to appear in “Duke Mathematical Journal”.

GRADIENT ESTIMATES FOR A CLASS OF PARABOLIC SYSTEMS

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Abstract. We establish local Calderón & Zygmund type estimates for a class of parabolic problems whose model is the non-homogeneous, degenerate/singular parabolic p-Laplacean system

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F)\ ,$$

proving that

$$F \in L^q_{\rm loc} \Longrightarrow Du \in L^q_{\rm loc} \qquad \forall\ q \geq p\ .$$

We also treat systems with discontinuous coefficients of VMO type.

1. Introduction

The aim of this paper is to present Calderón-Zygmund type estimates for weak solutions to a class of degenerate/singular parabolic systems, and equations, a prominent model example of which is the non-homogeneous, parabolic p-Laplacean system

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F)\ , \qquad\qquad p > \frac{2n}{n+2}\ , \tag{1}$$

considered in the cylindrical domain $C := \Omega \times [0,T)$. Here $\Omega \subset \mathbb{R}^n$ is a bounded open set, and $u \in C^0((0,T); L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$, $N \geq 1$, while $F \in L^p(C,\mathbb{R}^{nN})$. Such a system is degenerate when $p > 2$, and singular when $p < 2$; the lower bound on the exponent p assumed in (1) is standard in the theory of the parabolic p-Laplacean operator, and unavoidable for the type of regularity we are considering here.

For system (1) we prove that

$$F \in L^q_{\rm loc}(C,\mathbb{R}^{nN}) \quad \Longrightarrow \quad Du \in L^q_{\rm loc}(C,\mathbb{R}^{nN}) \tag{2}$$

for any $q \geq p$. In the elliptic, stationary case

$$\operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F)\ , \tag{3}$$

the result in (2) was essentially obtained by T. Iwaniec [14] in the scalar case ($N = 1$), and by DiBenedetto & Manfredi [10] for systems ($N > 1$). The extension to anisotropic elliptic equations with possibly discontinuous, vanishing mean oscillation (VMO) coefficients has been achieved by Kinnunen & Zhou [19, 20], while a class of general non-linear elliptic equations and systems in divergence form, under non-standard growth conditions, has been treated by the authors [1].

There has recently been a great deal of work concerning the integrability properties of weak and “very weak” solutions to systems similar to (1), see [16, 17, 18]. In particular, in a recent, interesting paper [16], Kinnunen & Lewis proved higher integrability of the spatial gradient for solutions of general non-linear parabolic systems with p-growth including (1), introducing a localization method to overcome the lack of homogeneity of parabolic systems with p-growth when $p \neq 2$. They come up with a sort of reverse-type Hölder inequality. The new ingredient offered by these authors is a suitable application of DiBenedetto's intrisic geometry method for degenerate/singular parabolic systems [8], in the setting of Gehring's type estimates. Subsequently, Misawa [22] considered higher integrability of the gradient of solutions to (1), assuming $F \in L^\infty$, and therefore in L^q for every $q > 1$.

In this paper, by means of a new technique, we will use the result of Kinnunen & Lewis, and partially some methods adapted from [5] and [1], to be finally able to prove the natural integrability result in (2).

A main difficulty of the problem is that no use of classical Harmonic Analysis tools can be made here: system (1) is non-linear in the gradient, and therefore the use of singular integrals is ruled out, while, since it is degenerate/singular, and scales differently in space and time, no maximal function operator is naturally associated with the problem. We shall therefore adopt again an intrinsic geometry viewpoint, arguing directly on certain Calderón-Zygmund type covering arguments, and completely avoiding the use of the maximal function operator, or of other Harmonic Analysis principles as the "good-λ inequality" one. A peculiar aspect of our work, that allows to treat the general situation considered here, is that instead of using the $C^{1,\alpha}$ estimates for the homogeneous ($F \equiv 0$) p-Laplacean systems, as done in [10, 14, 22] both for the elliptic and the parabolic case, we shall only use the $C^{0,1}$ estimates [8], which immediately exhibit the right scaling properties when considered on "intrinsic cylinders", and perfectly fit in this context. This is a natural attempt, since we want to prove L^q estimates for Du, whose limit case is indeed given by the $C^{0,1}$ estimates; anyway, the proof is quite delicate. An approach to gradient estimates for equations in divergence form, making use of $C^{0,1}$ estimates, and working via maximal functions, has been introduced in the elliptic, homogeneous case by Caffarelli & Peral in [5]; such an approach works for (homogeneous) parabolic equations only when $p = 2$ [24], again for the reasons explained above. As already mentioned, it is worth pointing out that we cannot use here the so called "good-λ-inequality" principle; we shall rather replace it with a new, direct argument, that we like to call the "large-M-inequality" principle, see (81) below. We like to mention that, apart from the different scaling procedures adopted for the singular and degenerate cases, the proof offered here does not distinguish between the cases $p < 2$ and $p \geq 2$.

Our results will cover a more general class of degenerate/singular parabolic systems, of the type

$$u_t - \operatorname{div}(a(z)|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F) , \tag{4}$$

whose coefficients $a(z) \equiv a(x,t)$ may be discontinuous in a VMO/BMO fashion, see Section 2, for which we still prove (2); furthermore, extensions involving operators different from the p-Laplacean are outlined in Section 5. We shall also derive natural, and neat, local estimates for solutions, in the form of certain non-homogeneous reverse-type Hölder inequalities, see (9); here the non-homogeneity of the estimates precisely reflects that of the system (space/time) via the "scaling deficit exponent" d introduced in (10) below.

The problem of deriving Calderón-Zygmund type estimates for elliptic and parabolic equations, eventually with discontinuous coefficients, is a classical one, and already has a long tradition. In the elliptic and scalar case, it has been usually faced via Harmonic Analysis tools such as: non-linear commutators [6], or Riezs transform [15], or the maximal function operator [19]; see also [12, 23]. Parabolic equations with coefficients of VMO/BMO type have been treated only in the linear case, and in particular, again when $p = 2$, making use of Harmonic Analysis tools such as non-linear commutators [3], and, more recently, of the maximal function operator [4]; needless to say, such ingredients are not available in the case of the evolutionary p-Laplacean operator.

Acknowledgments. The work of the authors was partially supported by MIUR, via the project "Calcolo delle Variazioni" (PIN 2004).

2. Results

General notation. We establish some notation, in addition to what was given in the Introduction. By "cylinder" $Q_z(\theta, \varrho) \subset \mathbb{R}^{n+1}$, "centered" at the point $z \equiv (x, t) \in \mathbb{R}^{n+1}$, with $\theta, \varrho > 0$, we will always mean a set of the type $Q_z(\theta, \varrho) = B_x(\varrho) \times (t-\theta, t+\theta)$, where, as usual, $B_x(\varrho) := \{y \in \mathbb{R}^N \ : \ |x - y| < \varrho\}$; with abuse of terminology such cylinders will be also called "cubes" in the following. As a partial exception we will write B^1 to denote the unit ball centered at the origin of $\mathbb{R}^N$. When not essential, the center of a cylinder will not be specified i.e.: $Q(\theta, \varrho) \equiv Q_z(\theta, \varrho)$. In the case of the standard parabolic cylinders, i.e. when $\theta = \varrho^2 = R^2$, we shall simply write $Q_R \equiv Q(R^2, R)$. The parabolic boundary $\partial_p Q$ of a cylinder $Q_{\tilde{z}}(\theta, \varrho)$ is the union of the lower base $B_{\tilde{x}}(\varrho) \times \{\tilde{t} - \theta\}$ and the side surface $\{|x - \tilde{x}| = \varrho\} \times [\tilde{t} - \theta, \tilde{t} + \theta]$. Adopting a usual convention, c will denote a constant whose value may change in any two occurrences, and only the relevant dependences will be specified, as e.g. $c(\gamma, p)$; particular constants will be denoted by c_1, $\tilde{c}$ and the like. For the Lebesgue measure of a measurable set A, we shall employ either of the notations $|A| = \text{meas}(A)$; then we define the mean value on a cylinder $Q \subset \mathbb{R}^{n+1}$ of an integrable function $v \in L^1(Q)$ by

$$(v)_Q \equiv \fint_Q v \, dx := \frac{1}{|Q|} \int_Q v \, dx \ .$$

When $Q = Q_R$ we will also employ the notation $(v)_R \equiv (v)_{Q_R}$.

Strong VMO/BMO functions. Here we shall define the class of coefficients $a(z) \equiv a(x, t)$ we are using when treating systems of the type (4). In order to preserve the basic parabolicity properties of the systems, and allowing a degeneration caused only by the presence of the factor $|Du|^{p-2}$ in (4), we shall always assume that the function $a : C \to \mathbb{R}$ satisfies

$$0 < \nu \leq a(z) \leq L < \infty \ , \qquad \forall \ z \in C \ . \tag{5}$$

Definition 1. *We say that a function $a(z)$ satisfies the strong VMO condition if*

$$\lim_{R \to 0} \omega(R) = 0 \tag{6}$$

where

$$\omega(R) := \sup_{Q \subset\subset C} \fint_Q |a(z) - (a)_Q|\, dz \tag{7}$$

and the supremum is taken among all cylinders of the type $Q_z(\theta, \varrho)$ with $\varrho \leq R$ and $\theta \leq R^2$. We say that the function $a(z)$ satisfies the strong BMO condition if

$$[a]_{BMO} := \sup_{R>0} \omega(R) < \infty\ .$$

As a remark to this definition, to adapt to the nonlinear parabolic structure we are allowed to pick more cylinders with respect to a usual elliptic-style VMO/BMO definition [25], allowing for the size of the space radius ϱ of the cylinder Q to be unrelated to the time height θ. This class includes for instance all continuous coefficients $a(z)$, and it is large enough to include many possibly discontinuous functions. For instance, in (4) we may take $a(x,t) = b(x)c(t)$, where both $b(x)$ and $c(t)$ are usual VMO/BMO functions, in Ω and $[0,T)$ respectively, and satisfying (5). The strong VMO/BMO condition is in our opinion the natural one in order to treat situations as in (4). Indeed, when dealing with partial differential equations, especially elliptic and parabolic ones, the notion of VMO/BMO is usually given using a family of cubes or cylinders that are relevant both for the scaling properties and for the geometry of the equation. Since the works of DiBenedetto, see [8] and related references, it is known that the natural class of cylinders $Q(\theta, \varrho)$ occurring in connection with (1) is the one having the ratio ϱ/θ depending on the solution u itself, via quantities like for instance $|(Du)_Q|^{p-2}$, which are a priori arbitrary, and not related to the coefficient $a(z)$. Therefore, when treating such problems, we have to allow for a larger freedom in the choice of the suitable VMO/BMO-like definition. Anyway, the class considered here is already, implicitly used in [22].

Main results. When $F \in L^p(C, \mathbb{R}^{nN})$ is a vector field, following [8] pages 17 and 215, a weak solution to system (4) is a map

$$u \in C^0((0,T); L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$$

such that for every $0 < t_1 < t_2 < T$

$$\int_\Omega u\varphi(x,t)\, dx\Big|_{t_1}^{t_2} - \int_\Omega \int_{t_1}^{t_2} u\varphi_t + a(z)\langle |Du|^{p-2}Du, D\varphi\rangle\, dz$$
$$= -\int_\Omega \int_{t_1}^{t_2} \langle |F|^{p-2}F, D\varphi\rangle\, dz$$

for every test function $\varphi \in W^{1,2}_{\text{loc}}(0,T; L^2(\Omega, \mathbb{R}^N)) \cap L^p_{\text{loc}}(0,T; W^{1,p}_0(\Omega, \mathbb{R}^N))$. When dealing with weak solutions we shall always adopt the formulation via Steklov averages; see again DiBenedetto's book [8], pages 11 and 21.

Theorem 1. *Let $u \in C^0((0,T); L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution to* (4)*, where*

$$p > \frac{2n}{n+2} \tag{8}$$

and the function $a : C \to \mathbb{R}$ satisfies (5) *and is strongly* VMO. *Assume that $|F|^p \in L^q_{\rm loc}(C)$ for some $q > 1$. Then $|Du|^p \in L^q_{\rm loc}(C)$. Moreover there exists a constant $c \equiv c(n, N, p, \nu, L, q, \omega(\cdot)) > 1$ such that if $Q_{2R} \subset\subset C$ then*

$$\left(\fint_{Q_R} |Du|^{pq}\, dz\right)^{\frac{1}{q}} \le c \left[\fint_{Q_{2R}} |Du|^p\, dz + \left(\fint_{Q_{2R}} |F|^{pq}\, dz + 1\right)^{\frac{1}{q}}\right]^d , \tag{9}$$

where

$$1 \le d := \begin{cases} \dfrac{p}{2} & \textit{if } p \ge 2 \\ \dfrac{2p}{p(n+2) - 2n} & \textit{if } p < 2 \,. \end{cases} \tag{10}$$

We also have a result concerning coefficients $a(z)$ which are not necessarily VMO, but rather have suitably small BMO semi-norm:

Theorem 2. *Let $u \in C^0((0,T); L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution to* (4)*, where the function $a : C \to \mathbb{R}$ satisfies* (5)*, and p is as in* (8)*. Fix $q > 1$, and assume that $|F|^p \in L^q_{\rm loc}(C)$. For every $q > 1$ there exists a number $\varepsilon \equiv \varepsilon(n, N, p, \nu, L, q) > 0$ such that if $[a]_{BMO} \le \varepsilon$ then $|Du|^p \in L^q_{\rm loc}(C)$. Moreover, there exists a constant $c \equiv c(n, N, p, \nu, L, q) > 1$ such that* (9) *holds for every $Q_{2R} \subset\subset C$, with d as in* (10)*.*

Remark 1. The exponent d outside the square bracket in (9) prevents the estimate to be homogeneous, and of reverse-Hölder type. The occurrence of d is absolutely natural, and reflects the non-homogeneity of system (4), due to the fact that the evolutionary part of the system scales differently from the diffusion one: multiplying a solution by a constant does not yield another solution, even when $F \equiv 0$. Of course, $d = 1$ if and only if $p = 2$, and the system is not degenerate/singular; moreover, $d \nearrow \infty$ when $p \searrow \frac{2n}{n+2}$; for more comments on the dependence of the constant c on the number q, see Remark 3 below. Finally, we notice that it is possible to apply the method presented here also the elliptic case (3): this would yield a true, homogeneous reverse-Hölder type inequality, that is (9) with $d = 1$.

3. Preliminary material

In this section we collect some known results that will be crucial in the sequel, and we operate a few manipulations on known estimates in order to get them in the exact form we shall later need. We start with the higher integrability result of Kinnunen & Lewis [16], which is essential here to treat the case where the coefficient function $a(z)$ is not continuous. The version reported here is adapted to our setting from the more general right hand side structure in [16], for the equivalence see (88) with $f \equiv h_1$ in the notation of [16], formula (2.3); also, [16] asserts that (11) holds for some $\delta_0 > 0$, but the fact that then it holds for all $\delta < \delta_0$ may be deduced from their proof, after formula (4.13): indeed the only condition on δ_0 is that it must be small.

Theorem 3. *Let $u \in C^0((0,T); L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution to* (4)*, where* (8) *is in force and the function $a : C \to \mathbb{R}$ satisfies* (5)*. Assume* *that* $|F|^p \in L^q_{\rm loc}(C)$ *for some* $q > 1$. *Then there exists* $\delta_0 \equiv \delta_0(n, N, p, \nu, L)$ *with* $0 < \delta_0 < p(q-1)$, *such that* $|Du| \in L^{p+\delta}_{\rm loc}(C)$ *for every* $0 < \delta \le \delta_0$. *Moreover, there exists a constant* $c \equiv c(n, p, L, \nu) > 1$ *such that if* $Q_{z_0}(R^p, R) \subset\subset C$, *then*

$$\left(\fint_{Q_{z_0}(\frac{R^p}{2^p}, \frac{R}{2})} |Du|^{p+\delta} \, dz \right)^{\frac{1}{p+\delta}} \le cR^{\sigma p - 1} \left(\fint_{Q_{z_0}(R^p, R)} |Du|^p \, dz \right)^{\sigma} + \frac{c}{R} + c \left(\fint_{Q_{z_0}(R^p, R)} |F|^{p+\delta} \, dz \right)^{\frac{1}{p+\delta}} , \quad (11)$$

where

$$\sigma := \frac{2+\delta}{2(p+\delta)} \, . \quad (12)$$

In the remainder of the paper we shall eventually take $\delta < \delta_0$ in order to have

$$s := p + \delta \le \min\{(p+pq)/2, p+1\} < pq \, , \quad (13)$$

and we notice that this implies

$$\frac{pq}{pq - s} \le \frac{2q}{q-1} \, . \quad (14)$$

The first two lemmas are a consequence of the fundamental L^∞ gradient estimates of DiBenedetto [8, 9].

Lemma 1. *Let* $v \in C^0((t_1, t_2); L^2(A, \mathbb{R}^N)) \cap L^p(t_1, t_2; W^{1,p}(A, \mathbb{R}^N))$ *be a weak solution to*

$$v_t - {\rm div}(\tilde{a}|Dv|^{p-2}Dv) = 0 \qquad in \quad A \times [t_1, t_2) \, , \quad (15)$$

where $A \subset \mathbb{R}^n$ *is an open set,* $t_1 < t_2$, $p \ge 2$, *and* $\nu \le \tilde{a} \le L$. *Assume*

$$\fint_{Q(\lambda^{2-p}\varrho^2, \varrho)} |Dv|^p \, dz \le c_1 \lambda^p \quad (16)$$

for some $\lambda > 0$ *and some cylinder* $Q(\lambda^{2-p}\varrho^2, \varrho) \subset\subset A \times [t_1, t_2)$, *where* c_1 *is a given positive constant. Then there exists a constant* $c > 0$, *depending only on* n, N, p, ν, L *and* c_1, *such that*

$$\sup_{Q(\frac{1}{2}\lambda^{2-p}\varrho^2, \frac{1}{2}\varrho)} |Dv| \le c\lambda \, . \quad (17)$$

Proof. From Theorem 5.1, Chapter 8 in [8], and in particular estimate (5.1), page 238, taking $\sigma = 3/4$, we have that if $Q(\theta, \gamma) \subset\subset A \times [t_1, t_2)$ is a non-degenerate cylinder, then

$$\sup_{Q(\frac{\theta}{2}, \frac{\gamma}{2})} |Dv| \le c(n, N, p, \nu, L) \sqrt{\frac{\theta}{\gamma^2}} \left(\fint_{Q(\theta, \gamma)} |Dv|^p \, dz \right)^{\frac{1}{2}} \, . \quad (18)$$

Then we take $\theta = \lambda^{2-p}\gamma^2$ and $\gamma = \varrho$, so that $\sqrt{\theta/\gamma^2} = \lambda^{\frac{2-p}{2}}$. Using this fact in the previous inequality, and finally using (16), we immediately obtain (17). □

In the case $p < 2$ the estimate one is allowed to use is different, so we need another statement (and another proof, albeit very similar).

Lemma 2. *Let $v \in C^0((t_1,t_2); L^2(A,\mathbb{R}^N)) \cap L^p(t_1,t_2; W^{1,p}(A,\mathbb{R}^N))$ be a weak solution to* (15)*, where $A \subset \mathbb{R}^n$ is an open set, $t_1 < t_2$, $\nu \le \tilde{a} \le L$, and $p < 2$ satisfies* (8)*. Assume*

$$\fint_{Q(\varrho^2, \lambda^{\frac{p-2}{2}}\varrho)} |Dv|^p \, dz \le c_1 \lambda^p \tag{19}$$

for some $\lambda > 0$ and some cylinder $Q(\varrho^2, \lambda^{\frac{p-2}{2}}\varrho) \subset\subset A \times [t_1, t_2)$, where c_1 is a given positive constant. Then there exists a constant $c > 0$, depending only on n, N, p, ν, L and c_1, such that

$$\sup_{Q(\frac{1}{2}\varrho^2, \frac{1}{2}\lambda^{\frac{p-2}{2}}\varrho)} |Dv| \le c\lambda \, . \tag{20}$$

Proof. This time we use Theorem 5.2, Chapter 8 in [8], and in particular estimate (5.3), page 239, where we can take $r = p$, since p is assumed to satisfy (8). Taking again $\sigma = 3/4$, we have that if $Q(\theta, \gamma) \subset\subset A \times [t_1, t_2)$ is a non-degenerate cylinder, then

$$\sup_{Q(\frac{\theta}{2}, \frac{\gamma}{2})} |Dv| \le c(n, N, p, \nu, L) \left(\frac{\gamma^2}{\theta}\right)^{\frac{n}{p(n+2)-2n}} \left(\fint_{Q(\theta,\gamma)} |Dv|^p \, dz\right)^{\frac{2}{p(n+2)-2n}} \, . \tag{21}$$

Then we take $\gamma = \lambda^{\frac{p-2}{2}}\varrho$ and $\theta = \varrho$, so that $\sqrt{\gamma^2/\theta} = \lambda^{p-2}$. Using this fact in the previous inequality, and finally using (19), we immediately obtain (20). □

The next twin lemmas show how solutions to (4) satisfy real reverse-Hölder inequalities when considered on cylinders built according to the "intrinsic geometry".

Lemma 3. *Let $u \in C^0((0,T); L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega,\mathbb{R}^N))$ be a weak solution to* (4)*, where $p \ge 2$, and the function $a : C \to \mathbb{R}$ satisfies* (5)*. Assume*

$$\left(\fint_{Q(\lambda^{2-p}\varrho^2, \varrho)} |Du|^p \, dz\right)^{\frac{1}{p}} \le c_1 \lambda \tag{22}$$

and

$$\lambda \le c_2 \left(\fint_{Q(\lambda^{2-p}\varrho^2, \varrho)} |Du|^p \, dz\right)^{\frac{1}{p}} + c_2 \left(\fint_{Q(\lambda^{2-p}\varrho^2, \varrho)} M^s |F|^s \, dz\right)^{\frac{1}{s}} \tag{23}$$

hold for some $\lambda > 0$ and some cylinder $Q(\lambda^{2-p}\varrho^2, \varrho) \subset\subset C$, where $s > p$ is defined in (13) *via Theorem 3, c_1 and c_2 are two given positive constants and $M \ge 1$. Then there exists a constant $c_3 \equiv c_3(n, N, p, \nu, L, c_1, c_2)$ such that*

$$\begin{aligned} &\left(\fint_{Q(\frac{1}{2^p}\lambda^{2-p}\varrho^2, \frac{1}{2}\varrho)} |Du|^s \, dz\right)^{\frac{1}{s}} \\ \le\ & c_3 \left(\fint_{Q(\lambda^{2-p}\varrho^2, \varrho)} |Du|^p \, dz\right)^{\frac{1}{p}} + c_3 \left(\fint_{Q(\lambda^{2-p}\varrho^2, \varrho)} (1 + M^s|F|^s) \, dz\right)^{\frac{1}{s}} \, . \end{aligned} \tag{24}$$

Proof. Without loss of generality we may assume that the cylinder $Q(\lambda^{2-p}\varrho^2,\varrho)$ is centered at the origin. Let us consider the re-scaled maps

$$\tilde{u}(x,t) := \frac{u(\varrho x, \lambda^{2-p}\varrho^2 t)}{\varrho\lambda}\ , \qquad \tilde{F}(x,t) := \frac{F(\varrho x, \lambda^{2-p}\varrho^2 t)}{\lambda}$$

with $(x,t) \in Q_1$, and the re-scaled coefficients $\tilde{a}(x,t) := a(\varrho x, \lambda^{2-p}\varrho^2 t)$. It is easy to check that $\tilde{u} \in C^0((0,1); L^2(B_1,\mathbb{R}^N)) \cap L^p(0,1; W^{1,p}(B_1,\mathbb{R}^N))$ is a weak solution to the system

$$\tilde{u}_t - \operatorname{div}(\tilde{a}(z)|D\tilde{u}|^{p-2}D\tilde{u}) = \operatorname{div}(|\tilde{F}|^{p-2}\tilde{F}) \qquad \text{in } \ Q_1\ .$$

Therefore we may apply Theorem 3, and in particular estimate (11), in order to get that there exists a constant c, only depending on n, N, p, ν, L, such that

$$\left(\fint_{Q\left(\frac{1}{2^p},\frac{1}{2}\right)} |D\tilde{u}|^s\,dz\right)^{\frac{1}{s}} \le c\left(\fint_{Q_1} |D\tilde{u}|^p\,dz\right)^{\sigma} + c\left(\fint_{Q_1} |\tilde{F}|^s\,dz\right)^{\frac{1}{s}} + c\,. \tag{25}$$

where, according to (12), $\sigma = (2-p+s)/2s$. Scaling back in (25) yields

$$\begin{aligned}&\left(\fint_{Q(\frac{1}{2^p}\lambda^{2-p}\varrho^2,\frac{1}{2}\varrho)} |Du|^s\,dz\right)^{\frac{1}{s}}\\ &\le\ c\lambda^{1-\sigma p}\left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p\,dz\right)^{\sigma} + c\left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |F|^s\,dz\right)^{\frac{1}{s}} + c\lambda\,. \end{aligned}\tag{26}$$

Here $c \equiv c(n,N,p,\nu,L)$. But using (22) and (23) we have

$$\begin{aligned}\lambda^{1-\sigma p}&\left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p\,dz\right)^{\sigma} \le c\lambda\\ &\le c\left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p\,dz\right)^{\frac{1}{p}} + c\left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} M^s|F|^s\,dz\right)^{\frac{1}{s}}\ ,\end{aligned}$$

where $c \equiv c(c_1,c_2)$. Finally, (24) follows connecting the last inequalities to (26). $\square$

Lemma 4. *Let $u \in C^0((0,T); L^2(\Omega,\mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega,\mathbb{R}^N))$ be a weak solution to* (4)*, where $2n/(n+2) < p \le 2$, and the function $a : C \to \mathbb{R}$ satisfies* (5)*. Assume*

$$\left(\fint_{Q(\varrho^2,\lambda^{\frac{p-2}{2}}\varrho)} |Du|^p\,dz\right)^{\frac{1}{p}} \le c_1\lambda$$

and

$$\lambda \le c_2\left(\fint_{Q(\varrho^2,\lambda^{\frac{p-2}{2}}\varrho)} |Du|^p\,dz\right)^{\frac{1}{p}} + c_2\left(\fint_{Q(\varrho^2,\lambda^{\frac{p-2}{2}}\varrho)} M^s|F|^s\,dz\right)^{\frac{1}{s}}$$

hold for some $\lambda > 0$ and some cylinder $Q(\varrho^2,\lambda^{\frac{p-2}{2}}\varrho) \subset\subset C$, where $s > p$ is defined in (13) *via Theorem 3, c_1 and c_2 are two given positive constants and $M \ge 1$. Then there* *exists a constant* $c_3 \equiv c_3(n, N, p, \nu, L, c_1, c_2)$ *such that*

$$\left(\fint_{Q(\frac{1}{2^p}\varrho^2, \frac{1}{2}\lambda^{\frac{p-2}{2}}\varrho)} |Du|^s \, dz \right)^{\frac{1}{s}}$$

$$\leq c_3 \left(\fint_{Q(\varrho^2, \lambda^{\frac{p-2}{2}}\varrho)} |Du|^p \, dz \right)^{\frac{1}{p}} + c_3 \left(\fint_{Q(\varrho^2, \lambda^{\frac{p-2}{2}}\varrho)} (1 + M^s |F|^s) \, dz \right)^{\frac{1}{s}} .$$

Proof. Again we assume that $Q(\varrho^2, \lambda^{\frac{p-2}{2}}\varrho)$ is centered at the origin. This time we consider the re-scaled maps

$$\tilde{u}(x,t) := \frac{u(\lambda^{\frac{p-2}{2}}\varrho x, \varrho^2 t)}{\varrho \lambda^{\frac{p}{2}}} , \qquad \tilde{F}(x,t) := \frac{F(\lambda^{\frac{p-2}{2}}\varrho x, \varrho^2 t)}{\lambda}$$

with $(x,t) \in Q_1$, and the re-scaled coefficients $\tilde{a}(x,t) := a(\lambda^{\frac{p-2}{2}}\varrho x, \varrho^2 t)$. The remainder of the proof now follows exactly as in the previous lemma. □

We conclude the section with a couple of elementary results: the first can be promptly adapted from Lemma 2.2 in [7], the second can be found in [8], with slight modifications.

Lemma 5. *Let* $p > 1$ *and* $\mu \in [0, 1]$*; there exists a constant* $c \equiv c(n, N, p)$ *such that if* $v, w \in \mathbb{R}^{nN}$ *then:*

$$(\mu^2 + |A|^2)^{\frac{p}{2}} \leq c(\mu^2 + |B|^2)^{\frac{p}{2}} + c(\mu^2 + |B|^2 + |A|^2)^{\frac{p-2}{2}} |B - A|^2 .$$

Lemma 6. *Let* $1 < p < \infty$ *and* $\mu \in [0, 1]$. *There exists a constant* $c \equiv c(n, N, p)$, *independent of* μ, *such that for any* $A, B \in \mathbb{R}^{nN}$

$$(\mu^2 + |B|^2 + |A|^2)^{\frac{p-2}{2}} |B - A|^2 \leq c \langle (\mu^2 + |B|^2)^{\frac{p-2}{2}} B - (\mu^2 + |A|^2)^{\frac{p-2}{2}} A, B - A \rangle.$$

When $\mu = |A| = |B| = 0$, and $p < 2$, the quantities involved in the previous inequality are meant to be 0.

4. Proof of Theorems 1 and 2

By an approximation argument, in Step 1 we will reduce the proof of Theorem 1 to proving (9) when the solution has locally bounded gradient; then we will devote the remaining steps to this last task.

Step 1: Approximation. We first show how to approximate the solution u of (4), in a neighbourhood of a given cylinder, with a sequence u_ε of solutions to similar problems, whose gradients are bounded. Let $Q_{2R} \equiv (t_0 - (2R)^2, t_0 + (2R)^2) \times B_{x_0}(2R) \subset\subset C$ be as in the statement, and let $Q_{2\bar{R}} \subset\subset C$ be a cylinder, concentric with Q_{2R}, with $\bar{R} > R$. Let $\phi_1 : \mathbb{R}^n \to \mathbb{R}$, and $\phi_2 : \mathbb{R} \to \mathbb{R}$ be two standard mollifiers with compact support in B^1 and $(-1, 1)$ respectively, and for all $\varepsilon < \frac{1}{2}\,\mathrm{dist}(Q_{2\bar{R}}, \partial C)$ and $(x,t) \in Q_{2\bar{R}}$ define

$$F_\varepsilon(x,t) := \int_{Q_1} F(x + \varepsilon y, t + \varepsilon s)\phi_1(y)\phi_2(s) dy\, ds$$

and

$$a_\varepsilon(x,t) := \int_{Q_1} a(x+\varepsilon y, t+\varepsilon s)\phi_1(y)\phi_2(s)dy\, ds \ .$$

Clearly, $F_\varepsilon \in C^\infty(Q_{2\bar{R}}, \mathbb{R}^{nN})$, and $a_\varepsilon \in C^\infty(Q_{2\bar{R}})$. Moreover,

$$F_\varepsilon \to F \qquad \text{strongly in} \quad L^{pq}(Q_{2\bar{R}}, \mathbb{R}^{nN}) \ , \tag{27}$$

$$a_\varepsilon \to a \qquad \text{strongly in} \quad L^{t}(Q_{2\bar{R}}, \mathbb{R}^{nN}) \quad \forall\ t < \infty \ . \tag{28}$$

Finally, the new functions a_ε satisfy (5). Now we define the map

$$\begin{aligned} u_\varepsilon \in\ & C^0((t_0-(2\bar{R})^2, t_0+(2\bar{R})^2); L^2(B_{x_0}(\bar{R}), \mathbb{R}^N)) \\ & \cap L^p(t_0-(2\bar{R})^2, t_0+(2\bar{R})^2; W^{1,p}(B_{x_0}(\bar{R}), \mathbb{R}^N)) \ , \end{aligned}$$

as the unique solution to the following Cauchy-Dirichlet problem:

$$\begin{cases} (u_\varepsilon)_t - \operatorname{div}(a_\varepsilon(z)|Du_\varepsilon|^{p-2}Du_\varepsilon) = \operatorname{div}(|F_\varepsilon|^{p-2}F_\varepsilon) & \text{in } Q_{2\bar{R}} \\ u_\varepsilon \equiv u & \text{on } \partial_p Q_{2\bar{R}} \ . \end{cases} \tag{29}$$

The existence of such u_ε follows from the theory of monotone operators, or via Galerkin approximation, see [21]; for such problems and their exact meaning see [8], pages 20-21, and page 296. Our aim is now to show that

$$Du_\varepsilon \to Du \qquad \text{strongly in} \qquad L^p(Q_{2\bar{R}}, \mathbb{R}^{nN}) \ . \tag{30}$$

Using the fact that both u and u_ε are weak solutions, we have that

$$\begin{aligned} & (u_\varepsilon - u)_t - \operatorname{div}(a_\varepsilon(z)(|Du_\varepsilon|^{p-2}Du_\varepsilon - |Du|^{p-2}Du)) \\ & = \ \operatorname{div}((a_\varepsilon(z) - a(z))|Du|^{p-2}Du) + \operatorname{div}(|F_\varepsilon|^{p-2}F_\varepsilon - |F|^{p-2}F) \ . \end{aligned}$$

Now we test the previous identity with the map $u_\varepsilon - u$, which is possible modulo Steklov averages (for the definition see [8], pages 11 and 21); note that this is an admissible test map since $u \equiv u_\varepsilon$ on $\partial_p Q_{2\bar{R}}$. After a simple computation we arrive at

$$\begin{aligned} & \sup_{\substack{t_0-(2\bar{R})^2 \le t \\ < t_0+(2\bar{R})^2}} \int_{B_{x_0}(2\bar{R})} |u_\varepsilon(x,t) - u(x,t)|^2\, dx \\ & \qquad + \int_{Q_{2\bar{R}}} a_\varepsilon(z)\langle |Du_\varepsilon|^{p-2}Du_\varepsilon - |Du|^{p-2}Du, Du_\varepsilon - Du\rangle\, dz \\ \le \ & c\left| \int_{Q_{2\bar{R}}} (a_\varepsilon(z) - a(z))\langle |Du|^{p-2}Du, Du_\varepsilon - Du\rangle\, dz \right| \\ & \qquad + c\left| \int_{Q_{2\bar{R}}} \langle |F_\varepsilon|^{p-2}F_\varepsilon - |F|^{p-2}F, Du_\varepsilon - Du\rangle\, dz \right| \ , \end{aligned} \tag{31}$$

and therefore, using (5),

$$\int_{Q_{2\bar{R}}} |Du_\varepsilon|^p\, dz$$

$$\leq \; c\int_{Q_{2\bar R}} [|a_\varepsilon(z)|+|a(z)|]\left(|Du|^{p-1}|Du_\varepsilon|+|Du_\varepsilon|^{p-1}|Du|+|Du|^p\right)\,dz$$
$$+c\int_{Q_{2\bar R}} (|F_\varepsilon|+|F|)^{p-1}\,(|Du_\varepsilon|+|Du|)\,dz\,.$$

Finally using Young's inequality in a standard way, and the definitions of F_ε and a_ε, we get

$$\int_{Q_{2\bar R}} |Du_\varepsilon|^p\,dz \leq c\int_C |Du|^p+|F|^p\,dz \leq c_1\,. \tag{32}$$

Now we go back to (31). In the following we shall use the expression

$$(A,B)\mapsto (|A|^2+|B|^2)^{\frac{p-2}{2}}|B-A|^2 \qquad\qquad A,B\in\mathbb{R}^{nN}\,,$$

which is already defined in Lemma 6 and that involves a singularity when $|A|=|B|=0$ and $p<2$. In this case the meaning of the previous quantity was defined as 0. Using Lemma 6 with $\mu=0$, together with (5), we find

$$\int_{Q_{2\bar R}} (|Du_\varepsilon|^2+|Du|^2)^{\frac{p-2}{2}}|Du_\varepsilon-Du|^2\,dz$$
$$\leq c(n,N,p,\nu)\int_{Q_{2\bar R}} a_\varepsilon(z)\langle|Du_\varepsilon|^{p-2}Du_\varepsilon-|Du|^{p-2}Du, Du_\varepsilon-Du\rangle\,dz\,,$$

and from (31) we have

$$\int_{Q_{2\bar R}} (|Du_\varepsilon|^2+|Du|^2)^{\frac{p-2}{2}}|Du_\varepsilon-Du|^2\,dz$$
$$\leq \; c\int_{Q_{2\bar R}} |a_\varepsilon(z)-a(z)||Du|^{p-1}|Du_\varepsilon-Du|\,dz$$
$$+c\int_{Q_{2\bar R}} \left||F_\varepsilon|^{p-2}F_\varepsilon-|F|^{p-2}F\right||Du_\varepsilon-Du|\,dz\,, \tag{33}$$

with $c\equiv c(n,N,p,\nu,L)$. Using Young's inequality with $\delta\in(0,1)$ we find

$$\int_{Q_{2\bar R}} (|Du_\varepsilon|^2+|Du|^2)^{\frac{p-2}{2}}|Du_\varepsilon-Du|^2\,dz$$
$$\leq \; c(\delta)\int_{Q_{2\bar R}} |a_\varepsilon(z)-a(z)|^{\frac{p}{p-1}}|Du|^p\,dz+\delta\int_{Q_{2\bar R}} |Du_\varepsilon|^p+|Du|^p\,dz$$
$$+c(\delta)\int_{Q_{2\bar R}} \left||F_\varepsilon|^{p-2}F_\varepsilon-|F|^{p-2}F\right|^{\frac{p}{p-1}}\,dz\,, \tag{34}$$

with the constants c depending also on n,N,p,ν,L. We have now, recalling that s is the higher integrability exponent defined in (13),

$$\int_{Q_{2\bar R}} |a_\varepsilon(z)-a(z)|^{\frac{p}{p-1}}|Du|^p\,dz$$

$$\leq \left(\int_{Q_{2\bar{R}}} |a_\varepsilon(z)-a(z)|^{\frac{ps}{(p-1)(s-p)}}\,dz\right)^{\frac{s-p}{s}} \cdot \left(\int_{Q_{2\bar{R}}} |Du|^s\,dz\right)^{\frac{p}{s}} \stackrel{(28)}{\to} 0 \tag{35}$$

as $\varepsilon \to 0$. By a dominated convergence argument and (27) we directly have

$$\int_{Q_{2\bar{R}}} \left||F_\varepsilon|^{p-2}F_\varepsilon - |F|^{p-2}F\right|^{\frac{p}{p-1}}\,dz \to 0 \tag{36}$$

as $\varepsilon \to 0$. Keeping into account (32), connecting (34)–(36), and finally letting $\delta \to 0$, we obtain

$$\lim_{\varepsilon\to 0} \int_{Q_{2\bar{R}}} (|Du_\varepsilon|^2+|Du|^2)^{\frac{p-2}{2}}|Du_\varepsilon - Du|^2\,dz = 0\;. \tag{37}$$

Now, if $p<2$, using Hölder's inequality and again (32)

$$\begin{aligned}
&\int_{Q_{2\bar{R}}} |Du_\varepsilon - Du|^p\,dz \\
&\leq \left(\int_{Q_{2\bar{R}}} |Du_\varepsilon|^p + |Du|^p\,dz\right)^{\frac{1}{2}} \cdot \left(\int_{Q_{2\bar{R}}} (|Du_\varepsilon|^2+|Du|^2)^{\frac{p-2}{2}}|Du_\varepsilon - Du|^2\,dz\right)^{\frac{1}{2}}\;,
\end{aligned}$$

therefore

$$\lim_{\varepsilon\to 0} \int_{Q_{2\bar{R}}} |Du_\varepsilon - Du|^p\,dz = 0\;, \tag{38}$$

which proves (30) in the case $p<2$. If instead $p \geq 2$, going back to (33) and using Young's inequality we get

$$\int_{Q_{2\bar{R}}} |Du_\varepsilon - Du|^p\,dz \leq \int_{Q_{2\bar{R}}} (|Du_\varepsilon|^2+|Du|^2)^{\frac{p-2}{2}}|Du_\varepsilon - Du|^2\,dz$$

and (38) follows from (37). Now we finally show how the validity of (9) in the general case follows from the case when Du is bounded. Therefore let us assume that (9) holds whenever Du is bounded. We consider the maps $\{u_\varepsilon\}$ defined in (29); the regularity theory for the parabolic p-Laplacean systems applies, see [8] chapter 8, and therefore $Du_\varepsilon \in L^\infty(Q_{2R},\mathbb{R}^{nN})$. Then by (9)

$$\begin{aligned}
\left(⨍_{Q_R} |Du|^{pq}\,dz\right)^{\frac{1}{q}} &\leq \liminf_{\varepsilon\to 0} \left(⨍_{Q_R} |Du_\varepsilon|^{pq}\,dz\right)^{\frac{1}{q}} \\
&\leq c\lim_{\varepsilon\to 0}\left[⨍_{Q_{2R}} |Du_\varepsilon|^p\,dz + \left(⨍_{Q_{2R}} |F_\varepsilon|^{pq}\,dz + 1\right)^{\frac{1}{q}}\right]^d \\
&= c\left[⨍_{Q_{2R}} |Du|^p\,dz + \left(⨍_{Q_{2R}} |F|^{pq}\,dz + 1\right)^{\frac{1}{q}}\right]^d\;,
\end{aligned}$$

where we used (30) and Fatou's lemma to manage for the left hand side. The remainder of the proof will be therefore dedicated to proving (9) under the additional assumption that Du is bounded. Once (9) is proved in the general case, the full statement $|Du|^p \in L^q_{\rm loc}(C)$ follows via a standard covering argument.

Step 2: A stopping time argument. We start with the case $p \geq 2$. We define $\lambda_0 > 1$ according to

$$\lambda_0^{\frac{1}{d}} := \left(\fint_{Q_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} + \left(\fint_{Q_{2R}} M^s |F|^s \, dz \right)^{\frac{1}{s}} + 1 \, . \tag{39}$$

where number d was defined in (10). The number $M > 1$ will be chosen later, *in a universal way*, that will only depend on the fixed parameters n, p, ν, L. Now pick any two numbers γ, λ such that

$$\frac{R}{2^{8p}} \leq \gamma \leq \frac{R}{2} \, , \qquad\qquad B\lambda_0 := 2^{10(n+2)t} \lambda_0 \leq \lambda \, . \tag{40}$$

We check that for all $z_0 \in Q_R$

$$\left(\fint_{Q_{z_0}(\lambda^{2-p}\gamma^2, \gamma)} |Du|^p \, dz \right)^{\frac{1}{p}} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\gamma^2, \gamma)} M^s |F|^s \, dz \right)^{\frac{1}{s}} < \lambda \, . \tag{41}$$

Indeed, we first remark that if $z_0 \in Q_R$ and $\varrho < R$, since $\lambda > \lambda_0 > 1$ and $p \geq 2$, then $Q_z(\lambda^{2-p}\varrho^2, \varrho) \subset Q_{2R}$, so that in particular when γ satisfies (40)

$$\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\lambda^{2-p}\gamma^2, \gamma)|} > 1 \, . \tag{42}$$

Then

$$\begin{aligned}
&\left(\fint_{Q_{z_0}(\lambda^{2-p}\gamma^2, \gamma)} |Du|^p \, dz \right)^{\frac{1}{p}} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\gamma^2, \gamma)} M^s |F|^s \, dz \right)^{\frac{1}{s}} \\
&\leq \left(\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\lambda^{2-p}\gamma^2, \gamma)|} \right)^{\frac{1}{p}} \cdot \left(\fint_{Q_{2R}} |Du|^p \, dz \right)^{\frac{1}{p}} \\
&\qquad + \left(\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\lambda^{2-p}\gamma^2, \gamma)|} \right)^{\frac{1}{s}} \cdot \left(\fint_{Q_{2R}} M^s |F|^s \, dz \right)^{\frac{1}{s}} \\
&\overset{(13),(39),(40),(42)}{<} (2^{10(n+2)p} \lambda^{p-2})^{\frac{1}{p}} \lambda_0^{\frac{1}{d}} \overset{(10)}{=} (2^{10(n+2)p} \lambda^{p-2})^{\frac{1}{p}} \lambda_0^{\frac{2}{p}} \\
&\overset{(40)}{\leq} \lambda \, .
\end{aligned}$$

Now, with λ as in (40), take a point $z_0 \in Q_R$ such that $|Du(z_0)| > \lambda$. By Lebesgue's differentiation theorem, for almost every such point we have

$$\lim_{\varrho \to 0} \left\{ \left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2, \varrho)} |Du|^p \, dz \right)^{\frac{1}{p}} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2, \varrho)} M^s |F|^s \, dz \right)^{\frac{1}{s}} \right\} > \lambda \, . \tag{43}$$

Assume that for some $\varrho > 0$ satisfying $\varrho \leq R/2$

$$\left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p\,dz\right)^{\frac{1}{p}} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2,\varrho)} M^s|F|^s\,dz\right)^{\frac{1}{s}} > \lambda\ .$$

We remark that some such ϱ exist by (43); since for $\varrho \geq R/2^{8p}$ the opposite inequality holds by (40),(41), necessarily we conclude that $\varrho < R/2^{8p}$. Therefore we can select a radius $\varrho_{z_0} \leq R/2$ to be the largest for which

$$\left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2,\varrho_{z_0})} |Du|^p\,dz\right)^{\frac{1}{p}} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2,\varrho_{z_0})} M^s|F|^s\,dz\right)^{\frac{1}{s}} = \lambda\ , \tag{44}$$

in the sense that if $R/2 \geq \varrho > \varrho_{z_0}$ then

$$\left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p\,dz\right)^{\frac{1}{p}} + \left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho^2,\varrho)} M^s|F|^s\,dz\right)^{\frac{1}{s}} < \lambda\ .$$

By the above argumentation it must be

$$\varrho_{z_0} < R/2^{8p}\ . \tag{45}$$

Since $\lambda > 1$, and $p \geq 2$, we immediately have that

$$Q_{z_0}(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2,(2^{jp}\varrho_{z_0})) \subset Q((2R)^2,(2R)) \qquad\qquad j \in \{0,\ldots,5\}\ .$$

Moreover we observe that for $j \in \{0,\ldots,5\}$ we have

$$\begin{aligned}\frac{\lambda}{8^{jp}} \leq &\left(\fint_{Q_{z_0}(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2,(2^{jp}\varrho_{z_0}))} |Du|^p\,dz\right)^{\frac{1}{p}} \\ &+ \left(\fint_{Q_{z_0}(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2,(2^{jp}\varrho_{z_0}))} M^s|F|^s\,dz\right)^{\frac{1}{s}} \leq \lambda\ .\end{aligned} \tag{46}$$

Indeed the right hand side inequality just follows from the choice of ϱ_{z_0}, while as for the left hand side, the sum of the integrals appearing in (46) can be estimated from below as follows:

$$\begin{aligned}&\left(\fint_{Q_{z_0}(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2,(2^{jp}\varrho_{z_0}))} |Du|^p\,dz\right)^{\frac{1}{p}} \\ &\qquad + \left(\fint_{Q_{z_0}(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2,(2^{jp}\varrho_{z_0}))} M^s|F|^s\,dz\right)^{\frac{1}{s}} \\ \geq\ &\left(\frac{|Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2,\varrho_{z_0})|}{|Q_{z_0}(\lambda^{2-p}(2^{jp}\varrho_{z_0})^2,(2^{jp}\varrho_{z_0}))|}\right)^{\frac{1}{p}} \cdot \left[\left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2,\varrho_{z_0})} |Du|^p\,dz\right)^{\frac{1}{p}}\right.\end{aligned}$$

$$+\left(\fint_{Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2,\varrho_{z_0})} M^s|F|^s\,dz\right)^{\frac{1}{s}}\Bigg] \overset{(44)}{=} \frac{\lambda}{8^{jp}}\ .$$

Now let us consider the level set

$$E(\lambda) := \{z \in Q_R\ :\ |Du(z)| > \lambda\}\ .$$

For a.e. $z_0 \in E(\lambda)$ we can find a cube $Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2, \varrho_{z_0}) \subset Q_{2R}$ as constructed above, and in particular such that (46) holds for $j \in \{0, \dots, 5\}$. Therefore, applying Vitali's covering theorem, we find a family of disjoint cubes $\{Q_i^0\}$ of the type considered up to now:

$$Q_i^0 \equiv Q_{z_i}(\lambda^{2-p}\varrho_{z_i}^2, \varrho_{z_i}) \subset Q_{2R}\ , \qquad z_i \in E(\lambda)\ , \tag{47}$$

such that

$$E(\lambda) \subset \bigcup_{i\in\mathbb{N}} Q_i^1 \cup \text{negligible set}\ .$$

Here we have denoted

$$Q_i^1 \equiv Q_{z_i}(\lambda^{2-p}(2^{3p}\varrho_{z_i})^2, (2^{3p}\varrho_{z_i}))\ .$$

For future convenience we also introduce

$$Q_i^2 \equiv Q_{z_i}(\lambda^{2-p}(2^{4p}\varrho_{z_i})^2, (2^{4p}\varrho_{z_i}))$$

and

$$Q_i^3 \equiv Q_{z_i}(\lambda^{2-p}(2^{5p}\varrho_{z_i})^2, (2^{5p}\varrho_{z_i}))\ .$$

We now deal with the case $p < 2$. The basic change with respect to the case $p \geq 2$, and following the sub-quadratic scaling introduced by DiBenedetto (see [8] page 80), is to use this time cubes of the type $Q_z(\varrho^2, \lambda^{\frac{p-2}{2}}\varrho)$. In this case λ_0 is still defined as in (39), and γ, λ are again picked according to (40). With $z_0 \in Q_R$, once again we have

$$\left(\fint_{Q_{z_0}(\gamma^2,\lambda^{\frac{p-2}{2}}\gamma)} |Du|^p\,dz\right)^{\frac{1}{p}} + \left(\fint_{Q_{z_0}(\gamma^2,\lambda^{\frac{p-2}{2}}\gamma)} M^s|F|^s\,dz\right)^{\frac{1}{s}} < \lambda\ . \tag{48}$$

The equivalent of (42) in this case is

$$\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\gamma^2, \lambda^{\frac{p-2}{2}}\gamma)|} > 1\ . \tag{49}$$

Then we have

$$\begin{aligned}
&\left(\fint_{Q_{z_0}(\gamma^2,\lambda^{\frac{p-2}{2}}\gamma)} |Du|^p\,dz\right)^{\frac{1}{p}} + \left(\fint_{Q_{z_0}(\gamma^2,\lambda^{\frac{p-2}{2}}\gamma)} M^s|F|^s\,dz\right)^{\frac{1}{s}}\\
&\quad\leq \left(\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\gamma^2, \lambda^{\frac{p-2}{2}}\gamma)|}\right)^{\frac{1}{p}} \cdot \left(\fint_{Q_{2R}} |Du|^p\,dz\right)^{\frac{1}{p}}\\
&\qquad + \left(\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\gamma^2, \lambda^{\frac{p-2}{2}}\gamma)|}\right)^{\frac{1}{s}} \cdot \left(\fint_{Q_{2R}} M^s|F|^s\,dz\right)^{\frac{1}{s}}
\end{aligned}$$

$$\stackrel{(13),(39),(40),(49)}{\leq} \left[2^{10(n+2)p}\lambda^{n\left(\frac{2-p}{2}\right)}\right]^{\frac{1}{p}} \lambda_0^{\frac{1}{d}} \stackrel{(10)}{=} \left[2^{10(n+2)p}\lambda^{n\left(\frac{2-p}{2}\right)}\right]^{\frac{1}{p}} \lambda_0^{\frac{p(n+2)-2n}{2p}}$$

$$\stackrel{(8),(40)}{\leq} \lambda\ ,$$

and (48) is proved. An important remark to be made is that, beside needing it for Lemmata 2 & 4, and Kinnunen and Lewis' Theorem 3, this is the only point where we need (8). For the remainder we can proceed exactly as for the case $p \geq 2$, but using the cubes of the type $Q_{z_0}(\varrho_{z_0}^2, \lambda^{\frac{p-2}{2}}\varrho_{z_0}) \subset Q_{2R}$, instead of those of the type $Q_{z_0}(\lambda^{2-p}\varrho_{z_0}^2, \varrho_{z_0})$. At the end we come up with a family of disjoint cubes $\{Q_i^0\}$ of the type

$$Q_i^0 \equiv Q_{z_i}(\varrho_{z_i}^2, \lambda^{\frac{p-2}{2}}\varrho_{z_i}) \subset Q_{2R}\ , \qquad z_i \in E(\lambda)\ , \tag{50}$$

such that (45) holds and having the fundamental property that for $j \in \{0, \ldots, 5\}$ and all i

$$\frac{\lambda}{8^{jp}} \leq \left(\fint_{Q_{z_i}((2^{jp}\varrho_{z_i})^2, \lambda^{\frac{p-2}{2}}(2^{jp}\varrho_{z_i}))} |Du|^p\, dz\right)^{\frac{1}{p}} + \left(\fint_{Q_{z_i}((2^{jp}\varrho_{z_i})^2, \lambda^{\frac{p-2}{2}}(2^{jp}\varrho_{z_i}))} M^s|F|^s\, dz\right)^{\frac{1}{s}} \leq \lambda\ , \tag{51}$$

and such that

$$E(\lambda) \subset \bigcup_{i\in\mathbb{N}} Q_i^1 \cup \text{negligible set}\ .$$

Accordingly, in the case $p < 2$ we are denoting

$$Q_i^1 \equiv Q_{z_i}((2^{3p}\varrho_{z_i})^2, \lambda^{\frac{p-2}{2}}(2^{3p}\varrho_{z_i}))\ ,$$

$$Q_i^2 \equiv Q_{z_i}((2^{4p}\varrho_{z_i})^2, \lambda^{\frac{p-2}{2}}(2^{4p}\varrho_{z_i}))$$

and

$$Q_i^3 \equiv Q_{z_i}((2^{5p}\varrho_{z_i})^2, \lambda^{\frac{p-2}{2}}(2^{5p}\varrho_{z_i}))\ .$$

From now, for the remainder of the proof, when dealing with cubes of the type $Q_i^0, \ldots, Q_i^3$, we shall implicitly understand which kind we are using, depending on p.

Step 3: Comparison maps. When $p \geq 2$, on the cube Q_i^2 centered at $z_i := (x_i, t_i)$ we define the map

$$\begin{aligned} v_i \in\ & C^0((t_i - 2^{8p}\lambda^{2-p}\varrho_i^2, t_i + 2^{8p}\lambda^{2-p}\varrho_i^2); L^2(B_{x_i}(2^{4p}\varrho_i), \mathbb{R}^N)) \\ & \cap L^p(t_i - 2^{8p}\lambda^{2-p}\varrho_i^2, t_i + 2^{8p}\lambda^{2-p}\varrho_i^2; W^{1,p}(B_{x_i}(2^{4p}\varrho_i), \mathbb{R}^N)) \end{aligned}$$

as the unique solution to the Cauchy-Dirichlet problem

$$\begin{cases} (v_i)_t - \operatorname{div}(a_i|Dv_i|^{p-2}Dv_i) = 0 & \text{in } Q_i^2 \\ v_i \equiv u & \text{on } \partial_p Q_i^2 \end{cases} \tag{52}$$

(see again [8], pages 20-21, and page 296), where

$$a_i := \fint_{Q_i^2} a(z)\, dz \ .$$

We remark that due to (5)

$$\nu \leq a_i \leq L \ . \tag{53}$$

We are going to find some estimates on v_i. Using the fact that both u and v_i are solutions we have

$$\begin{aligned}(u - v_i)_t &- \operatorname{div}(a_i(|Du|^{p-2}Du - |Dv_i|^{p-2}Dv_i)) \\ &= \operatorname{div}((a(z) - a_i)|Du|^{p-2}Du) + \operatorname{div}(|F|^{p-2}F)\end{aligned}$$

in the weak sense. Now we shall proceed formally, as in Step 1, by testing the previous equality with the map $\varphi = u - v_i$, which may be justified via Steklov averages. Again it is crucial that u and v_i agree on the parabolic boundary $\partial_p Q_i^2$. As in Step 1, we obtain the equivalent of (31):

$$\begin{aligned}&\int_{Q_i^2} a_i \langle |Du|^{p-2}Du - |Dv_i|^{p-2}Dv_i, Du - Dv_i\rangle\, dz \\ &\leq \ c\left|\int_{Q_i^2} (a_i(z) - a(z))\langle |Du|^{p-2}Du, Du - Dv_i\rangle\, dz\right| \\ &\qquad + c\left|\int_{Q_i^2} \langle |F|^{p-2}F, Du - Dv_i\rangle\, dz\right| \ ,\end{aligned} \tag{54}$$

and, using (53), the equivalent of (32):

$$\int_{Q_i^2} |Dv_i|^p\, dz \leq c(n,N,p,\nu,L) \int_{Q_i^2} |Du|^p + |F|^p\, dz \ . \tag{55}$$

Therefore, recalling that $M \geq 1$ and using Hölder's inequality, we find

$$\fint_{Q_i^2} |Dv_i|^p\, dz \leq c \fint_{Q_i^2} |Du|^p\, dz + c\left(\fint_{Q_i^2} M^s |F|^s\, dz\right)^{\frac{p}{s}} \overset{(46)}{\leq} c\lambda^p \ , \tag{56}$$

where $c \equiv c(n,N,p,\nu,L)$. Now remark that

$$Q_i^1 \subset Q_{z_i}\left(\frac{\lambda^{2-p}(2^{4p}\varrho_i)^2}{2}, \frac{(2^{4p}\varrho_i)}{2}\right) \ ,$$

therefore by (56) we can apply Lemma 1, with $Q = Q_i^2$, to get that there exists an absolute constant A_1, depending only on n, N, p, ν, L, such that

$$A_1 \geq 1 \ , \qquad \sup_{Q_i^1} |Dv_i| \leq A_1\lambda \ . \tag{57}$$

When $p < 2$ we can proceed in a completely analogous way, invoking Lemma 2 instead of Lemma 1 and using the right kind of cubes, and (57) follows again. We remark in particular that (54)–(57) hold both for $p \geq 2$ and $p < 2$.

Now we want to get an estimate for the integral

$$\int_{Q_i^2} (|Du|^2 + |Dv_i|^2)^{\frac{p-2}{2}} |Du - Dv_i|^2 \; dz \; .$$

Using Lemma 6 we have

$$\begin{aligned}
&\frac{\nu}{c(n,N,p)} \int_{Q_i^2} (|Du|^2 + |Dv_i|^2)^{\frac{p-2}{2}} |Du - Dv_i|^2 \, dz \\
&\overset{(53)}{\leq} \quad a_i \int_{Q_i^2} \langle |Du|^{p-2}Du - |Dv_i|^{p-2}Dv_i, Du - Dv_i \rangle \, dz \\
&\overset{(54)}{\leq} \quad c \int_{Q_i^2} |a(z) - a_i| |Du|^{p-1} |Du - Dv_i| \, dz + c \int_{Q_i^2} |F|^{p-1} |Du - Dv_i| \, dz \; . \qquad (58)
\end{aligned}$$

Using Young's inequality, with $\delta \in (0,1)$ we have

$$\begin{aligned}
&\int_{Q_i^2} |a(z) - a_i| |Du|^{p-1} |Du - Dv_i| \, dz \\
&\quad \leq \quad \frac{c}{\delta^{\frac{1}{p-1}}} \int_{Q_i^2} |a(z) - a_i|^{\frac{p}{p-1}} |Du|^p \, dz + \delta \int_{Q_i^2} |Du|^p + |Dv_i|^p \, dz \\
&\quad \overset{(55)}{\leq} \quad \frac{c}{\delta^{\frac{1}{p-1}}} \int_{Q_i^2} |a(z) - a_i|^{\frac{p}{p-1}} |Du|^p \, dz + c\delta \int_{Q_i^2} |Du|^p + |F|^p \, dz \; , \qquad (59)
\end{aligned}$$

and

$$\begin{aligned}
\int_{Q_i^2} |F|^{p-1} |Du - Dv_i| \, dz \quad &\leq \quad \frac{c}{\delta^{\frac{1}{p-1}}} \int_{Q_i^2} |F|^p \, dz + \delta \int_{Q_i^2} |Du|^p + |Dv_i|^p \, dz \\
&\overset{(55)}{\leq} \quad \frac{c}{\delta^{\frac{1}{p-1}}} \int_{Q_i^2} |F|^p \, dz + c\delta \int_{Q_i^2} |Du|^p \, dz
\end{aligned}$$

with $c \equiv c(n, N, p, \nu, L)$. Connecting the last two inequalities with (58) we finally have

$$\begin{aligned}
&\int_{Q_i^2} (|Du|^2 + |Dv_i|^2)^{\frac{p-2}{2}} |Du - Dv_i|^2 \, dz \\
&\quad \leq \quad \frac{c}{\delta^{\frac{1}{p-1}}} \int_{Q_i^2} |a(z) - a_i|^{\frac{p}{p-1}} |Du|^p \, dz + \frac{c}{\delta^{\frac{1}{p-1}}} \int_{Q_i^2} |F|^p \, dz + c\delta \int_{Q_i^2} |Du|^p \, dz \; , \qquad (60)
\end{aligned}$$

with $c \equiv c(n, N, p, \nu, L)$, and $\delta \in (0,1)$ yet to be chosen.

We estimate the first integral appearing in the right hand side of (60). Using Hölder's inequality we have

$$\int_{Q_i^2} |a(z) - a_i|^{\frac{p}{p-1}} |Du|^p \, dz \leq \left(\fint_{Q_i^2} |a(z) - a_i|^b \, dz \right)^{\frac{s-p}{s}} \left(\fint_{Q_i^2} |Du|^s \, dz \right)^{\frac{p}{s}} |Q_i^2| \; ,$$

where we have set

$$b := \frac{p}{p-1} \frac{s}{s-p} > 1 \; .$$

We remark that

$$\left(\fint_{Q_i^2} |a(z)-a_i|^b\,dz\right)^{\frac{s-p}{s}} \le (2L)^{\frac{(b-1)(s-p)}{s}}[\omega(R)]^{\frac{s-p}{s}}$$

as a consequence of (5),(7),(45),(53), while

$$\left(\fint_{Q_i^2} |Du|^s\,dz\right)^{\frac{p}{s}} \le c\fint_{Q_i^3} |Du|^p\,dz + c\left(\fint_{Q_i^3}(1+M^s|F|^s)\,dz\right)^{\frac{p}{s}}$$

as a consequence of Lemma 3 or Lemma 4. Merging the last three estimates with (60) we finally obtain the estimate we were looking for:

$$\begin{aligned}
&\int_{Q_i^2} (|Du|^2+|Dv_i|^2)^{\frac{p-2}{2}}|Du-Dv_i|^2\,dz\\
&\le\ c\left\{\frac{[\omega(R)]^{\frac{s-p}{s}}}{\delta^{\frac{1}{p-1}}}+\delta\right\}\int_{Q_i^3}|Du|^p\,dz\\
&\qquad +\frac{c[\omega(R)]^{\frac{s-p}{s}}}{\delta^{\frac{1}{p-1}}}\left(\fint_{Q_i^3}(1+M^s|F|^s)\,dz\right)^{\frac{p}{s}}|Q_i^0|+\frac{c}{\delta^{\frac{1}{p-1}}}\int_{Q_i^3}|F|^p\,dz\ , \qquad (61)
\end{aligned}$$

where the constant c depends on the data n,N,p,ν,L, we estimated $|Q_i^3|\le 4^{10np}|Q_i^0|$ and $\delta\in(0,1)$ is yet to be chosen.

Step 4: Estimates on cubes. Lemma 5 with $\mu=0$ implies

$$|Du|^p \le c_l|Dv_i|^p + c_l(|Du|^2+|Dv_i|^2)^{\frac{p-2}{2}}|Du-Dv_i|^2$$

where $c_l\equiv c_l(n,p)$ is the constant appearing in the lemma. Accordingly, we fix the constant

$$A := (1+2c_l)A_1\ , \qquad (62)$$

where A_1 is the constant appearing in (57). In this way A depends only on the data n,N,p,ν,L. We have that

$$\begin{aligned}
&|\{z\in Q_i^1\ :\ |Du|^p>A\lambda^p\}|\\
&\le\ |\{z\in Q_i^1\ :\ (|Du|^2+|Dv_i|^2)^{\frac{p-2}{2}}|Du-Dv_i|^2>A_1\lambda^p\}|\\
&\qquad +|\{z\in Q_i^1\ :\ |Dv_i|^p>A_1\lambda^p\}| \qquad\qquad (\overset{(57),(62)}{=}0)\\
&=\ |\{z\in Q_i^1\ :\ (|Du|^2+|Dv_i|^2)^{\frac{p-2}{2}}|Du-Dv_i|^2>A_1\lambda^p\}|\ ,
\end{aligned}$$

so that

$$|\{z\in Q_i^1\ :\ |Du|^p>A\lambda^p\}| \le \frac{1}{A_1\lambda^p}\int_{Q_i^1}(|Du|^2+|Dv_i|^2)^{\frac{p-2}{2}}|Du-Dv_i|^2\,dx\ ,$$

and using (61),(62)

$$|\{z\in Q_i^1\ :\ |Du|^p>A\lambda^p\}|\ \le\ \frac{c}{A\lambda^p}\left\{\frac{[\omega(R)]^{\frac{s-p}{s}}}{\delta^{\frac{1}{p-1}}}+\delta\right\}\int_{Q_i^3}|Du|^p\,dz$$

$$
+\frac{c[\omega(R)]^{\frac{s-p}{s}}}{A\lambda^p\delta^{\frac{1}{p-1}}}\left(\fint_{Q_i^3}(1+M^s|F|^s)\,dz\right)^{\frac{p}{s}}|Q_i^0|
+\frac{c}{A\lambda^p\delta^{\frac{1}{p-1}}}\int_{Q_i^3}|F|^p\,dz\,. \tag{63}
$$

We shall now carefully estimate these three integrals; we remark that the constant c just seen depends only on n, p, ν, L. Since we will later backtrack to find the exact dependence on q, we will be careful to let every constant c be independent of q; given (13), when not essential we will majorize constants as e.g. 2^s by $c = c(p)$. We first provide an estimate for $|Q_i^0|$; by (44),(47) or the analogous for $p < 2$, either of the following inequalities must be true:

$$
\left(\frac{\lambda}{2}\right)^p \le \frac{1}{|Q_i^0|}\int_{Q_i^0}|Du|^p\,dz \qquad \text{or} \qquad \left(\frac{\lambda}{2}\right)^s \le \frac{1}{|Q_i^0|}\int_{Q_i^0}M^s|F|^s\,dz\,.
$$

In any case

$$
|Q_i^0| \le \frac{2^p}{\lambda^p}\int_{Q_i^0}|Du|^p\,dz + \frac{2^s}{\lambda^s}\int_{Q_i^0}M^s|F|^s\,dz\,. \tag{64}
$$

We now split the last integral as follows, for some $\gamma > 0$:

$$
\begin{aligned}
\frac{1}{\lambda^s}\int_{Q_i^0}M^s|F|^s\,dz &= \frac{1}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}M^s|F|^s\,dz + \frac{1}{\lambda^s}\int_{Q_i^0\cap\{|F|\le\gamma\lambda\}}M^s|F|^s\,dz \\
&\le \frac{1}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}M^s|F|^s\,dz + M^s\gamma^s|Q_i^0|\,.
\end{aligned} \tag{65}
$$

Choosing

$$
\gamma^s := \frac{1}{2^{s+1}M^s}\,, \tag{66}
$$

connecting (65) to (64) and reabsorbing $|Q_i^0|/2$ we find the estimate for $|Q_i^0|$ we were interested in:

$$
|Q_i^0| \le \frac{2^{p+1}}{\lambda^p}\int_{Q_i^0}|Du|^p\,dz + \frac{2^{s+1}}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}M^s|F|^s\,dz\,. \tag{67}
$$

Now we gain a further estimate, again splitting with some $\tau > 0$:

$$
\begin{aligned}
\frac{1}{\lambda^p}\int_{Q_i^0}|Du|^p\,dz &= \frac{1}{\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}}|Du|^p\,dz + \frac{1}{\lambda^p}\int_{Q_i^0\cap\{|Du|\le\tau\lambda\}}|Du|^p\,dz \\
&\le \frac{1}{\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}}|Du|^p\,dz + \tau^p|Q_i^0| \\
&\overset{(67)}{\le} \frac{1}{\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}}|Du|^p\,dz + \frac{2^{p+1}\tau^p}{\lambda^p}\int_{Q_i^0}|Du|^p\,dz \\
&\qquad + \frac{2^{s+1}\tau^p}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}M^s|F|^s\,dz\,.
\end{aligned} \tag{68}
$$

Choosing

$$\tau^p := \frac{1}{2^{p+2}}\ , \tag{69}$$

and reabsorbing the second-last integral into the left hand side of (68), we conclude with

$$\frac{1}{\lambda^p}\int_{Q_i^0}|Du|^p\,dz \leq \frac{2}{\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}}|Du|^p\,dz + \frac{2^{s-p}}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}M^s|F|^s\,dz\ .$$

In particular from (67) we deduce

$$|Q_i^0| \leq \frac{2^{p+2}}{\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}}|Du|^p\,dz + \frac{2^{s+2}}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}M^s|F|^s\,dz\ . \tag{70}$$

Then we have

$$\begin{aligned}
\frac{1}{\lambda^p}\int_{Q_i^3}|Du|^p\,dz &= \frac{|Q_i^3|}{\lambda^p}\fint_{Q_i^3}|Du|^p\,dz \\
&\overset{(46)/(51)}{\leq} |Q_i^3| \\
&\leq 2^{5p(n+2)}|Q_i^0| \\
&\overset{(70)}{\leq} \frac{c}{\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}}|Du|^p\,dz \\
&\qquad + \frac{c}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}M^s|F|^s\,dz
\end{aligned} \tag{71}$$

(as we said, we omitted a 2^s here). Similarly, using Hölder's inequality,

$$\begin{aligned}
\frac{1}{\lambda^p}\int_{Q_i^3}|F|^p\,dz &= \frac{|Q_i^3|}{M^p\lambda^p}\fint_{Q_i^3}M^p|F|^p\,dz \\
&\leq \frac{c|Q_i^0|}{M^p\lambda^p}\left(\fint_{Q_i^3}M^s|F|^s\,dz\right)^{\frac{p}{s}} \\
&\overset{(46)/(51)}{\leq} \frac{c|Q_i^0|}{M^p} \\
&\overset{(70)}{\leq} \frac{c}{M^p\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}}|Du|^p\,dz \\
&\qquad + \frac{c}{M^p\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}}M^s|F|^s\,dz\ .
\end{aligned} \tag{72}$$

Finally, since $\lambda > 1$, we have

$$\begin{aligned}
&\frac{|Q_i^0|}{\lambda^p}\left(\fint_{Q_i^3}(1+M^s|F|^s)\,dz\right)^{\frac{p}{s}} \\
&\overset{(46)/(51)}{\leq} \quad |Q_i^0|
\end{aligned}$$

$$\stackrel{(70)}{\leq} \quad \frac{c}{\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}} |Du|^p\,dz + \frac{c}{\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}} M^s|F|^s\,dz\,. \tag{73}$$

Connecting (71)–(73) to (63), we have the estimate on the cubes we were looking for:

$$\begin{aligned}
&|\{z\in Q_i^1\ :\ |Du|^p > A\lambda^p\}| \\
&\leq \frac{c}{A\lambda^p}\left\{\frac{[\omega(R)]^{\frac{s-p}{s}}}{\delta^{\frac{1}{p-1}}} + \frac{1}{M^p\delta^{\frac{1}{p-1}}} + \delta\right\}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}} |Du|^p\,dz \\
&\qquad + \frac{cM^s}{A\lambda^s}\left\{\frac{[\omega(R)]^{\frac{s-p}{s}}}{\delta^{\frac{1}{p-1}}} + \frac{1}{M^p\delta^{\frac{1}{p-1}}} + \delta\right\}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}} |F|^s\,dz\,.
\end{aligned}$$

Define

$$G \equiv G(\delta, M, R) := \frac{[\omega(R)]^{\frac{s-p}{s}}}{\delta^{\frac{1}{p-1}}} + \frac{1}{M^p\delta^{\frac{1}{p-1}}} + \delta \tag{74}$$

and the estimate above may be rewritten

$$\begin{aligned}
&|\{z\in Q_i^1\ :\ |Du|^p > A\lambda^p\}| \\
&\leq G(\delta, M, R)\left\{\frac{c}{A\lambda^p}\int_{Q_i^0\cap\{|Du|>\tau\lambda\}} |Du|^p\,dz + \frac{cM^s}{A\lambda^s}\int_{Q_i^0\cap\{|F|>\gamma\lambda\}} |F|^s\,dz\right\}\,.
\end{aligned} \tag{75}$$

Here the constant c only depends on n, N, p, ν, L, while γ and τ have been chosen in (66) and (69), respectively. The constant $\delta \in (0,1)$ is yet to be chosen.

Step 5: Final estimate. Here we are going to prove the validity of (9) provided

$$R \leq R_0\,, \tag{76}$$

where $R_0 \equiv R_0(n, N, p, \nu, L, \omega(\cdot)) > 0$ is a radius we will determine in (82) below. The general case follows via a standard covering argument.

Using the fact that the cubes $\{Q_i^0\}$ are disjoint, summing up on $i\in\mathbb{N}$ in (75) we have by (47)/(50), since $A > 1$ by (57),(62),

$$\begin{aligned}
&|\{z\in Q_R\ :\ |Du| > A^{\frac{1}{p}}\lambda\}| \\
&\leq \sum_i |\{z\in Q_i^1\ :\ |Du|^p > A\lambda^p\}| \\
&\leq G(\delta, M, R)\left\{\frac{c}{A\lambda^p}\int_{Q_R\cap\{|Du|>\tau\lambda\}} |Du|^p\,dz + \frac{cM^s}{A\lambda^s}\int_{Q_R\cap\{|F|>\gamma\lambda\}} |F|^s\,dz\right\}\,.
\end{aligned}$$

The previous inequality holds for every $\lambda \geq B\lambda_0$, recall (40). Therefore, integrating with respect to λ yields

$$\begin{aligned}
&\int_{B\lambda_0}^{\infty} \lambda^{pq-1}|\{z\in Q_R\ :\ |Du| > A^{\frac{1}{p}}\lambda\}|\,d\lambda \\
&\leq \frac{cG(\delta, M, R)}{A}\int_{B\lambda_0}^{\infty} \lambda^{pq-p-1}\int_{Q_R\cap\{|Du|>\tau\lambda\}} |Du|^p\,dz\,d\lambda
\end{aligned}$$

$$+\frac{cM^sG(\delta,M,R)}{A}\int_{B\lambda_0}^{\infty}\lambda^{pq-s-1}\int_{Q_R\cap\{|F|>\gamma\lambda\}}|F|^s\,dz\,d\lambda\,. \tag{77}$$

We recall that for any measurable function $g\geq 0$ and any $\beta>\alpha>1$

$$\int g^{\beta}\,dx=(\beta-\alpha)\int_0^{+\infty}t^{\beta-\alpha-1}\int_{\{x:g(x)>t\}}g^{\alpha}\,dx\,dt\ : \tag{78}$$

then we have, using in a standard way Fubini's theorem,

$$\begin{aligned}
&\int_{Q_R}|Du|^{pq}\,dz\\
&= \quad pq\int_0^{\infty}(A^{\frac{1}{p}}\lambda)^{pq-1}|\{z\in Q_R\ :\ |Du|>A^{\frac{1}{p}}\lambda\}|\,d(A^{\frac{1}{p}}\lambda)\\
&= \quad pq\int_0^{B\lambda_0}(\text{same})\,d\lambda+pq\int_{B\lambda_0}^{\infty}(\text{same})\,d\lambda\\
&\leq \quad A^qB^{pq}\lambda_0^{pq}|Q_R|+pqA^q\int_{B\lambda_0}^{\infty}\lambda^{pq-1}|\{z\in Q_R\ :\ |Du|>A^{\frac{1}{p}}\lambda\}|\,d\lambda\\
&\overset{(62),(77)}{\leq} \quad A^qB^{pq}\lambda_0^{pq}|Q_R|\\
&\qquad +\frac{cpqA^qG(\delta,M,R)}{\tau^{pq-p}}\int_0^{\infty}(\tau\lambda)^{pq-p-1}\int_{Q_R\cap\{|Du|>\tau\lambda\}}|Du|^p\,dz\,d(\tau\lambda)\\
&\qquad +\frac{cpqA^qM^sG(\delta,M,R)}{\gamma^{pq-s}}\int_0^{\infty}(\gamma\lambda)^{pq-s-1}\int_{Q_R\cap\{|F|>\gamma\lambda\}}|F|^s\,dz\,d(\gamma\lambda)\\
&\overset{(66),(69),(78)}{\leq} \quad A^qB^{pq}\lambda_0^{pq}|Q_R|\\
&\qquad +c\frac{q}{q-1}2^{(p+2)q}A^qG(\delta,M,R)\int_{Q_R}|Du|^{pq}\,dz\\
&\qquad +c\frac{pq}{pq-s}A^q2^{pq}M^{pq}G(\delta,M,R)\int_{Q_R}|F|^{pq}\,dz\\
&\overset{(14)}{\leq} \quad \tilde{c}A^qB^{pq}\lambda_0^{pq}|Q_R| \qquad\qquad (79)\\
&\qquad +\tilde{c}2^{(p+2)q}A^q\frac{q}{q-1}G(\delta,M,R)\left\{\int_{Q_R}|Du|^{pq}\,dz+M^{pq}\int_{Q_R}|F|^{pq}\,dz\right\}\ ,
\end{aligned}$$

where $\tilde{c}\equiv\tilde{c}(n,N,p,\nu,L)$, and the dependence on q is explicitly stated. We remark that nothing depends on λ_0 except the first term at the right hand side.

Now we look again at (74) and we first choose $\delta>0$ small enough in order to have

$$\tilde{c}\delta\leq\frac{q-1}{6qA^q2^{(p+2)q}}\ . \tag{80}$$

Since $\tilde{c}$ depends only on n,N,p,ν,L,q, this fixes also δ as a number depending only on n,N,p,ν,L,q. We can therefore select $M\equiv M(\tilde{c},\delta)\equiv M(n,N,p,\nu,L,q)>1$ large enough in order to have ("large-M-inequality" principle)

$$\frac{\tilde{c}}{M^p} \leq \frac{(q-1)\delta^{\frac{1}{p-1}}}{6qA^q 2^{(p+2)q}} \,. \tag{81}$$

We may finally select $R_0 \equiv R_0(n, N, p, \nu, L, q, \omega(\cdot)) > 0$, small enough to ensure

$$\tilde{c}[\omega(R_0)]^{\frac{s-p}{s}} \leq \frac{(q-1)\delta^{\frac{1}{p-1}}}{6qA^q 2^{(p+2)q}} \,. \tag{82}$$

Therefore, taking R as we said in (76), and keeping into account (74) and the last three inequalities, we obtain

$$\tilde{c} 2^{(p+2)q} A^q \frac{q}{q-1} G(\delta, M, R) \leq \frac{1}{2} \,. \tag{83}$$

Plugging this last inequality in (79), we may reabsorb the second-last integral of (79) in the left hand side, and finally passing to averages we get by (40)

$$\left(\fint_{Q_R} |Du|^{pq} \, dz \right)^{\frac{1}{q}} \leq c\lambda_0^p + c \left(\fint_{Q_R} |F|^{pq} \, dz \right)^{\frac{1}{q}} \,, \tag{84}$$

where now c depends also on q, i.e. $c \equiv c(n, N, p, \nu, L, q)$. Now we recall the choice of λ_0 in (39), that M has been chosen in (81) depending on n, N, p, ν, L, q, and we finally apply Hölder's inequality to have

$$\begin{aligned} \lambda_0^p &\leq c \left[\fint_{Q_{2R}} |Du|^p \, dz + \left(\fint_{Q_{2R}} |F|^s \, dz \right)^{\frac{p}{s}} + 1 \right]^d \\ &\leq c \left[\fint_{Q_{2R}} |Du|^p \, dz + \left(\fint_{Q_{2R}} |F|^{pq} \, dz \right)^{\frac{1}{q}} + 1 \right]^d \,, \end{aligned} \tag{85}$$

where again $c \equiv c(n, N, p, \nu, L, q)$, and d is defined in (10). Joining the last estimate and (84) yields (9) for all $R \leq R_0$; at this stage the constant c in (9) depends only on n, N, p, ν, L, q, and not on $\omega(\cdot)$: it is only R_0 which depends on $\omega(\cdot)$ through (82). The case $R \geq R_0$ follows from a standard covering argument; at this point c also depends on $\omega(\cdot)$, via the covering coefficients who depend on R_0. The proof of Theorem 1 is now complete.

By carefully looking at the proof of Theorem 1 we can immediately infer the statement of Theorem 2. Indeed, we first chose δ in (80), and M in (81), then, when dealing with (82), we may choose the number $\varepsilon \equiv \varepsilon(n, N, p, \nu, L, q) > 0$ in the statement of Theorem 2 small enough to have

$$\tilde{c}\varepsilon^{\frac{s-p}{s}} < \frac{(q-1)\delta^{\frac{1}{p-1}}}{6qA^q 2^{(p+2)q}}$$

and then we just estimate as follows:

$$\tilde{c}[\omega(R_0)]^{\frac{s-p}{s}} \leq \tilde{c}[a]_{BMO}^{\frac{s-p}{s}} < \frac{(q-1)\delta^{\frac{1}{p-1}}}{6qA^q 2^{(p+2)q}} \,. \tag{86}$$

The last inequality allows to recover (83), and the remainder of the proof works without any further change. This also implies that the constant c in the final estimate (9) depends only on n, N, p, ν, L, q.

Remark 2. The previous proof deserves some comments. The stopping time argument in Step 2 clearly depends on the choice of the positive quantity M, via the stopping time radius ϱ_{z_0} determined by the first occurrence of the equality (44). On the other hand the constant M is chosen at the very end, in (81). Actually the proof should be read backwards: the choice of M is only influenced by the constant $\tilde{c}$ appearing in (79). In turn $\tilde{c}$ is universal, in the sense that it only depends on n, N, p, ν, L, q, and not on the cubes Q_i^0 determined in Step 2. Therefore, once the choice of $\tilde{c}$, and therefore of M, is done in the universal way dictated by (79)–(81), we can restart from Step 2, finding the family of cubes $\{Q_i^0\}$ with a fixed value of M, and then proceed toward the end through Steps 3–5.

Remark 3. The technique developed here allows to get a rather precise dependence of the constant c in estimate (9) on the integrability parameter q. We shall observe interesting similarities with the estimates obtained in the elliptic case (3) via maximal function techniques in [1, 10, 14]; this is not surprising, since the local use of estimates (18) and (21), in combination with Vitali's covering lemma, in some sense emulates the use of the maximal function. For the sake of simplicity we shall confine ourselves to the model problem (1). We go back to (79); since $a(z) \equiv 1$, then we have $\omega(R) \equiv 0$; then, taking δ and M in such a way to obtain equalities in (80) and (81), respectively, and combining this with (79), recalling that B only depends on n, p by (40) we obtain

$$\left(\fint_{Q_R} |Du|^{pq} \, dz \right)^{\frac{1}{q}} \leq cA\lambda_0^p + c \left(\frac{q}{q-1} A^q 2^{(p+2)q} \right)^{\frac{p}{p-1}} \left(\fint_{Q_R} |F|^{pq} \, dz \right)^{\frac{1}{q}} ,$$

and therefore, by (85)

$$\begin{aligned} &\left(\fint_{Q_R} |Du|^{pq} \, dz \right)^{\frac{1}{q}} \\ &\leq \; cA \left[\fint_{Q_{2R}} |Du|^p \, dz + \left(\frac{q}{q-1} A^q 2^{(p+2)q} \right)^{\frac{p}{p-1}} \left(\fint_{Q_{2R}} |F|^{pq} \, dz \right)^{\frac{1}{q}} + 1 \right]^d . \end{aligned} \quad (87)$$

The previous a priori estimate reveals the same asymptotic behavior for $q \searrow 1$ of the constant appearing in the Hardy-Littlewood maximal function estimate (see [2]), and it is the same appearing also in the a priori elliptic estimates of [1, 10, 14]. On the other hand this is harmless: when q is approaching 1, a priori estimates bounding the L^{pq} norm of Du by the L^{pq} norm of F are simply given by Theorem 3, that works for any small choice of δ, with all constants remaining bounded as $\delta \searrow 0$, see [16]. Note that the scaling of estimates (9) and (87) is in perfect accordance with that of the known a priori estimates for the evolutionary p-Laplacean operator: when $F \equiv 0$, letting $q \nearrow \infty$ in (87) we obtain, up to an absolute constant, the sup estimates (18) and (21) for $\theta = \gamma^2 = R^2$.

5. A few possible extensions

For the sake of simplicity, and in order the emphasize the main ideas, we have up to now restricted ourself to the analysis of the model cases (1) and (4); nevertheless the methods presented in this paper immediately apply to several more general situations.

As for the right hand side, as we already mentioned we could have considered instead of (4) the system

$$u_t - \operatorname{div}(a(z)|Du|^{p-2}Du) = \operatorname{div} f \ ,$$

which is equivalent to (4) through

$$f = |F|^{p-1}\frac{F}{|F|} \qquad \longleftrightarrow \qquad F = |f|^{\frac{1}{p-1}}\frac{f}{|f|} \tag{88}$$

and (besides complicating the exponents in the proof) leads to the more awkward statement "if $f \in L^{pq/(p-1)}_{\mathrm{loc}}$ then $|Du|^p \in L^q_{\mathrm{loc}}$", together with an equally awkward estimate of the type (9).

In what follows we shall describe other parabolic problems to which our techniques apply, sketching the main modifications to the proof, as far as the estimates in Steps 2-5 from Section 4 are concerned. The approximation part of Step 1 can be easily reconstructed as in Section 4.

The vectorial case $N > 1$, and different operators. The reader will recognize that the main property of the evolutionary p-Laplacean operator used in the proof of Theorem 1, apart from the obvious monotonicity and growth properties used in the comparison estimates of Steps 1 and 3 from Section 4, is the possibility to get the explicit L^∞ bounds (18) and (21). In turn, these are used to get the fundamental Lemmas 1 and 2, and eventually the crucial estimate (56) on the comparison map v_i. This observation allows to extend our results to a family of degenerate parabolic systems whose special structure allows for the L^∞ bounds (18) and (21), and to which the results of Theorems 1 and 2 extend.

We may consider systems of the type

$$u_t - \operatorname{div}[a(z)g(|Du|)Du] = \operatorname{div}(|F|^{p-2}F) \ . \tag{89}$$

The function a is in VMO, and the assumptions on the function $g : \mathbb{R}^+ \to \mathbb{R}^+$, which is in $C^1(\mathbb{R}\setminus\{0\})$, are different depending on p:

$$\text{if } p \geq 2 \ , \qquad g'(s) \geq 0 \qquad \forall \quad s > 0 \tag{90}$$

$$\nu s^{p-2} \leq g(s) \leq L s^{p-2} \qquad \forall \quad s > 0 \tag{91}$$

$$\frac{|g'(s)|s}{g(s)} \leq \begin{cases} L & \text{if } p \geq 2 \\ \theta(<1) & \text{if } p < 2 \end{cases} \qquad \forall \quad s > 0 \tag{92}$$

$$\langle g(|w_2|)w_2 - g(|w_1|)w_1, w_2 - w_1\rangle \geq \nu(\mu^2 + |w_1|^2 + |w_2|^2)^{\frac{p-2}{2}}|w_2 - w_1|^2 \ , \tag{93}$$

for all $w_1, w_2 \in \mathbb{R}^{nN}$. Under the previous assumptions, the conditions stated at page 217 in [8] are satisfied, and the solutions to the comparison system

$$v_t - \operatorname{div}[\tilde{a}g(|Dv|)Dv] = 0$$

satisfy the gradient bounds (18) and (21) for $p \geq 2$ and $p < 2$ respectively. Accordingly, the main modification in the proof of Section 4 is the use of the comparison system

$$\begin{cases} (v_i)_t - \operatorname{div}[a_i g(|Dv_i|)Dv_i] = 0 & \text{in } Q_i^2 \\ v_i \equiv u & \text{on } \partial_p Q_i^2 \end{cases} \tag{94}$$

instead of (52). Then one gets the upper bound (56), and the remainder of the proof follows thanks to the growth, ellipticity and monotonicity assumptions (91)–(93), thereby replacing the use of Lemma 6. Finally, the integrability results of Theorems 1 and 2 follow.

Another left hand side structure we can treat with the methods proposed here is the one already considered in [19] for the elliptic case. Let us consider a $n^2 \times N^2$ tensor $A(z) \equiv \{A_{i,j}^{\alpha,\beta}(z)\}$, defined on C, and whose entries are strongly VMO in the sense of Definition 1: assume the tensor $A(z)$ satisfies the following ellipticity and boundedness conditions:

$$\nu|\lambda|^2 \leq \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} A_{i,j}^{\alpha,\beta}(z)\lambda_i^\alpha \lambda_j^\beta \leq L|\lambda|^2 \,, \qquad \forall\, \lambda \in \mathbb{R}^{nN}, \qquad \forall\, x \in C \,.$$

Then the result of Theorem 1 also holds for solutions to the system

$$u_t - \operatorname{div}(\langle A(z)Du, Du\rangle^{\frac{p-2}{2}} Du) = \operatorname{div}(|F|^{p-2}F) \,. \tag{95}$$

Accordingly, assuming the tensor $\{A_{i,j}^{\alpha,\beta}(z)\}$ to have BMO entries, the analog of Theorem 2 for solutions of (95) also follows. The proofs for (95) are very much similar to those already considered for Theorems 1 and 2.

We can also consider systems of the type

$$u_t - \operatorname{div}[\tilde{g}(z, |Du|)Du] = \operatorname{div}(|F|^{p-2}F) \,, \tag{96}$$

with $\tilde{g} : C \times \mathbb{R}^+ \to \mathbb{R}^+$. This time we assume that the function

$$w \in \mathbb{R}^{nN} \mapsto \tilde{g}(z_0, |w|) \in \mathbb{R} \,,$$

satisfies the assumptions required on the function g appearing in (89), that is (90)–(93), uniformly with respect to $z_0 \in C$. Moreover the following type of continuity (or rather, limited discontinuity) assumption is required with respect to the variable z:

$$|\tilde{g}(z_2, |w|)) - \tilde{g}(z_1, |w|)| \leq L\omega(|z_2 - z_1|)(1 + |w|)^{p-2} \tag{97}$$

for every $w \in \mathbb{R}^{nN}$ and $z_1, z_2 \in C$, where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded, non-decreasing function such that

$$\lim_{R \to 0} \omega(R) = \ell \geq 0 \,. \tag{98}$$

If $\ell = 0$ then $\tilde{g}$ is continuous. In this case, the only differences are the following: first, the comparison function v_i is now defined as the unique solution to the new comparison system

$$\begin{cases} (v_i)_t - \operatorname{div}[\tilde{g}(z_i, |Dv_i|)Dv_i] = 0 & \text{in } Q_i^2 \\ v_i \equiv u & \text{on } \partial_p Q_i^2 \,, \end{cases}$$

where we recall that z_i is the center of the cylinder Q_i. Once again the L^∞-bound in (56) follows for v_i, thanks to the assumptions satisfied by $w \mapsto \tilde{g}(z_0, w)$. Second, estimate (59) must be worked out directly using (97), and it is not necessary to use Hölder's inequality there; in particular, the use of the higher integrability result of Theorem 3 can be avoided, and instead of the quantity at the left hand side of (45), one can use the simpler

$$\left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} |Du|^p \, dz\right)^{\frac{1}{p}} + \left(\fint_{Q(\lambda^{2-p}\varrho^2,\varrho)} M^p |F|^p \, dz\right)^{\frac{1}{p}} .$$

Then, following the proof of Section 4, we finally come to:

Theorem 4. *Let $u \in C^0((0,T); L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution to* (96)*, where p satisfies* (8)*, the function $\tilde{g}$ is as described above, and $\omega(\cdot)$ satisfies* (98) *with $\ell = 0$. Assume that $|F|^p \in L^q_{\mathrm{loc}}(C)$ for some $q > 1$. Then $|Du|^p \in L^q_{\mathrm{loc}}(C)$. Moreover, estimate* (9) *holds, where d is as in* (10)*.*

The appropriate analog of Theorem 2 is instead the following:

Theorem 5. *Let $u \in C^0((0,T); L^2(\Omega, \mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution to* (96)*, where p satisfies* (8)*, the function $\tilde{g}$ is as described above, and $\omega(\cdot)$ satisfies* (98)*. Fix $q > 1$, and assume that $|F|^p \in L^q_{\mathrm{loc}}(C)$. Then there exists a number $\varepsilon \equiv \varepsilon(n, N, p, \nu, L, q) > 0$ such that if $\ell \leq \varepsilon$ then $|Du|^p \in L^q_{\mathrm{loc}}(C)$. Moreover, there exists a constant $c \equiv c(n, N, p, \nu, L, q) > 1$, such that if $Q_{2R} \subset\subset C$, then* (9) *holds, with d as in* (10)*.*

The scalar case $N = 1$. In the scalar case the L^∞ bounds (18) and (21) are true for general parabolic equations with no additional structure properties as dependence upon Du specified via the quantity $|Du|$; this is a peculiarity of case $N = 1$, see again [8]. Therefore we shall consider a general parabolic equation of the type

$$u_t - \operatorname{div}[a(z)A(Du)] = \operatorname{div}(|F|^{p-2}F) , \tag{99}$$

where the vector field $A : \mathbb{R}^N \to \mathbb{R}^N$ is $C^1(\mathbb{R} \setminus \{0\})$, and satisfies the following growth and ellipticity assumptions:

$$|A(w)| + |DA(w)|(\mu^2 + |w|^2)^{\frac{1}{2}} \leq L(\mu^2 + |w|^2)^{\frac{p-1}{2}} , \tag{100}$$

$$DA(w)\lambda \otimes \lambda \geq \nu(\mu^2 + |w|^2)^{\frac{p-2}{2}} |\lambda|^2 \tag{101}$$

for every $w, \lambda \in \mathbb{R}^n$, where, as usual, $0 < \nu \leq L$, and $\mu \in [0, 1]$. It is then standard to verify that assuming (101) implies the existence of a constant $c \equiv c(n, p, \nu) > 0$ such that the monotonicity condition

$$\langle A(w_2) - A(w_1), w_2 - w_1 \rangle \geq c(\mu^2 + |w_1|^2 + |w_2|^2)^{\frac{p-2}{2}} |w_2 - w_1|^2 ,$$

holds for all $w_1, w_2 \in \mathbb{R}^n$. Under assumptions (100) and (101), the results of Theorem 1 and 2 hold for general weak solutions to equation (99), with the same proof given in

Section 4; the only change comes again when considering (52), which is now replaced by the equation

$$\begin{cases} (v_i)_t - \operatorname{div}[a_i A(Dv_i)] = 0 & \text{in } Q_i^2 \\ v_i \equiv u & \text{on } \partial_p Q_i^2 \ . \end{cases}$$

For weak solutions to the parabolic equation (99), Theorems 1 and 2 hold with exactly the same proof of Section 4. As in the vectorial case, we can also consider more general equations such as

$$u_t - \operatorname{div} \tilde{A}(z, Du) = \operatorname{div}(|F|^{p-2}F) \ . \qquad (102)$$

Here we assume that for every $z \in C$ the vector field $w \mapsto \tilde{A}(z, w)$ satisfies assumptions (100),(101) uniformly with respect to $z \in C$. Moreover, the map $\tilde{A} : C \times \mathbb{R}^n \to \mathbb{R}^n$ is required to satisfy the continuity property

$$|\tilde{A}(z_2, w) - \tilde{A}(z_1, w)| \leq \omega(|z_2 - z_1|)(1 + |w|)^{p-1} \ ,$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is the usual bounded, non-decreasing function. At this point, the results of Theorems 4 and 5 follows for weak solutions to (102). Note that any type of modulus of continuity is allowed [11].

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