilon - |F|^{p-2} F, Du_\varepsilon - Du \rangle \, dz \Big|, \end{aligned} \tag{31}$$

and therefore, using (5),

$$\begin{aligned} \int_{Q_{2\bar{R}}} |Du_\varepsilon|^p \, dz \\ \le c \int_{Q_{2\bar{R}}} \big[|a_\varepsilon(z)| + |a(z)| \big] (|Du|^{p-1} |Du_\varepsilon| + |Du_\varepsilon|^{p-1} |Du| + |Du|^p) \, dz \\ + c \int_{Q_{2\bar{R}}} (|F_\varepsilon| + |F|)^{p-1} (|Du_\varepsilon| + |Du|) \, dz. \end{aligned}$$

Finally, using Young's inequality in a standard way and the definitions of F_ε and a_ε, we get

$$\int_{Q_{2\bar{R}}} |Du_\varepsilon|^p \, dz \le c \int_C |Du|^p + |F|^p \, dz \le c_1. \tag{32}$$

Now we go back to (31). In the following, we use the expression

$$(A, B) \mapsto (|A|^2 + |B|^2)^{(p-2)/2} |B - A|^2, \quad A, B \in \mathbb{R}^{nN},$$

which is already defined in Lemma 6 and involves a singularity when $|A| = |B| = 0$ and $p < 2$. In this case, the meaning of the previous quantity was defined as zero. Using Lemma 6 with $\mu = 0$, together with (5), we find

$$\begin{aligned} \int_{Q_{2\bar{R}}} (|Du_\varepsilon|^2 + |Du|^2)^{(p-2)/2} |Du_\varepsilon - Du|^2 \, dz \\ \le c(n, N, p, \nu) \int_{Q_{2\bar{R}}} a_\varepsilon(z) \langle |Du_\varepsilon|^{p-2} Du_\varepsilon - |Du|^{p-2} Du, Du_\varepsilon - Du \rangle \, dz, \end{aligned}$$

and from (31) we have

$$\int_{Q_{2\bar{R}}} (|Du_\varepsilon|^2 + |Du|^2)^{(p-2)/2} |Du_\varepsilon - Du|^2 \, dz$$
$$\leq c \int_{Q_{2\bar{R}}} |a_\varepsilon(z) - a(z)| |Du|^{p-1} |Du_\varepsilon - Du| \, dz$$
$$+ c \int_{Q_{2\bar{R}}} \big| |F_\varepsilon|^{p-2} F_\varepsilon - |F|^{p-2} F \big| |Du_\varepsilon - Du| \, dz \tag{33}$$

with $c \equiv c(n, N, p, \nu, L)$. Using Young's inequality with $\delta \in (0, 1)$, we find

$$\int_{Q_{2\bar{R}}} (|Du_\varepsilon|^2 + |Du|^2)^{(p-2)/2} |Du_\varepsilon - Du|^2 \, dz$$
$$\leq c(\delta) \int_{Q_{2\bar{R}}} |a_\varepsilon(z) - a(z)|^{p/(p-1)} |Du|^p \, dz + \delta \int_{Q_{2\bar{R}}} |Du_\varepsilon|^p + |Du|^p \, dz$$
$$+ c(\delta) \int_{Q_{2\bar{R}}} \big| |F_\varepsilon|^{p-2} F_\varepsilon - |F|^{p-2} F \big|^{p/(p-1)} \, dz \tag{34}$$

with the constants c depending also on n, N, p, ν, L. Now, recalling that s is the higher integrability exponent defined in (13), we have

$$\int_{Q_{2\bar{R}}} |a_\varepsilon(z) - a(z)|^{p/(p-1)} |Du|^p \, dz$$
$$\leq \left(\int_{Q_{2\bar{R}}} |a_\varepsilon(z) - a(z)|^{ps/((p-1)(s-p))} \, dz \right)^{(s-p)/s} \cdot \left(\int_{Q_{2\bar{R}}} |Du|^s \, dz \right)^{p/s} \overset{(28)}{\to} 0 \tag{35}$$

as $\varepsilon \to 0$. By a dominated convergence argument and (27), we directly have

$$\int_{Q_{2\bar{R}}} \big| |F_\varepsilon|^{p-2} F_\varepsilon - |F|^{p-2} F \big|^{p/(p-1)} \, dz \to 0 \tag{36}$$

as $\varepsilon \to 0$. Taking into account (32), connecting (34)–(36), and finally letting $\delta \to 0$, we obtain

$$\lim_{\varepsilon \to 0} \int_{Q_{2\bar{R}}} (|Du_\varepsilon|^2 + |Du|^2)^{(p-2)/2} |Du_\varepsilon - Du|^2 \, dz = 0. \tag{37}$$

Now, if $p < 2$, using the Hölder inequality and again (32),

$$\int_{Q_{2\bar{R}}} |Du_\varepsilon - Du|^p \, dz$$
$$\leq \left(\int_{Q_{2\bar{R}}} |Du_\varepsilon|^p + |Du|^p \, dz \right)^{1/2} \cdot \left(\int_{Q_{2\bar{R}}} (|Du_\varepsilon|^2 + |Du|^2)^{(p-2)/2} |Du_\varepsilon - Du|^2 \, dz \right)^{1/2};$$

therefore

$$\lim_{\varepsilon\to 0}\int_{Q_{2\bar R}} |Du_\varepsilon - Du|^p\,dz = 0, \tag{38}$$

which proves (30) in the case $p < 2$. If, instead, $p \geq 2$, going back to (33) and using Young's inequality, we get

$$\int_{Q_{2\bar R}} |Du_\varepsilon - Du|^p\,dz \leq \int_{Q_{2\bar R}} (|Du_\varepsilon|^2 + |Du|^2)^{(p-2)/2}|Du_\varepsilon - Du|^2\,dz,$$

and (38) follows from (37). Now we finally show how the validity of (9) in the general case follows from the case when Du is bounded. Therefore, let us assume that (9) holds whenever Du is bounded. We consider the maps $\{u_\varepsilon\}$ defined in (29); the regularity theory for the parabolic p-Laplacian systems applies (see [8, Chapter 8]), and therefore $Du_\varepsilon \in L^\infty(Q_{2R}, \mathbb{R}^{nN})$. Then, by (9),

$$\begin{aligned}\left(\fint_{Q_R} |Du|^{pq}\,dz\right)^{1/q} &\leq \liminf_{\varepsilon\to 0}\left(\fint_{Q_R} |Du_\varepsilon|^{pq}\,dz\right)^{1/q}\\ &\leq c\lim_{\varepsilon\to 0}\left[\fint_{Q_{2R}} |Du_\varepsilon|^p\,dz + \left(\fint_{Q_{2R}} |F_\varepsilon|^{pq}\,dz + 1\right)^{1/q}\right]^d\\ &= c\left[\fint_{Q_{2R}} |Du|^p\,dz + \left(\fint_{Q_{2R}} |F|^{pq}\,dz + 1\right)^{1/q}\right]^d,\end{aligned}$$

where we used (30) and Fatou's lemma to manage for the left-hand side. The remainder of the proof is therefore dedicated to proving (9) under the additional assumption that Du is bounded. Once (9) is proved in the general case, the full statement $|Du|^p \in L^q_{\mathrm{loc}}(C)$ follows via a standard covering argument.

Step 2: A stopping-time argument. We start with the case $p \geq 2$. We define $\lambda_0 > 1$ according to

$$\lambda_0^{1/d} := \left(\fint_{Q_{2R}} |Du|^p\,dz\right)^{1/p} + \left(\fint_{Q_{2R}} M^s|F|^s\,dz\right)^{1/s} + 1, \tag{39}$$

where the number d was defined in (10). The number $M > 1$ is chosen later, *in a universal way* that depends only on the fixed parameters n, p, ν, L. Now pick any two numbers γ, λ such that

$$\frac{R}{2^{8p}} \leq \gamma \leq \frac{R}{2}, \qquad B\lambda_0 := 2^{10(n+2)t}\lambda_0 \leq \lambda. \tag{40}$$

We check that for all $z_0 \in Q_R$,

$$\left(\fint_{Q_{z_0}(\lambda^{2-p}\gamma^2,\gamma)} |Du|^p\,dz\right)^{1/p} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\gamma^2,\gamma)} M^s|F|^s\,dz\right)^{1/s} < \lambda. \tag{41}$$

Indeed, we first remark that if $z_0 \in Q_R$ and $\varrho < R$, since $\lambda > \lambda_0 > 1$ and $p \geq 2$, then $Q_z(\lambda^{2-p}\varrho^2,\varrho) \subset Q_{2R}$, so that, in particular, when γ satisfies (40),

$$\frac{|Q((2R)^2,2R)|}{|Q_{z_0}(\lambda^{2-p}\gamma^2,\gamma)|} > 1. \tag{42}$$

Then

$$\begin{aligned}
&\left(\fint_{Q_{z_0}(\lambda^{2-p}\gamma^2,\gamma)} |Du|^p\,dz\right)^{1/p} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\gamma^2,\gamma)} M^s|F|^s\,dz\right)^{1/s}\\
&\quad\leq \left(\frac{|Q((2R)^2,2R)|}{|Q_{z_0}(\lambda^{2-p}\gamma^2,\gamma)|}\right)^{1/p}\cdot\left(\fint_{Q_{2R}} |Du|^p\,dz\right)^{1/p}\\
&\qquad+\left(\frac{|Q((2R)^2,2R)|}{|Q_{z_0}(\lambda^{2-p}\gamma^2,\gamma)|}\right)^{1/s}\cdot\left(\fint_{Q_{2R}} M^s|F|^s\,dz\right)^{1/s}\\
&\overset{(13),(39),(40),(42)}{<} (2^{10(n+2)p}\lambda^{p-2})^{1/p}\lambda_0^{1/d} \overset{(10)}{=} (2^{10(n+2)p}\lambda^{p-2})^{1/p}\lambda_0^{2/p} \overset{(40)}{\leq} \lambda.
\end{aligned}$$

Now, with λ as in (40), take a point $z_0 \in Q_R$ such that $|Du(z_0)| > \lambda$. By Lebesgue's differentiation theorem, for almost every such point we have

$$\lim_{\varrho\to 0}\left\{\left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p\,dz\right)^{1/p} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2,\varrho)} M^s|F|^s\,dz\right)^{1/s}\right\} > \lambda. \tag{43}$$

Assume that for some $\varrho > 0$ satisfying $\varrho \leq R/2$,

$$\left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p\,dz\right)^{1/p} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2,\varrho)} M^s|F|^s\,dz\right)^{1/s} > \lambda.$$

We note that some such ϱ exist by (43); since, for $\varrho \geq R/2^{8p}$, the opposite inequality holds by (40) and (41), necessarily we conclude that $\varrho < R/2^{8p}$. Therefore we can select a radius $\varrho_{z_0} \leq R/2$ to be the largest for which

$$\left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2,\varrho_{z_0})} |Du|^p\,dz\right)^{1/p} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2,\varrho_{z_0})} M^s|F|^s\,dz\right)^{1/s} = \lambda, \tag{44}$$

in the sense that if $R/2 \geq \varrho > \varrho_{z_0}$, then

$$\left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p\,dz\right)^{1/p} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2,\varrho)} M^s|F|^s\,dz\right)^{1/s} < \lambda.$$

By this argumentation, it must be

$$\varrho_{z_0} < \frac{R}{2^{8p}}. \tag{45}$$

Since $\lambda > 1$ and $p \geq 2$, we immediately have

$$Q_{z_0}\big(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2, (2^{jp}\varrho_{z_0})\big) \subset Q\big((2R)^2, (2R)\big), \quad j \in \{0, \dots, 5\}.$$

Moreover, we observe that for $j \in \{0, \dots, 5\}$, we have

$$\begin{aligned}\frac{\lambda}{8^{jp}} \leq \Big(&\fint_{Q_{z_0}(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2, (2^{jp}\varrho_{z_0}))} |Du|^p \, dz\Big)^{1/p} \\ &+ \Big(\fint_{Q_{z_0}(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2, (2^{jp}\varrho_{z_0}))} M^s |F|^s \, dz\Big)^{1/s} \leq \lambda. \end{aligned} \tag{46}$$

Indeed, the right-hand-side inequality just follows from the choice of ϱ_{z_0}, while as for the left-hand side, the sum of the integrals appearing in (46) can be estimated from below as follows:

$$\begin{aligned}&\Big(\fint_{Q_{z_0}(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2, (2^{jp}\varrho_{z_0}))} |Du|^p \, dz\Big)^{1/p} + \Big(\fint_{Q_{z_0}(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2, (2^{jp}\varrho_{z_0}))} M^s |F|^s \, dz\Big)^{1/s} \\ &\geq \Big(\frac{|Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2, \varrho_{z_0})|}{|Q_{z_0}(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2, (2^{jp}\varrho_{z_0}))|}\Big)^{1/p} \cdot \Big[\Big(\fint_{Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2, \varrho_{z_0})} |Du|^p \, dz\Big)^{1/p} \\ &\quad + \Big(\fint_{Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2, \varrho_{z_0})} M^s |F|^s \, dz\Big)^{1/s}\Big] \overset{(44)}{=} \frac{\lambda}{8^{jp}}.\end{aligned}$$

Now let us consider the level set

$$E(\lambda) := \big\{z \in Q_R \, : \, |Du(z)| > \lambda\big\}.$$

For a.e. $z_0 \in E(\lambda)$, we can find a cube $Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2, \varrho_{z_0}) \subset Q_{2R}$ as constructed above and, in particular, such that (46) holds for $j \in \{0, \dots, 5\}$. Therefore, applying Vitali's covering theorem, we find a family of disjoint cubes $\{Q_i^0\}$ of the type considered up to now:

$$Q_i^0 \equiv Q_{z_i}(\lambda^{2-p}\varrho_{z_i}^2, \varrho_{z_i}) \subset Q_{2R}, \quad z_i \in E(\lambda), \tag{47}$$

such that

$$E(\lambda) \subset \bigcup_{i \in \mathbb{N}} Q_i^1 \cup \text{negligible set}.$$

Here, we have denoted

$$Q_i^1 \equiv Q_{z_i}\big(\lambda^{2-p}(2^{3p}\varrho_{z_i})^2, (2^{3p}\varrho_{z_i})\big).$$

For future convenience, we also introduce

$$Q_i^2 \equiv Q_{z_i}\big(\lambda^{2-p}(2^{4p}\varrho_{z_i})^2, (2^{4p}\varrho_{z_i})\big)$$

and

$$Q_i^3 \equiv Q_{z_i}\big(\lambda^{2-p}(2^{5p}\varrho_{z_i})^2, (2^{5p}\varrho_{z_i})\big).$$

We now deal with the case $p < 2$. The basic change with respect to the case $p \geq 2$, and following the subquadratic scaling introduced by DiBenedetto in [8, page 80], is to use cubes of the type $Q_z(\varrho^2, \lambda^{(p-2)/2}\varrho)$. In this case, λ_0 is still defined as in (39), and γ, λ are again picked according to (40). With $z_0 \in Q_R$, once again, we have

$$\Big(\fint_{Q_{z_0}(\gamma^2,\lambda^{(p-2)/2}\gamma)} |Du|^p\,dz\Big)^{1/p} + \Big(\fint_{Q_{z_0}(\gamma^2,\lambda^{(p-2)/2}\gamma)} M^s|F|^s\,dz\Big)^{1/s} < \lambda. \tag{48}$$

The equivalent of (42) in this case is

$$\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\gamma^2, \lambda^{(p-2)/2}\gamma)|} > 1. \tag{49}$$

Then we have

$$\begin{aligned}
&\Big(\fint_{Q_{z_0}(\gamma^2,\lambda^{(p-2)/2}\gamma)} |Du|^p\,dz\Big)^{1/p} + \Big(\fint_{Q_{z_0}(\gamma^2,\lambda^{(p-2)/2}\gamma)} M^s|F|^s\,dz\Big)^{1/s}\\
&\qquad \leq \Big(\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\gamma^2, \lambda^{(p-2)/2}\gamma)|}\Big)^{1/p} \cdot \Big(\fint_{Q_{2R}} |Du|^p\,dz\Big)^{1/p}\\
&\qquad\quad + \Big(\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\gamma^2, \lambda^{(p-2)/2}\gamma)|}\Big)^{1/s} \cdot \Big(\fint_{Q_{2R}} M^s|F|^s\,dz\Big)^{1/s}\\
&\overset{(13),(39),(40),(49)}{\leq} [2^{10(n+2)p}\lambda^{n((2-p)/2)}]^{1/p}\lambda_0^{1/d} \overset{(10)}{=} [2^{10(n+2)p}\lambda^{n((2-p)/2)}]^{1/p}\lambda_0^{(p(n+2)-2n)/(2p)}\\
&\qquad \overset{(8),(40)}{\leq} \lambda,
\end{aligned}$$

and (48) is proved. An important remark to be made is that, beside needing it for Lemmas 2 and 4 and Kinnunen and Lewis's theorem (see Theorem 3), this is the only point where we need (8). For the remainder, we can proceed exactly as for the case $p \geq 2$ but using the cubes of type $Q_{z_0}(\varrho_{z_0}^2, \lambda^{(p-2)/2}\varrho_{z_0}) \subset Q_{2R}$ instead of those of type $Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2, \varrho_{z_0})$. At the end, we come up with a family of disjoint cubes $\{Q_i^0\}$ of the type

$$Q_i^0 \equiv Q_{z_i}(\varrho_{z_i}^2, \lambda^{(p-2)/2}\varrho_{z_i}) \subset Q_{2R}, \quad z_i \in E(\lambda), \tag{50}$$

such that (45) holds and having the fundamental property that for $j \in \{0, \dots, 5\}$ and all i,

$$\begin{aligned}\frac{\lambda}{8^{jp}} \leq \Big(&\fint_{Q_{z_i}((2^{jp}\varrho_{z_i})^2, \lambda^{(p-2)/2}(2^{jp}\varrho_{z_i}))} |Du|^p \, dz\Big)^{1/p} \\ &+ \Big(\fint_{Q_{z_i}((2^{jp}\varrho_{z_i})^2, \lambda^{(p-2)/2}(2^{jp}\varrho_{z_i}))} M^s|F|^s \, dz\Big)^{1/s} \leq \lambda,\end{aligned} \tag{51}$$

and such that

$$E(\lambda) \subset \bigcup_{i \in \mathbb{N}} Q_i^1 \cup \text{negligible set.}$$

Accordingly, in the case $p < 2$, we are denoting

$$\begin{aligned}Q_i^1 &\equiv Q_{z_i}\big((2^{3p}\varrho_{z_i})^2, \lambda^{(p-2)/2}(2^{3p}\varrho_{z_i})\big), \\ Q_i^2 &\equiv Q_{z_i}\big((2^{4p}\varrho_{z_i})^2, \lambda^{(p-2)/2}(2^{4p}\varrho_{z_i})\big),\end{aligned}$$

and

$$Q_i^3 \equiv Q_{z_i}\big((2^{5p}\varrho_{z_i})^2, \lambda^{(p-2)/2}(2^{5p}\varrho_{z_i})\big).$$

From now on, for the remainder of the proof, when dealing with cubes of type $Q_i^0, \dots, Q_i^3$, we implicitly understand which kind we are using, depending on p.

Step 3: Comparison maps. When $p \geq 2$, on the cube Q_i^2 centered at $z_i := (x_i, t_i)$, we define the map

$$\begin{aligned}v_i \in C^0\big(&(t_i - 2^{8p}\lambda^{2-p}\varrho_i^2, t_i + 2^{8p}\lambda^{2-p}\varrho_i^2); L^2(B_{x_i}(2^{4p}\varrho_i), \mathbb{R}^N)\big) \\ &\cap L^p\big(t_i - 2^{8p}\lambda^{2-p}\varrho_i^2, t_i + 2^{8p}\lambda^{2-p}\varrho_i^2; W^{1,p}(B_{x_i}(2^{4p}\varrho_i), \mathbb{R}^N)\big)\end{aligned}$$

as the unique solution to the Cauchy-Dirichlet problem

$$\begin{cases}(v_i)_t - \operatorname{div}(a_i|Dv_i|^{p-2}Dv_i) = 0 & \text{in } Q_i^2, \\ v_i \equiv u & \text{on } \partial_p Q_i^2\end{cases} \tag{52}$$

(see again [8, pages 20–21, 296]), where

$$a_i := \fint_{Q_i^2} a(z)\,dz.$$

We note that, due to (5),

$$\nu \le a_i \le L. \tag{53}$$

We now find some estimates on v_i. Using the fact that both u and v_i are solutions, we have

$$\begin{aligned}(u - v_i)_t &- \operatorname{div}\big(a_i(|Du|^{p-2}Du - |Dv_i|^{p-2}Dv_i)\big)\\ &= \operatorname{div}\big((a(z) - a_i)|Du|^{p-2}Du\big) + \operatorname{div}(|F|^{p-2}F),\end{aligned}$$

in the weak sense. Now we proceed formally, as in Step 1, by testing the previous equality with the map $\varphi = u - v_i$, which may be justified via Steklov averages. Again, it is crucial that u and v_i agree on the parabolic boundary $\partial_p Q_i^2$. As in Step 1, we obtain the equivalent of (31),

$$\begin{aligned}&\int_{Q_i^2} a_i\langle |Du|^{p-2}Du - |Dv_i|^{p-2}Dv_i, Du - Dv_i\rangle\,dz\\ &\quad\le c\left|\int_{Q_i^2}\big(a_i(z) - a(z)\big)\langle |Du|^{p-2}Du, Du - Dv_i\rangle\,dz\right|\\ &\qquad + c\left|\int_{Q_i^2}\langle |F|^{p-2}F, Du - Dv_i\rangle\,dz\right|,\end{aligned} \tag{54}$$

and, using (53), the equivalent of (32),

$$\int_{Q_i^2} |Dv_i|^p\,dz \le c(n, N, p, \nu, L)\int_{Q_i^2} |Du|^p + |F|^p\,dz. \tag{55}$$

Therefore, recalling that $M \ge 1$ and using the Hölder inequality, we find

$$\fint_{Q_i^2} |Dv_i|^p\,dz \le c\fint_{Q_i^2} |Du|^p\,dz + c\Big(\fint_{Q_i^2} M^s|F|^s\,dz\Big)^{p/s} \overset{(46)}{\le} c\lambda^p, \tag{56}$$

where $c \equiv c(n, N, p, \nu, L)$. Now note that

$$Q_i^1 \subset Q_{z_i}\Big(\frac{\lambda^{2-p}(2^{4p}\varrho_i)^2}{2}, \frac{(2^{4p}\varrho_i)}{2}\Big);$$

therefore, by (56), we can apply Lemma 1, with $Q = Q_i^2$, to get that there exists an absolute constant A_1, depending only on n, N, p, ν, L, such that

$$A_1 \geq 1, \qquad \sup_{Q_i^1} |Dv_i| \leq A_1 \lambda. \tag{57}$$

When $p < 2$, we can proceed in a completely analogous way, invoking Lemma 2 instead of Lemma 1 and using the right kind of cubes, and (57) follows again. We note in particular that (54)–(57) hold for both $p \geq 2$ and $p < 2$.

Now we want to get an estimate for the integral

$$\int_{Q_i^2} (|Du|^2 + |Dv_i|^2)^{(p-2)/2} |Du - Dv_i|^2 \, dz.$$

Using Lemma 6, we have

$$\begin{aligned}
&\frac{\nu}{c(n, N, p)} \int_{Q_i^2} (|Du|^2 + |Dv_i|^2)^{(p-2)/2} |Du - Dv_i|^2 \, dz \\
&\quad \overset{(53)}{\leq} a_i \int_{Q_i^2} \langle |Du|^{p-2} Du - |Dv_i|^{p-2} Dv_i, Du - Dv_i \rangle \, dz \\
&\quad \overset{(54)}{\leq} c \int_{Q_i^2} |a(z) - a_i| |Du|^{p-1} |Du - Dv_i| \, dz + c \int_{Q_i^2} |F|^{p-1} |Du - Dv_i| \, dz.
\end{aligned} \tag{58}$$

Using Young's inequality, with $\delta \in (0, 1)$ we have

$$\begin{aligned}
&\int_{Q_i^2} |a(z) - a_i| |Du|^{p-1} |Du - Dv_i| \, dz \\
&\quad \leq \frac{c}{\delta^{1/(p-1)}} \int_{Q_i^2} |a(z) - a_i|^{p/(p-1)} |Du|^p \, dz + \delta \int_{Q_i^2} |Du|^p + |Dv_i|^p \, dz \\
&\quad \overset{(55)}{\leq} \frac{c}{\delta^{1/(p-1)}} \int_{Q_i^2} |a(z) - a_i|^{p/(p-1)} |Du|^p \, dz + c\delta \int_{Q_i^2} |Du|^p + |F|^p \, dz
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
\int_{Q_i^2} |F|^{p-1} |Du - Dv_i| \, dz &\leq \frac{c}{\delta^{1/(p-1)}} \int_{Q_i^2} |F|^p \, dz + \delta \int_{Q_i^2} |Du|^p + |Dv_i|^p \, dz \\
&\overset{(55)}{\leq} \frac{c}{\delta^{1/(p-1)}} \int_{Q_i^2} |F|^p \, dz + c\delta \int_{Q_i^2} |Du|^p \, dz
\end{aligned}$$

with $c \equiv c(n, N, p, \nu, L)$. Connecting the last two inequalities with (58), we finally have

$$\begin{aligned}\int_{Q_i^2} (|Du|^2 + |Dv_i|^2)^{(p-2)/2} |Du - Dv_i|^2 \, dz \\ \leq \frac{c}{\delta^{1/(p-1)}} \int_{Q_i^2} |a(z) - a_i|^{p/(p-1)} |Du|^p \, dz \\ + \frac{c}{\delta^{1/(p-1)}} \int_{Q_i^2} |F|^p \, dz + c\delta \int_{Q_i^2} |Du|^p \, dz \end{aligned} \tag{60}$$

with $c \equiv c(n, N, p, \nu, L)$, and $\delta \in (0, 1)$ not yet chosen.

We estimate the first integral appearing in the right-hand side of (60). Using the Hölder inequality, we have

$$\int_{Q_i^2} |a(z) - a_i|^{p/(p-1)} |Du|^p \, dz \leq \left(\fint_{Q_i^2} |a(z) - a_i|^b \, dz \right)^{(s-p)/s} \left(\fint_{Q_i^2} |Du|^s \, dz \right)^{p/s} |Q_i^2|,$$

where we have set

$$b := \frac{p}{p-1} \frac{s}{s-p} > 1.$$

We note that

$$\left(\fint_{Q_i^2} |a(z) - a_i|^b \, dz \right)^{(s-p)/s} \leq (2L)^{(b-1)(s-p)/s} [\omega(R)]^{(s-p)/s}$$

as a consequence of (5), (7), (45), and (53), while

$$\left(\fint_{Q_i^2} |Du|^s \, dz \right)^{p/s} \leq c \fint_{Q_i^3} |Du|^p \, dz + c \left(\fint_{Q_i^3} (1 + M^s |F|^s) \, dz \right)^{p/s}$$

as a consequence of Lemma 3 or Lemma 4. Merging the last three estimates with (60), we finally obtain the estimate that we were looking for:

$$\begin{aligned}\int_{Q_i^2} (|Du|^2 + |Dv_i|^2)^{(p-2)/2} |Du - Dv_i|^2 \, dz \\ \leq c \left\{ \frac{[\omega(R)]^{(s-p)/s}}{\delta^{1/(p-1)}} + \delta \right\} \int_{Q_i^3} |Du|^p \, dz \\ + \frac{c[\omega(R)]^{(s-p)/s}}{\delta^{1/(p-1)}} \left(\fint_{Q_i^3} (1 + M^s |F|^s) \, dz \right)^{p/s} |Q_i^0| + \frac{c}{\delta^{1/(p-1)}} \int_{Q_i^3} |F|^p \, dz, \end{aligned} \tag{61}$$

where the constant c depends on the data n, N, p, ν, L, we estimated $|Q_i^3| \leq 4^{10np} |Q_i^0|$, and $\delta \in (0, 1)$ is not yet chosen.

Step 4: Estimates on cubes. Lemma 5 with $\mu = 0$ implies

$$|Du|^p \le c_l |Dv_i|^p + c_l(|Du|^2 + |Dv_i|^2)^{(p-2)/2}|Du - Dv_i|^2,$$

where $c_l \equiv c_l(n, p)$ is the constant appearing in the lemma. Accordingly, we fix the constant

$$A := (1 + 2c_l)A_1, \tag{62}$$

where A_1 is the constant appearing in (57). In this way, A depends only on the data n, N, p, ν, L. We have

$$\begin{aligned}
&\left|\left\{z \in Q_i^1 \;:\; |Du|^p > A\lambda^p\right\}\right| \\
&\le \left|\left\{z \in Q_i^1 \;:\; (|Du|^2 + |Dv_i|^2)^{(p-2)/2}|Du - Dv_i|^2 > A_1\lambda^p\right\}\right| \\
&\quad + \left|\left\{z \in Q_i^1 \;:\; |Dv_i|^p > A_1\lambda^p\right\}\right| \quad (\overset{(57),(62)}{=} 0) \\
&= \left|\left\{z \in Q_i^1 \;:\; (|Du|^2 + |Dv_i|^2)^{(p-2)/2}|Du - Dv_i|^2 > A_1\lambda^p\right\}\right|,
\end{aligned}$$

so that

$$\left|\left\{z \in Q_i^1 \;:\; |Du|^p > A\lambda^p\right\}\right| \le \frac{1}{A_1\lambda^p}\int_{Q_i^1}(|Du|^2 + |Dv_i|^2)^{(p-2)/2}|Du - Dv_i|^2\,dx,$$

and using (61) and (62),

$$\begin{aligned}
\left|\left\{z \in Q_i^1 \;:\; |Du|^p > A\lambda^p\right\}\right| &\le \frac{c}{A\lambda^p}\left\{\frac{[\omega(R)]^{(s-p)/s}}{\delta^{1/(p-1)}} + \delta\right\}\int_{Q_i^3}|Du|^p\,dz \\
&\quad + \frac{c[\omega(R)]^{(s-p)/s}}{A\lambda^p\delta^{1/(p-1)}}\left(\fint_{Q_i^3}(1 + M^s|F|^s)\,dz\right)^{p/s}|Q_i^0| \\
&\quad + \frac{c}{A\lambda^p\delta^{1/(p-1)}}\int_{Q_i^3}|F|^p\,dz.
\end{aligned} \tag{63}$$

We now carefully estimate these three integrals; we note that the constant c just seen depends only on n, p, ν, L. Since we later backtrack to find the exact dependence on q, we are careful to let every constant c be independent of q; given (13), when not essential, we majorize constants as, for example, 2^s by $c = c(p)$. We first provide an estimate for $|Q_i^0|$; by (44), (47), or the analogous expressions for $p < 2$, either of the following inequalities must be true:

$$\left(\frac{\lambda}{2}\right)^p \le \frac{1}{|Q_i^0|}\int_{Q_i^0}|Du|^p\,dz \qquad \text{or} \qquad \left(\frac{\lambda}{2}\right)^s \le \frac{1}{|Q_i^0|}\int_{Q_i^0}M^s|F|^s\,dz.$$

In any case,

$$|Q_i^0| \leq \frac{2^p}{\lambda^p}\int_{Q_i^0} |Du|^p \, dz + \frac{2^s}{\lambda^s}\int_{Q_i^0} M^s |F|^s \, dz. \tag{64}$$

We now split the last integral as follows: for some $\gamma > 0$,

$$\begin{aligned}\frac{1}{\lambda^s}\int_{Q_i^0} M^s|F|^s \, dz &= \frac{1}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}} M^s|F|^s \, dz + \frac{1}{\lambda^s}\int_{Q_i^0\cap\{|F|\leq\gamma\lambda\}} M^s|F|^s \, dz \\ &\leq \frac{1}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}} M^s|F|^s \, dz + M^s\gamma^s|Q_i^0|. \end{aligned} \tag{65}$$

Choosing

$$\gamma^s := \frac{1}{2^{s+1}M^s}, \tag{66}$$

connecting (65) to (64), and reabsorbing $|Q_i^0|/2$, we find the estimate for $|Q_i^0|$ in which we are interested:

$$|Q_i^0| \leq \frac{2^{p+1}}{\lambda^p}\int_{Q_i^0} |Du|^p \, dz + \frac{2^{s+1}}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}} M^s|F|^s \, dz. \tag{67}$$

Now we gain a further estimate, again splitting with some $\tau > 0$:

$$\begin{aligned}\frac{1}{\lambda^p}\int_{Q_i^0} |Du|^p \, dz &= \frac{1}{\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}} |Du|^p \, dz + \frac{1}{\lambda^p}\int_{Q_i^0\cap\{|Du|\leq\tau\lambda\}} |Du|^p \, dz \\ &\leq \frac{1}{\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}} |Du|^p \, dz + \tau^p|Q_i^0| \\ &\overset{(67)}{\leq} \frac{1}{\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}} |Du|^p \, dz + \frac{2^{p+1}\tau^p}{\lambda^p}\int_{Q_i^0} |Du|^p \, dz \\ &\quad + \frac{2^{s+1}\tau^p}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}} M^s|F|^s \, dz. \end{aligned} \tag{68}$$

Choosing

$$\tau^p := \frac{1}{2^{p+2}} \tag{69}$$

and reabsorbing the next-to-last integral into the left-hand side of (68), we conclude with

$$\frac{1}{\lambda^p}\int_{Q_i^0} |Du|^p \, dz \leq \frac{2}{\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}} |Du|^p \, dz + \frac{2^{s-p}}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}} M^s|F|^s \, dz.$$

In particular, from (67) we deduce

$$|Q_i^0| \leq \frac{2^{p+2}}{\lambda^p} \int_{Q_i^0 \cap \{|Du|>\tau\lambda\}} |Du|^p \, dz + \frac{2^{s+2}}{\lambda^s} \int_{Q_i^0 \cap \{|F|>\gamma\lambda\}} M^s |F|^s \, dz. \tag{70}$$

Then we have

$$\begin{aligned} \frac{1}{\lambda^p} \int_{Q_i^3} |Du|^p \, dz &= \frac{|Q_i^3|}{\lambda^p} \fint_{Q_i^3} |Du|^p \, dz \\ &\overset{(46)/(51)}{\leq} |Q_i^3| \\ &\leq 2^{5p(n+2)} |Q_i^0| \\ &\overset{(70)}{\leq} \frac{c}{\lambda^p} \int_{Q_i^0 \cap \{|Du|>\tau\lambda\}} |Du|^p \, dz \\ &\quad + \frac{c}{\lambda^s} \int_{Q_i^0 \cap \{|F|>\gamma\lambda\}} M^s |F|^s \, dz. \end{aligned} \tag{71}$$

(As we said, we omitted a 2^s here.) Similarly, using the Hölder inequality,

$$\begin{aligned} \frac{1}{\lambda^p} \int_{Q_i^3} |F|^p \, dz = \frac{|Q_i^3|}{M^p \lambda^p} \fint_{Q_i^3} M^p |F|^p \, dz &\leq \frac{c|Q_i^0|}{M^p \lambda^p} \Big(\fint_{Q_i^3} M^s |F|^s \, dz \Big)^{p/s} \\ \overset{(46)/(51)}{\leq} \frac{c|Q_i^0|}{M^p} \overset{(70)}{\leq} \frac{c}{M^p \lambda^p} &\int_{Q_i^0 \cap \{|Du|>\tau\lambda\}} |Du|^p \, dz \\ + \frac{c}{M^p \lambda^s} &\int_{Q_i^0 \cap \{|F|>\gamma\lambda\}} M^s |F|^s \, dz. \end{aligned} \tag{72}$$

Finally, since $\lambda > 1$, we have

$$\begin{aligned} \frac{|Q_i^0|}{\lambda^p} \Big(\fint_{Q_i^3} (1 + M^s |F|^s) \, dz \Big)^{p/s} &\overset{(46)/(51)}{\leq} |Q_i^0| \\ \overset{(70)}{\leq} \frac{c}{\lambda^p} \int_{Q_i^0 \cap \{|Du|>\tau\lambda\}} |Du|^p \, dz &+ \frac{c}{\lambda^s} \int_{Q_i^0 \cap \{|F|>\gamma\lambda\}} M^s |F|^s \, dz. \end{aligned} \tag{73}$$

Connecting (71)–(73) to (63), we have the estimate on the cubes which we were looking for:

$$\begin{aligned} &\Big| \Big\{ z \in Q_i^1 \ : \ |Du|^p > A\lambda^p \Big\} \Big| \\ &\quad \leq \frac{c}{A\lambda^p} \Big\{ \frac{[\omega(R)]^{(s-p)/s}}{\delta^{1/(p-1)}} + \frac{1}{M^p \delta^{1/(p-1)}} + \delta \Big\} \int_{Q_i^0 \cap \{|Du|>\tau\lambda\}} |Du|^p \, dz \\ &\qquad + \frac{cM^s}{A\lambda^s} \Big\{ \frac{[\omega(R)]^{(s-p)/s}}{\delta^{1/(p-1)}} + \frac{1}{M^p \delta^{1/(p-1)}} + \delta \Big\} \int_{Q_i^0 \cap \{|F|>\gamma\lambda\}} |F|^s \, dz. \end{aligned}$$

Define

$$G \equiv G(\delta, M, R) := \frac{[\omega(R)]^{(s-p)/s}}{\delta^{1/(p-1)}} + \frac{1}{M^p \delta^{1/(p-1)}} + \delta, \tag{74}$$

and the estimate may be rewritten as

$$\begin{aligned} &\left|\left\{z \in Q_i^1 \;:\; |Du|^p > A\lambda^p\right\}\right| \\ &\quad \le G(\delta, M, R)\left\{\frac{c}{A\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}}|Du|^p\,dz + \frac{cM^s}{A\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}|F|^s\,dz\right\}. \end{aligned} \tag{75}$$

Here, the constant c depends only on n, N, p, ν, L, while γ and τ have been chosen in (66) and (69), respectively. The constant $\delta \in (0, 1)$ is not yet chosen.

Step 5: Final estimate. Here, we prove the validity of (9), provided

$$R \le R_0, \tag{76}$$

where $R_0 \equiv R_0(n, N, p, \nu, L, \omega(\cdot)) > 0$ is a radius that we determine in (82). The general case follows via a standard covering argument.

Using the fact that the cubes $\{Q_i^0\}$ are disjoint, summing up on $i \in \mathbb{N}$ in (75) we have, by (47) or (50), since $A > 1$ by (57), (62),

$$\begin{aligned} &\left|\left\{z \in Q_R \;:\; |Du| > A^{1/p}\lambda\right\}\right| \\ &\quad \le \sum_i \left|\left\{z \in Q_i^1 \;:\; |Du|^p > A\lambda^p\right\}\right| \\ &\quad \le G(\delta, M, R)\left\{\frac{c}{A\lambda^p}\int_{Q_{2R}\cap\{|Du|>\tau\lambda\}}|Du|^p\,dz + \frac{cM^s}{A\lambda^s}\int_{Q_{2R}\cap\{|F|>\gamma\lambda\}}|F|^s\,dz\right\}. \end{aligned}$$

The previous inequality holds for every $\lambda \ge B\lambda_0$ (recall (40)). Therefore, integrating with respect to λ yields

$$\begin{aligned} &\int_{B\lambda_0}^{\infty}\lambda^{pq-1}\left|\left\{z \in Q_R \;:\; |Du| > A^{1/p}\lambda\right\}\right|d\lambda \\ &\quad \le \frac{cG(\delta, M, R)}{A}\int_{B\lambda_0}^{\infty}\lambda^{pq-p-1}\int_{Q_{2R}\cap\{|Du|>\tau\lambda\}}|Du|^p\,dz\,d\lambda \\ &\qquad + \frac{cM^sG(\delta, M, R)}{A}\int_{B\lambda_0}^{\infty}\lambda^{pq-s-1}\int_{Q_{2R}\cap\{|F|>\gamma\lambda\}}|F|^s\,dz\,d\lambda. \end{aligned} \tag{77}$$

We recall that for any measurable function $g \ge 0$ and any $\beta > \alpha > 1$,

$$\int g^\beta\,dx = (\beta - \alpha)\int_0^{+\infty} t^{\beta-\alpha-1}\int_{\{x:g(x)>t\}} g^\alpha\,dx\,dt; \tag{78}$$

then we have, using Fubini's theorem in a standard way,

$$
\begin{aligned}
&\int_{Q_R} |Du|^{pq}\,dz \\
&\quad = pq\int_0^\infty (A^{1/p}\lambda)^{pq-1}\big|\{z\in Q_R : |Du| > A^{1/p}\lambda\}\big|\,d(A^{1/p}\lambda) \\
&\quad = pq\int_0^{B\lambda_0} (\text{same})\,d\lambda + pq\int_{B\lambda_0}^\infty (\text{same})\,d\lambda \\
&\quad \le A^q B^{pq}\lambda_0^{pq}|Q_R| + pqA^q\int_{B\lambda_0}^\infty \lambda^{pq-1}\big|\{z\in Q_R : |Du| > A^{1/p}\lambda\}\big|\,d\lambda \\
&\quad \overset{(62),(77)}{\le} A^q B^{pq}\lambda_0^{pq}|Q_R| \\
&\qquad + \frac{cpqA^qG(\delta,M,R)}{\tau^{pq-p}}\int_0^\infty (\tau\lambda)^{pq-p-1}\int_{Q_{2R}\cap\{|Du|>\tau\lambda\}} |Du|^p\,dz\,d(\tau\lambda) \\
&\qquad + \frac{cpqA^qM^sG(\delta,M,R)}{\gamma^{pq-s}}\int_0^\infty (\gamma\lambda)^{pq-s-1}\int_{Q_{2R}\cap\{|F|>\gamma\lambda\}} |F|^s\,dz\,d(\gamma\lambda) \\
&\quad \overset{(66),(69),(78)}{\le} A^q B^{pq}\lambda_0^{pq}|Q_R| \\
&\qquad + c\frac{q}{q-1}2^{(p+2)q}A^qG(\delta,M,R)\int_{Q_{2R}} |Du|^{pq}\,dz \\
&\qquad + c\frac{pq}{pq-s}A^q2^{pq}M^{pq}G(\delta,M,R)\int_{Q_{2R}} |F|^{pq}\,dz \\
&\quad \overset{(14)}{\le} \tilde{c}A^qB^{pq}\lambda_0^{pq}|Q_R| \\
&\qquad + \tilde{c}2^{(p+2)q}A^q\frac{q}{q-1}G(\delta,M,R)\Big\{\int_{Q_{2R}} |Du|^{pq}\,dz + M^{pq}\int_{Q_{2R}} |F|^{pq}\,dz\Big\},
\end{aligned}
\tag{79}
$$

where $\tilde{c} \equiv \tilde{c}(n, N, p, \nu, L)$ and the dependence on q is explicitly stated. We note that nothing depends on λ_0 except the first term at the right-hand side.

Now we look again at (74), we fix ε_0 so that it only depends on n, and we first choose $\delta > 0$ small enough in order to have

$$
\tilde{c}\delta \le \frac{q-1}{6qA^q2^{(p+2)q}}\varepsilon_0. \tag{80}
$$

Since $\tilde{c}$ depends only on n, N, p, ν, L, q, this also fixes δ as a number depending only on n, N, p, ν, L, q. We can therefore select $M \equiv M(\tilde{c}, \delta) \equiv M(n, N, p, \nu, L, q) > 1$

large enough in order to have (*large-M-inequality principle*)

$$\frac{\tilde{c}}{M^p} \leq \frac{(q-1)\delta^{1/(p-1)}}{6qA^q 2^{(p+2)q}}\varepsilon_0. \tag{81}$$

We may finally select $R_0 \equiv R_0(n, N, p, \nu, L, q, \omega(\cdot)) > 0$ small enough to ensure

$$\tilde{c}[\omega(R_0)]^{(s-p)/s} \leq \frac{(q-1)\delta^{1/(p-1)}}{6qA^q 2^{(p+2)q}}\varepsilon_0. \tag{82}$$

Therefore, for every R satisfying (76), taking into account (74) and inequalities (80) – (82), we obtain

$$\tilde{c}2^{(p+2)q}A^q \frac{q}{q-1}G(\delta, M, R) \leq \frac{1}{2}\varepsilon_0. \tag{83}$$

Plugging this last inequality into (79) and finally passing to averages, we get, by (40),

$$\left(\fint_{Q_R} |Du|^{pq}\,dz\right)^{1/q} \leq c\lambda_0^p + c\left(\fint_{Q_{2R}} |F|^{pq}\,dz\right)^{1/q} + \left(\varepsilon_0 \fint_{Q_{2R}} |Du|^{pq}\,dz\right)^{1/q}, \tag{84}$$

where now c depends also on q; that is, $c \equiv c(n, N, p, \nu, L, q)$. Now we recall the choice of λ_0 in (39), that M has been chosen in (81) depending on n, N, p, ν, L, q, and we finally apply the Hölder inequality to have

$$\begin{aligned} \lambda_0^p &\leq c\left[\fint_{Q_{2R}} |Du|^p\,dz + \left(\fint_{Q_{2R}} |F|^s\,dz\right)^{p/s} + 1\right]^d \\ &\leq c\left[\fint_{Q_{2R}} |Du|^p\,dz + \left(\fint_{Q_{2R}} |F|^{pq}\,dz\right)^{1/q} + 1\right]^d, \end{aligned} \tag{85}$$

where again $c \equiv c(n, N, p, \nu, L, q)$ and d is defined in (10). Joining the last estimate and (84), and reabsorbing at the left-hand side the last integral in (84) by a covering and iteration argument (see [13, Corollary 6.1, Lemma 6.1]), we obtain (9) for all $R \leq R_0$. At this stage, the constant c in (9) depends only on n, N, p, ν, L, q and not on $\omega(\cdot)$; it is only R_0 that depends on $\omega(\cdot)$ through (82). The case $R \geq R_0$ follows from a standard covering argument; at this point, c also depends on $\omega(\cdot)$ via the covering coefficients that depend on R_0. The proof of Theorem 1 is now complete. □

Proof of Theorem 2

By carefully looking at the proof of Theorem 1, we can immediately infer the statement of Theorem 2. Indeed, we first chose δ in (80) and M in (81); then, when dealing with

(82), we may choose the number $\varepsilon \equiv \varepsilon(n, N, p, \nu, L, q) > 0$ in the statement of Theorem 2 small enough to have

$$\tilde{c}\varepsilon^{(s-p)/s} < \frac{(q-1)\delta^{1/(p-1)}}{6qA^q 2^{(p+2)q}}\varepsilon_0,$$

and then we just estimate as follows:

$$\tilde{c}[\omega(R_0)]^{(s-p)/s} \leq \tilde{c}[a]_{BMO}^{(s-p)/s} < \frac{(q-1)\delta^{1/(p-1)}}{6qA^q 2^{(p+2)q}}\varepsilon_0. \tag{86}$$

The last inequality allows us to recover (83), and the remainder of the proof works without any further change. This also implies that the constant c in the final estimate (9) depends only on n, N, p, ν, L, q. □

Remark 2

The proof of Theorem 1 deserves some comments. The stopping-time argument in Step 2 clearly depends on the choice of the positive quantity M, via the stopping-time radius ϱ_{z_0} determined by the first occurrence of the equality (44). On the other hand, the constant M is chosen at the very end, in (81). Actually, the proof should be read backward: the choice of M is influenced only by the constant $\tilde{c}$ appearing in (79). In turn, $\tilde{c}$ is universal, in the sense that it only depends on n, N, p, ν, L, q and not on the cubes Q_i^0 determined in Step 2. Therefore, once the choice of $\tilde{c}$, and therefore of M, is done in the universal way dictated by (79)–(81), we can restart from Step 2, finding the family of cubes $\{Q_i^0\}$ with a fixed value of M, and then proceed toward the end through Steps 3–5.

Remark 3

The technique developed here allows us to get a rather precise dependence of the constant c in estimate (9) on the integrability parameter q. We observe interesting similarities with the estimates obtained in the elliptic case (3) via maximal function techniques in [1], [10], and [14]; this is not surprising since the local use of estimates (18) and (21), in combination with Vitali's covering lemma, in some sense emulates the use of the maximal function. For the sake of simplicity, we confine ourselves to the model problem (1). We go back to (79); since $a(z) \equiv 1$, then we have $\omega(R) \equiv 0$. Then, taking δ and M in such a way as to obtain equalities in (80) and (81), respectively, and combining this with (79), recalling that B depends only on n, p by (40), we obtain

$$\left(\fint_{Q_R} |Du|^{pq}\,dz\right)^{1/q} \leq cA\lambda_0^p + c\left(\frac{q}{q-1}A^q 2^{(p+2)q}\right)^{p/(p-1)}\left(\fint_{Q_R} |F|^{pq}\,dz\right)^{1/q},$$

and therefore, by (85) and the covering argument,

$$\begin{aligned}&\Big(\fint_{Q_R} |Du|^{pq}\,dz\Big)^{1/q}\\&\quad\le cA\Big[\fint_{Q_{2R}} |Du|^p\,dz + \Big(\frac{q}{q-1}A^q 2^{(p+2)q}\Big)^{p/(p-1)}\Big(\fint_{Q_{2R}} |F|^{pq}\,dz\Big)^{1/q} + 1\Big]^d.\end{aligned} \tag{87}$$

The previous a priori estimate reveals the same asymptotic behavior for $q \searrow 1$ both of the constant appearing in the Hardy-Littlewood maximal function estimate (see [2]) and of the a priori elliptic estimates of [1], [10], and [14]. On the other hand, this is harmless: when q is approaching 1, a priori estimates bounding the L^{pq}-norm of Du by the L^{pq}-norm of F are simply given by Theorem 3, which works for any small choice of δ, with all constants remaining bounded as $\delta \searrow 0$ (see [16]). Note that the scaling of estimates (9) and (87) is in perfect accordance with that of the known a priori estimates for the evolutionary p-Laplacian operator: when $F \equiv 0$, letting $q \nearrow \infty$ in (87), we obtain, up to an absolute constant, the sup estimates (18) and (21) for $\theta = \gamma^2 = R^2$.

5. A few possible extensions

For the sake of simplicity, and in order to emphasize the main ideas, we have up to now restricted ourselves to the analysis of the model cases (1) and (4); nevertheless, the methods presented in this article immediately apply to several more general situations.

As for the right-hand side, as we already mentioned, we could have considered instead of (4) the system

$$u_t - \operatorname{div}\big(a(z)|Du|^{p-2}Du\big) = \operatorname{div} f,$$

which is equivalent to (4) through

$$f = |F|^{p-1}\frac{F}{|F|} \longleftrightarrow F = |f|^{1/(p-1)}\frac{f}{|f|} \tag{88}$$

and (beside complicating the exponents in the proof) leads to the more awkward statement "if $f \in L^{pq/(p-1)}_{\mathrm{loc}}$, then $|Du|^p \in L^q_{\mathrm{loc}}$," together with an equally awkward estimate of the type (9).

In what follows, we describe other parabolic problems to which our techniques apply, sketching the main modifications to the proof as far as the estimates in Steps 2–5 from Section 4 are concerned. The approximation part of Step 1 can be easily reconstructed as in Section 4.

The vectorial case $N > 1$, and different operators. The reader recognizes that the main property of the evolutionary p-Laplacian operator used in the proof of Theorem 1, apart from the obvious monotonicity and growth properties used in the comparison estimates of Steps 1 and 3 from Section 4, is the possibility to get the explicit L^∞-bounds (18) and (21). In turn, these are used to get the fundamental Lemmas 1 and 2 and, eventually, the crucial estimate (56) on the comparison map v_i. This observation allows us to extend our results to a family of degenerate parabolic systems whose special structure allows for the L^∞-bounds (18) and (21) and to which the results of Theorems 1 and 2 extend.

We may consider systems of the type

$$u_t - \operatorname{div}\big[a(z)g(|Du|)Du\big] = \operatorname{div}(|F|^{p-2}F). \tag{89}$$

The function a is in VMO, and the assumptions on the function $g : \mathbb{R}^+ \to \mathbb{R}^+$, which is in $C^1(\mathbb{R} \setminus \{0\})$, are different depending on p:

$$\text{if } p \geq 2, \qquad g'(s) \geq 0, \quad \forall\, s > 0, \tag{90}$$

$$\nu s^{p-2} \leq g(s) \leq L s^{p-2}, \quad \forall\, s > 0, \tag{91}$$

$$\frac{|g'(s)|s}{g(s)} \leq \begin{cases} L & \text{if } p \geq 2, \\ \theta(<1) & \text{if } p < 2, \end{cases} \qquad \forall\, s > 0, \tag{92}$$

$$\big\langle g(|w_2|)w_2 - g(|w_1|)w_1, w_2 - w_1\big\rangle \geq \nu(\mu^2 + |w_1|^2 + |w_2|^2)^{(p-2)/2}|w_2 - w_1|^2, \tag{93}$$

for all $w_1, w_2 \in \mathbb{R}^{nN}$. Under the previous assumptions, the conditions stated on [8, page 217] are satisfied, and the solutions to the comparison system

$$v_t - \operatorname{div}\big[\tilde{a}g(|Dv|)Dv\big] = 0$$

satisfy the gradient bounds (18) and (21) for $p \geq 2$ and $p < 2$, respectively. Accordingly, the main modification in the proof of Theorem 1 is the use of the comparison system

$$\begin{cases} (v_i)_t - \operatorname{div}\big[a_i g(|Dv_i|)Dv_i\big] = 0 & \text{in } Q_i^2, \\ v_i \equiv u & \text{on } \partial_p Q_i^2 \end{cases} \tag{94}$$

instead of (52). Then one gets the upper bound (56), and the remainder of the proof follows thanks to the growth, ellipticity, and monotonicity assumptions (91)–(93), thereby replacing the use of Lemma 6. Finally, the integrability results of Theorems 1 and 2 follow.

Another left-hand-side structure that we can treat with the methods proposed here is the one already considered in [19] for the elliptic case. Let us consider an $(n^2 \times N^2)$-tensor $A(z) \equiv \{A_{i,j}^{\alpha,\beta}(z)\}$, defined on C and whose entries are strongly VMO in the sense of Definition 1: assume that the tensor $A(z)$ satisfies the following ellipticity and boundedness conditions:

$$\nu|\lambda|^2 \le \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} A_{i,j}^{\alpha,\beta}(z)\lambda_i^\alpha \lambda_j^\beta \le L|\lambda|^2, \quad \forall\, \lambda \in \mathbb{R}^{nN},\ \forall\, x \in C.$$

Then the result of Theorem 1 also holds for solutions to the system

$$u_t - \operatorname{div}\Big(\langle A(z)Du, Du\rangle^{(p-2)/2} Du\Big) = \operatorname{div}(|F|^{p-2}F). \tag{95}$$

Accordingly, assuming the tensor $\{A_{i,j}^{\alpha,\beta}(z)\}$ to have BMO entries, the analog of Theorem 2 for solutions of (95) also follows. The proofs for (95) are very much similar to those already considered for Theorems 1 and 2.

We can also consider systems of the type

$$u_t - \operatorname{div}\big[\tilde{g}(z, |Du|)Du\big] = \operatorname{div}(|F|^{p-2}F) \tag{96}$$

with $\tilde{g} : C \times \mathbb{R}^+ \to \mathbb{R}^+$. This time, we assume that the function

$$w \in \mathbb{R}^{nN} \mapsto \tilde{g}(z_0, |w|) \in \mathbb{R}$$

satisfies the assumptions required on the function g appearing in (89), that is, (90)–(93), uniformly with respect to $z_0 \in C$. Moreover, the following type of continuity (or, rather, limited discontinuity) assumption is required with respect to the variable z:

$$\big|\tilde{g}(z_2, |w|) - \tilde{g}(z_1, |w|)\big| \le L\omega(|z_2 - z_1|)(1 + |w|)^{p-2} \tag{97}$$

for every $w \in \mathbb{R}^{nN}$ and $z_1, z_2 \in C$, where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded, nondecreasing function such that

$$\lim_{R \to 0} \omega(R) = \ell \ge 0. \tag{98}$$

If $\ell = 0$, then $\tilde{g}$ is continuous. In this case, the only differences are the following. First, the comparison function v_i is now defined as the unique solution to the new comparison system

$$\begin{cases}(v_i)_t - \operatorname{div}\big[\tilde{g}(z_i, |Dv_i|)Dv_i\big] = 0 & \text{in } Q_i^2,\\ v_i \equiv u & \text{on } \partial_p Q_i^2,\end{cases}$$

where we recall that z_i is the center of the cylinder Q_i. Once again, the L^∞-bound in (56) follows for v_i, thanks to the assumptions satisfied by $w \mapsto \tilde{g}(z_0, w)$. Second, estimate (59) must be worked out directly using (97), and it is not necessary to use the Hölder inequality there; in particular, the use of the higher integrability result of Theorem 3 can be avoided, and instead of the quantity at the left-hand side of (45), one can use the simpler

$$\left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p\,dz\right)^{1/p} + \left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} M^p|F|^p\,dz\right)^{1/p}.$$

Then, following the proof of Section 4, we finally come to the following theorem.

THEOREM 4
Let $u \in C^0((0,T); L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega,\mathbb{R}^N))$ be a weak solution to (96), where p satisfies (8), the function $\tilde{g}$ is as described above, and $\omega(\cdot)$ satisfies (98) with $\ell = 0$. Assume that $|F|^p \in L^q_{\mathrm{loc}}(C)$ for some $q > 1$. Then $|Du|^p \in L^q_{\mathrm{loc}}(C)$. Moreover, estimate (9) holds, where d is as in (10).

The appropriate analog of Theorem 2 is instead the following.

THEOREM 5
Let $u \in C^0((0,T); L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega,\mathbb{R}^N))$ be a weak solution to (96), where p satisfies (8), the function $\tilde{g}$ is as described above, and $\omega(\cdot)$ satisfies (98). Fix $q > 1$, and assume that $|F|^p \in L^q_{\mathrm{loc}}(C)$. Then there exists a number $\varepsilon \equiv \varepsilon(n,N,p,\nu,L,q) > 0$ such that if $\ell \le \varepsilon$, then $|Du|^p \in L^q_{\mathrm{loc}}(C)$. Moreover, there exists a constant $c \equiv c(n,N,p,\nu,L,q) > 1$ such that if $Q_{2R} \Subset C$, then (9) holds with d as in (10).

The scalar case $N = 1$. In the scalar case, the L^∞-bounds (18) and (21) are true for general parabolic equations with no additional structure properties as dependence upon Du specified via the quantity $|Du|$; this is a peculiarity of the case where $N = 1$ (again, see [8]). Therefore, we consider a general parabolic equation of the type

$$u_t - \operatorname{div}[a(z)A(Du)] = \operatorname{div}(|F|^{p-2}F), \tag{99}$$

where the vector field $A : \mathbb{R}^N \to \mathbb{R}^N$ is $C^1(\mathbb{R} \setminus \{0\})$ and satisfies the following growth and ellipticity assumptions:

$$|A(w)| + |DA(w)|(\mu^2 + |w|^2)^{1/2} \le L(\mu^2 + |w|^2)^{(p-1)/2}, \tag{100}$$

$$DA(w)\lambda \otimes \lambda \ge \nu(\mu^2 + |w|^2)^{(p-2)/2}|\lambda|^2, \tag{101}$$

for every $w, \lambda \in \mathbb{R}^n$, where, as usual, $0 < \nu \le L$ and $\mu \in [0, 1]$. It is then standard to verify that assuming (101) implies the existence of a constant $c \equiv c(n, p, \nu) > 0$ such that the monotonicity condition

$$\langle A(w_2) - A(w_1), w_2 - w_1 \rangle \ge c(\mu^2 + |w_1|^2 + |w_2|^2)^{(p-2)/2}|w_2 - w_1|^2$$

holds for all $w_1, w_2 \in \mathbb{R}^n$. Under assumptions (100) and (101), the results of Theorems 1 and 2 hold for general weak solutions to equation (99) with the same proofs given in Section 4; the only change comes again when considering (52), which is now replaced by the equation

$$\begin{cases} (v_i)_t - \operatorname{div}[a_i A(Dv_i)] = 0 & \text{in } Q_i^2, \\ v_i \equiv u & \text{on } \partial_p Q_i^2. \end{cases}$$

For weak solutions to the parabolic equation (99), Theorems 1 and 2 hold with exactly the same proofs as in Section 4. As in the vectorial case, we can also consider more general equations such as

$$u_t - \operatorname{div} \tilde{A}(z, Du) = \operatorname{div}(|F|^{p-2}F). \tag{102}$$

Here, we assume that for every $z \in C$, the vector field $w \mapsto \tilde{A}(z, w)$ satisfies assumptions (100) and (101) uniformly with respect to $z \in C$. Moreover, the map $\tilde{A} : C \times \mathbb{R}^n \to \mathbb{R}^n$ is required to satisfy the continuity property

$$|\tilde{A}(z_2, w) - \tilde{A}(z_1, w)| \le \omega(|z_2 - z_1|)(1 + |w|)^{p-1},$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is the usual bounded, nondecreasing function. At this point, the results of Theorems 4 and 5 follow for weak solutions to (102). Note that any type of modulus of continuity is allowed (see [11]).

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