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Graphs of maps between manifolds in trace spaces and with vanishing mean oscillation

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Abstract

We give a positive answer to a question raised by Alberti in connection with a recent result by Brezis and Nguyen. We show the existence of currents associated with graphs of maps in trace spaces that have vanishing mean oscillation. The degree of such maps may be written in terms of these currents, of which we give some structure properties. We also deal with relevant examples.

Keywords: Vanishing mean oscillation; Currents; Trace spaces

In this paper we give a positive answer to a question raised by Alberti in connection with a recent result by Brezis and Nguyen.

In recent years there has been a growing interest concerning the notion of *degree* for mappings which do not possess the classical regularity properties.

An example of this are the papers [8,9] in which Brezis and Nirenberg investigate the class VMO of functions with *vanishing mean oscillation*, i.e., functions whose mean oscillation on balls (that is, the average of the difference from the integral average) converges to zero with the radius of the ball:

$$\sup_{x_0} \fint_{B_r(x_0)} \left| u - (u)_{B_r(x_0)} \right| \to 0,$$

see Section 1 for more details.

Very recently, Brezis and Nguyen [7] dealt with *continuity properties* for the degree in the same framework. In particular, they show the continuity of the degree of maps from the n-dimensional unit sphere $\mathbb{S}^n$ to itself, under the joint convergence in the BMO and $W^{1-1/n,n}$ norms, where BMO is the spaces of functions whose mean oscillation on balls is just bounded.

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We recall that the degree of a smooth map $u : \mathbb{S}^n \to \mathbb{S}^n$ is defined by

$$\deg(u) := \frac{1}{|\mathbb{S}^n|} \int_{\mathbb{S}^n} \det \nabla u(x)\, d\sigma,$$

where $\det \nabla u = \det(\nabla u, u)$, viewing $(\nabla u, u)$ as a square matrix of order $n+1$. We thus have

$$\det \nabla u(x) = u^{\#}\omega_n(x), \quad \omega_n := \sum_{j=1}^{n+1} (-1)^{j-1} y^j\, \widehat{dy^j},$$

where $\widehat{dy^j} := dy^1 \wedge \cdots \wedge dy^{j-1} \wedge dy^{j+1} \wedge \cdots \wedge dy^{n+1}$. Therefore, by the *area formula* we get

$$\left|\mathbb{S}^n\right| \cdot \deg(u) = \int_{\mathbb{S}^n} u^{\#}\omega_n = \int_{\mathcal{G}_u} \omega_n,$$

where $\mathcal{G}_u$ is the graph of u. Therefore, in the smooth case the degree can be written in terms of the current associated to the graph of u.

Alberti analyzed the paper [7] in his report on Mathematical Reviews [2], and he raised some interesting questions which are the starting point of our work. In particular, he discussed the possible existence of a *current* T_u associated with the graph of a VMO-map u from $\mathbb{S}^n$ to $\mathbb{S}^n$ belonging also to the space $W^{1-1/n,n}(\mathbb{S}^n, \mathbb{R}^{n+1})$ of *traces* of Sobolev functions $W^{1,n}(B^{n+1}, \mathbb{R}^{n+1})$.

Using the classical extension U introduced by Gagliardo [11] and used e.g. by Bethuel and Demengel [6] for Sobolev classes between manifolds, we give a positive answer to this question by proving, see Theorem 3.1 below (where, as in the rest of the paper, we deal with the general case of maps between Riemannian manifolds $\mathcal{X}$ and $\mathcal{Y}$), that the current

$$T_u := (-1)^{n-1}(\partial G_U) \llcorner \left(\mathbb{S}^n \times \mathbb{S}^n\right)$$

is well-defined. Notice that the degree in the sense of Brezis and Nguyen [7] agrees with the action of our current T_u on the form ω_n, as in the smooth case. An important feature is that the average $(u)_{B_r(x_0)}$ has small distance from the target sphere $\mathbb{S}^n$, independently of the centers of the balls, provided that the radius is small; a similar argument was used also in the approximation theorem by Schoen and Uhlenbeck [16].

Our approach is therefore based on the extension of $u : \mathbb{S}^n_x \to \mathbb{S}^n_y$ to the unit ball B^{n+1} by suitable averages in the spirit of Gagliardo [11]; the VMO condition allows then to modify the extension in a neighbourhood of $\mathbb{S}^n_x$ in order to preserve the constraint on the image.

The current T_u associated to the graph of such a map u is an *integral flat chain*, but in general it may have infinite mass, see Example 3.9. The current T_u acts on n-forms defined in $\mathbb{S}^n_x \times \mathbb{S}^n_y$, and it decomposes as $T_u = T_{u(0)} + T_{u(1)} + \cdots + T_{u(n)}$, where $T_{u(j)}$ acts on forms with j vertical differentials dy. It turns out that u is a function of *bounded variation* if and only if the first non-trivial component $T_{u(1)}$ has finite mass, see Proposition 3.4. In addition, T_u is a *Cartesian current* in the sense of Giaquinta, Modica and Souček [12] provided it has finite mass, see Proposition 4.1. Moreover, something on the higher order components may be said in the case of Sobolev $W^{1,q}$-maps, see Remark 3.8.

We also show that if a sequence of maps in $W^{1-1/n,n} \cap \mathrm{VMO}$ converges strongly in BMO and in $W^{1-1/n,n}$, then the corresponding graphs weakly converge (in the sense of currents) to the graph of the limit map, see Theorem 5.2. This extends the continuity property of the degree by Brezis and Nguyen [7].

We now mention some open questions raised in this context. From the work of Giaquinta, Modica and Souček [12], it turns out that in order to deal with currents carried by graphs of non-smooth maps u, a fundamental property is the *approximate differentiability a.e.* A function of bounded variation is approximately differentiable a.e., compare [5], but we do not know if the same holds true for functions in trace spaces $W^{1-1/p,p}$.

More precisely, in dimension two there is a function f in $C^{1,\alpha}$ for each $0 < \alpha < 1$ that does not satisfy the so-called *weak Sard property*, see [3]. Correspondingly, the function $b = \nabla^{\perp} f$ belongs to the fractional Sobolev classes $W^{s,p}$ for each $p > 1$ and $0 < s < 1$. Therefore, such a function b does not belong to the class $t^{1,1}$ (functions with first order Taylor expansion in L^1-sense), see [4]. If it were the case, in fact, the corresponding function f had to satisfy the C^2-Lusin property and, definitely, the weak Sard property. Notice that the existence of a first order Taylor expansion in L^1-sense is a slightly stronger property than the approximate differentiability a.e. However, due to this example it is reasonable to conjecture the existence of maps in $W^{1-1/p,p}$ that are not a.e. approximately differentiable.

Another feature appears when analyzing the relevant example by S. Müller [14] about the singular part of the *distributional determinant* $\mathrm{Det}\,\nabla u$, see Section 3. It is based on a map $u \in C^{0,s}(B^2, \mathbb{R}^2)$, where s is the dimension of the Cantor "middle thirds" set, such that $u \in W^{1-1/2,2} \cap \mathrm{VMO}$. We show that the highest order component $T_{u(2)}$ of the corresponding current acts on forms of the type $\varphi(x)\,dy^1 \wedge dy^2$ exactly as the distributional determinant of u, hence

$$\big\langle T_u, \varphi(x)\,dy^1 \wedge dy^2\big\rangle = \langle \mathrm{Det}\,\nabla u, \varphi\rangle = \big\langle V' \otimes V', \varphi\big\rangle \quad \forall \varphi \in C_c^\infty\big(B^2\big)$$

where V is the Cantor–Vitali function. Therefore, the distribution $T_{u(2)}$ concentrates on the set $(C \times C) \times \mathbb{R}^2$, and this is in contrast with the "smooth" case of Sobolev maps $u \in W^{1,2}(B^2, \mathbb{R}^2)$, where the action is written by means of the *pointwise determinant* $\det \nabla u$, namely

$$\big\langle T_u, \phi(x,y)\,dy^1 \wedge dy^2\big\rangle = \int_{B^2} \phi\big(x, u(x)\big) \det \nabla u(x)\,dx \quad \forall \phi \in C_c^\infty\big(B^2 \times \mathbb{R}^2\big).$$

However, it seems a hard problem to completely determine the structure of the current T_u. According to the example by S. Müller, assume e.g. that $u \in W^{1,1}(B^2, \mathbb{R}^2)$ is a bounded function in VMO, with pointwise determinant in $L^1(B^2)$ and such that the distributional determinant is a finite measure. Even in this case, we are not able to determine if the corresponding current T_u has finite mass and, if this is the case, how to write explicitly the action of T_u on the forms $\phi(x,y)\,dy^1 \wedge dy^2$. This open problem seems to be strictly related to the study of minimal regularity properties under which the distributional determinant $\mathrm{Det}\,\nabla u_\varepsilon$ is (weakly) continuous, where u_ε is the average of u on the ball B_ε.

1. Notation and preliminary results

In this paper we deal with mappings $u : \mathcal{X} \to \mathcal{Y}$ defined between smooth, connected, compact Riemannian manifolds $\mathcal{X}$ and $\mathcal{Y}$ without boundary. Actually, we let $\mathcal{X} = \partial\mathcal{M}$ and $n := \dim \mathcal{X}$, the model case being $\mathcal{X} = \mathbb{S}^n$, the unit sphere in $\mathbb{R}^{n+1}$. By Nash–Moser theorem, we shall assume that $\mathcal{M}$ and $\mathcal{Y}$ are isometrically embedded into $\mathbb{R}^l$ and $\mathbb{R}^N$, respectively, and denote $\mathrm{i}_{\mathcal{M}} \hookrightarrow \mathbb{R}^l$, $\mathrm{i}_{\mathcal{Y}} \hookrightarrow \mathbb{R}^N$ such imbeddings. We shall equip $\mathcal{M}$, $\mathcal{X}$, and $\mathcal{Y}$ with the metric induced by the Euclidean norms on the ambient space.

For $x \in \mathcal{X}$ and $0 < h < r_0$, where $r_0 > 0$ is the injectivity radius of $\mathcal{X}$, denote by $B(x,h)$ the geodesic n-ball of radius h centered at $x \in \mathcal{X}$. For $0 < \delta < r_0$ small, let

$$\mathcal{M}_\delta := \big\{z \in \overline{\mathcal{M}} \;\big|\; \mathrm{dist}(z, \mathcal{X}) \leqslant \delta\big\}, \quad \mathcal{X} = \partial\mathcal{M}.$$

There exists $0 < d < r_0$ such that the nearest point projection $\Pi_{\mathcal{M}}$ from $\mathcal{M}_d$ onto $\mathcal{X}$ is well-defined, and hence we may consider the fibration

$$\Phi^{-1} : \mathcal{X} \times [0,d] \to \mathcal{M}_d$$

such that $\Phi(z) = (\Pi_{\mathcal{M}}(z), \mathrm{dist}(z, \mathcal{X}))$ for $z \in \mathcal{M}_d$. Moreover, let $\mathcal{O}$ denote a neighbourhood of $\mathcal{Y}$ in $\mathbb{R}^N$ such that the nearest point projection π from $\mathcal{O}$ onto $\mathcal{Y}$ is a smooth fibration.

Trace spaces. Let p be a given exponent, $1 < p < \infty$, and denote by $\mathfrak{p}$ the integer part of p. We recall, see e.g. [1], that the fractional Sobolev space $W^{1/p}(\mathcal{X}) := W^{1-1/p,p}(\mathcal{X})$ is the Banach space of L^p-functions $u : \mathcal{X} \to \mathbb{R}$ which have finite $W^{1-1/p,p}$-seminorm

$$|u|_{1/p,\mathcal{X}}^p := \int_{\mathcal{X}} \int_{\mathcal{X}} \frac{|u(x) - u(y)|^p}{|x-y|^{n+p-1}}\,dx\,dy$$

endowed with the norm

$$\|u\|_{1/p,\mathcal{X}}^p := \|u\|_{L^p(\mathcal{X})}^p + |u|_{1/p,\mathcal{X}}^p. \tag{1.1}$$

$W^{1/p}(\mathcal{X},\mathbb{R}^N)$ is the space of vector-valued maps $u=(u^1,\dots,u^N)$ such that $u^j\in W^{1/p}(\mathcal{X})$ for every $j=1,\dots,N$. Since $\mathcal{X}=\partial\mathcal{M}$ for some smooth manifold $\mathcal{M}$, then $W^{1/p}(\partial\mathcal{M},\mathbb{R}^N)$ can be characterized as the space of functions u that are *traces* of functions U in the Sobolev space $W^{1,p}(\mathcal{M},\mathbb{R}^N)$,

$$U_{|\mathcal{X}}=u \quad \text{as traces.}$$

For $\Omega=\mathcal{M}_\delta$, where $0<\delta\leqslant d$, or $\Omega=\mathcal{M}$, we shall denote

$$W^{1,p}(\Omega,\mathcal{Y}):=\bigl\{U\in W^{1,p}\bigl(\Omega,\mathbb{R}^N\bigr)\bigm| U(z)\in\mathcal{Y} \text{ for } \mathcal{H}^{n+1}\text{-a.e. } z\in\Omega\bigr\},$$
$$W^{1/p}(\mathcal{X},\mathcal{Y}):=\bigl\{u\in W^{1/p}\bigl(\mathcal{X},\mathbb{R}^N\bigr)\bigm| u(x)\in\mathcal{Y} \text{ for } \mathcal{H}^{n}\text{-a.e. } x\in\mathcal{X}\bigr\}.$$

The extension problem. Following Bethuel and Demengel [6], to each map $u\in W^{1/p}(\mathcal{X},\mathcal{Y})$ we associate a function $\widetilde{U}\in W^{1,p}(\mathcal{M}_d,\mathbb{R}^N)$ given by $\widetilde{U}=v\circ\Phi$, where

$$v(x,h):=\fint_{B(x,h)} u\,d\mathcal{H}^n,\quad (x,h)\in\mathcal{X}\times\,]0,d]. \tag{1.2}$$

It turns out that $\widetilde{U}\in W^{1,p}(\mathcal{M}_d,\mathbb{R}^N)$ and is smooth outside $\mathcal{X}$, with $\widetilde{U}_{|\mathcal{X}}=u$ in the sense of the traces, compare [11]. Moreover, setting $u_h(x):=v(x,h)$, we have:

Proposition 1.1. *$u_h\to u$ strongly in $W^{1/p}$ as $h\to 0^+$.*

Proof. Assume for simplicity $\mathcal{M}_d=\mathcal{X}\times[0,d]$ and $d=1$. Define

$$V_h(x,t):=v\bigl(x,\phi_h(t)\bigr),\quad \phi_h(t):=(1-h)t+h.$$

We have $D_xV_h(x,t)=D_xv(x,\phi_h(t))$ and $D_tV_h(x,t)=D_tv(x,\phi_h(t))(1-h)$, so that

$$\lim_{h\to0}\int_{\mathcal{M}_d}\bigl|DV_h(x,t)\bigr|^p\,dx\,dt=\int_{\mathcal{M}_d}\bigl|Dv(x,t)\bigr|^p\,dx\,dt.$$

Since $V_h(x,0)=u_h(x)$, the claim follows. □

In [6] Bethuel and Demengel showed that even in the critical case $p=n+1$ there exists a positive radius $\delta\in(0,d)$ such that for each $z\in\mathcal{M}_\delta$ we have $\widetilde{U}(z)\in\mathcal{O}$. In fact, they estimate the average

$$\fint_{B(x,h)}\bigl|v(x,h)-u(y)\bigr|^{n+1}\,d\mathcal{H}^n(y)\leqslant C\int_{B(x,h)}\int_{B(x,h)}\frac{|u(y')-u(y)|^{n+1}}{|y'-y|^{2n}}\,d\mathcal{H}^n\bigl(y'\bigr)\,d\mathcal{H}^n(y) \tag{1.3}$$

and hence, by absolute continuity, taking $h>0$ small they deduce that the distance of $v(x,h)$ from $\mathcal{Y}$ is small, whence $v(x,h)\in\mathcal{O}$. Therefore, *the mapping $U:=\pi\circ\widetilde{U}_{|\mathcal{M}_\delta}$ belongs to $W^{1,n+1}(\mathcal{M}_\delta,\mathcal{Y})$, is smooth outside $\mathcal{X}$ and satisfies $U_{|\mathcal{X}}=u$ in the sense of the traces*. As a consequence, they showed that the extension problem is reduced to a related problem for smooth maps from $\mathcal{X}$ into $\mathcal{Y}$. If e.g. $\mathcal{X}=\mathbb{S}^n$, each map $u\in W^{1/(n+1)}(\mathbb{S}^n,\mathcal{Y})$ is the trace of a Sobolev map $U\in W^{1,n+1}(B^{n+1},\mathcal{Y})$ provided that the nth-homotopy group $\pi_n(\mathcal{Y})$ is trivial.

The class VMO. We now see that a similar argument holds true in the subclass of VMO-functions. The BMO-seminorm of a function $u\in L^1(\mathcal{X},\mathbb{R}^N)$ is given by

$$\|u\|_{\mathrm{BMO}}:=\sup\bigl\{J_\varepsilon^u(x)\bigm| 0<\varepsilon<r_0,\ x\in\mathcal{X}\bigr\}$$

where we have set

$$J_\varepsilon^u(x):=\fint_{B(x,\varepsilon)}\bigl|u-\bar{u}_\varepsilon(x)\bigr|\,d\mathcal{H}^n,\quad \bar{u}_\varepsilon(x):=\fint_{B(x,\varepsilon)}u\,d\mathcal{H}^n$$

and u belongs to $\mathrm{BMO}(\mathcal{X},\mathbb{R}^N)$ if $u\in L^1(\mathcal{X},\mathbb{R}^N)$ and $\|u\|_{\mathrm{BMO}}<\infty$.

The class $\mathrm{VMO}(\mathcal{X}, \mathbb{R}^N)$ is given by the completion of the class of smooth maps from $\mathcal{X}$ into $\mathbb{R}^N$ with respect to the BMO-seminorm. Moreover, Saranson [15] showed that *a map u belongs to the class* $\mathrm{VMO}(\mathcal{X}, \mathbb{R}^N)$ *if and only if* $u \in \mathrm{BMO}(\mathcal{X}, \mathbb{R}^N)$ *and the limit* $J^u_\varepsilon(x) \to 0$ *as* $\varepsilon \to 0^+$ *uniformly in* $x \in \mathcal{X}$, see [8] for a proof of this criterion. Therefore, the estimate (1.3) yields that

$$W^{1/p}(\mathcal{X}, \mathcal{Y}) \subset \mathrm{VMO}\big(\mathcal{X}, \mathbb{R}^N\big) \quad \text{if } p - 1 \geqslant n := \dim \mathcal{X}.$$

Such an inclusion is false in general if $p < n + 1$. For example, taking

$$\mathcal{X} = \mathbb{S}^2, \qquad \mathcal{Y} = \mathbb{S}^1, \qquad u(x_1, x_2, x_3) := \frac{(x_1, x_2)}{|(x_1, x_2)|}, \tag{1.4}$$

then $u \in W^{1/p}(\mathbb{S}^2, \mathbb{S}^1)$ for each $p < 3$ but $u \notin \mathrm{VMO}$. We thus denote

$$W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y}) := W^{1/p}(\mathcal{X}, \mathcal{Y}) \cap \mathrm{VMO}\big(\mathcal{X}, \mathbb{R}^N\big).$$

Arguing as above, one readily obtains:

Proposition 1.2. *Let* $1 < p < \infty$ *and* $u \in W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y})$. *There exists a positive* δ *and a function* $U \in W^{1,p}(\mathcal{M}_\delta, \mathcal{Y})$ *that is smooth outside* $\mathcal{X}$ *and satisfies* $U_{|\mathcal{X}} = u$ *in the sense of the traces.*

Proof. Define $v(x, h)$ as in (1.2) and observe that

$$\mathrm{dist}\big(v(x, h), \mathcal{Y}\big) \leqslant \fint_{B(x,h)} \big|v(x, h) - u(y)\big|\, d\mathcal{H}^n(y) =: J^u_h(x).$$

Choose $\sigma > 0$ so that the tubular neighbourhood of $\mathcal{Y}$ radius $\sigma > 0$ is contained in $\mathcal{O}$. Since $J^u_h(x) \to 0$ as $h \to 0^+$ uniformly in x, it suffices to take $\delta > 0$ small enough in such a way that $J^u_h(x) < \sigma$ for each $0 < h < \delta$ and $x \in \mathcal{X}$, and define $U := \pi \circ \widetilde{U}$, where $\widetilde{U} := v \circ \Phi$. □

As a consequence, if e.g. $\mathcal{X} = \mathbb{S}^n$, as in [6] one readily obtains:

Corollary 1.3. *If* $\pi_n(\mathcal{Y}) = 0$, *for any* $p > 1$ *each map* $u \in W^{1/p} \cap \mathrm{VMO}(\mathbb{S}^n, \mathcal{Y})$ *is the trace of a smooth Sobolev map* $U \in W^{1,p}(B^{n+1}, \mathcal{Y})$.

For future use we finally point out the following

Corollary 1.4. *Let* $1 < p < \infty$ *and* $\{u_k\} \subset W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y})$ *such that* $\|u_k - u\|_{\mathrm{BMO}} \to 0$ *as* $k \to \infty$ *for some* $u \in W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y})$. *Then, in Proposition 1.2 we may choose* $\delta > 0$ *uniformly with respect to* $\{u_k\}$.

Proof. Observe that for each x and h

$$J^{u_k}_h(x) \leqslant \|u_k - u\|_{\mathrm{BMO}} + J^u_h(x).$$

Therefore, we find $\bar{k}$ and $\delta > 0$ such that for $k \geqslant \bar{k}$ we have $J^{u_k}_h(x) \leqslant \sigma$ for each $0 < h < \delta$ and $x \in \mathcal{X}$. The claim follows. □

A density result. Using the above arguments and a partition of unity on $\mathcal{X}$, Brezis and Nirenberg [8] proved the following density result:

Theorem 1.5 *(Brezis–Nirenberg). Let* $u \in W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y})$. *There exists a sequence of smooth maps* $\{u_k\} \subset W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y})$ *such that* $u_k \to u$ *strongly in* $W^{1/p}$ *and* $\|u_k - u\|_{\mathrm{BMO}} \to 0$ *as* $k \to \infty$.

2. Semi-currents carried by graphs

In this section we discuss the notion of (semi-)current G_u carried by the graph of a map u in a trace space $W^{1/p}(\mathcal{X},\mathcal{Y})$, according to [13]. If u has vanishing mean oscillation, it turns out that G_u satisfies a null-boundary condition that fails to hold outside the class VMO. We then show that the Jump part of the derivative of a map of bounded variation is zero, provided that the given map has vanishing mean oscillation.

Semi-currents. Every compactly supported smooth differential k-form $\omega \in \mathcal{D}^k(\mathcal{X}\times\mathcal{Y})$, where $k \leqslant n$, splits as a sum $\omega = \sum_{j=0}^{\underline{k}} \omega^{(j)}$, $\underline{k} := \min(k, M)$, where $M := \dim \mathcal{Y}$. Here the $\omega^{(j)}$'s are the k-forms that contain exactly j differentials in the vertical $\mathcal{Y}$ variables. For fixed $r = 1,\dots,\underline{k}$ we denote by $\mathcal{D}^{k,r}(\mathcal{X}\times\mathcal{Y})$ the subspace of $\mathcal{D}^k(\mathcal{X}\times\mathcal{Y})$ of k-forms of the type $\omega = \sum_{j=0}^{r} \omega^{(j)}$. The dual space of "semi-currents" is denoted by $\mathcal{D}_{k,r}(\mathcal{X}\times\mathcal{Y})$. Of course we have $\mathcal{D}_{k,\underline{k}} = \mathcal{D}_k$, the space of all k-currents. A similar notation holds by replacing $\mathcal{X}$ and $\mathcal{Y}$ with $\mathcal{M}$ (or $\mathcal{M}_d$) and $\mathbb{R}^N$, respectively.

Example 2.1. If $U \in W^{1,p}(\mathcal{M},\mathbb{R}^N)$, then G_U is a well-defined $(n+1,\mathfrak{p})$-current in $\mathcal{D}_{n+1,\mathfrak{p}}(\mathcal{M}\times\mathbb{R}^N)$ and, in an approximate sense, $G_U := (\mathrm{Id}_{\mathcal{M}} \bowtie U)_\# [\![\mathcal{M}]\!]$, where $f \bowtie g$ denotes the join map

$$(f \bowtie g)(x) := \big(f(x), g(x)\big).$$

For example, if $\omega = \gamma \wedge \eta \in \mathcal{D}^{n+1}(\mathcal{M}\times\mathbb{R}^N)$, where $\gamma \in \mathcal{D}^{n+1-h}(\mathcal{M})$, $\eta \in \mathcal{D}^h(\mathbb{R}^N)$, and $0 \leqslant h \leqslant \min\{n+1, N, \mathfrak{p}\}$, by the area formula we have

$$\langle G_U, \gamma\wedge\eta\rangle = [\![\mathcal{M}]\!]\big((\mathrm{Id}_{\mathcal{M}} \bowtie U)^\#(\gamma\wedge\eta)\big) = [\![\mathcal{M}]\!]\big(\gamma \wedge U^\#\eta\big) = \int_{\mathcal{M}} \gamma\wedge U^\#\eta.$$

Setting moreover

$$\|G_U\| := \sup\big\{G_U(\omega) \;\big|\; \omega \in \mathcal{D}^{n+1,\mathfrak{p}}\big(\mathcal{M}\times\mathbb{R}^N\big),\ \|\omega\| \leqslant 1\big\},$$

where $\|\omega\|$ is the *comass* norm of ω, by using the parallelogram inequality we infer that

$$\|G_U\| \leqslant C \int_{\mathcal{M}} \big(1+|DU|^p\big)\, d\mathcal{H}^{n+1} < \infty$$

for some absolute constant $C = C(n,p,\mathcal{M}) > 0$, not depending on U. As a consequence, if e.g. $U \in W^{1,p}(\mathcal{M}_d,\mathcal{Y})$ and $\dim\mathcal{Y} \leqslant \mathfrak{p}$, it turns out that G_U is an integer multiplicity (say i.m.) rectifiable in $\mathcal{R}_{n+1}(\mathcal{M}_d\times\mathcal{Y})$ with finite mass, $\mathbf{M}(G_U) = \|G_U\| < \infty$, compare [12].

Definition 2.2. To any map $u \in W^{1/p}(\mathcal{X},\mathcal{Y})$ we associate a Sobolev map $\mathrm{Ext}(u) \in W^{1,p}(\mathcal{M},\mathbb{R}^N)$ given by a minimizer of the infimum problem

$$\inf\left\{\int_{\mathcal{M}} |DU|^p\, d\mathcal{H}^{n+1} \;\middle|\; U \in W^{1,p}\big(\mathcal{M},\mathbb{R}^N\big),\ U_{|\mathcal{X}} = u\right\}.$$

The $(n,\mathfrak{p}-1)$-current G_u in $\mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathbb{R}^N)$ carried by the graph of u is given by

$$G_u := (-1)^{n-1}(\partial G_U) \llcorner \big(\mathcal{X}\times\mathbb{R}^N\big) \quad \text{on } \mathcal{D}^{n,\mathfrak{p}-1}\big(\mathcal{X}\times\mathbb{R}^N\big), \tag{2.1}$$

where $U := \mathrm{Ext}(u)$ and $G_U \in \mathcal{D}_{n+1,\mathfrak{p}}(\mathcal{M}\times\mathbb{R}^N)$ is defined as in Example 2.1.

More precisely, for each $\delta > 0$ we choose a cut-off function $\eta = \eta_\delta \in C^\infty([0,\delta],[0,1])$ such that $\eta(t) = 1$ for $0 \leqslant t \leqslant \delta/4$, $\eta(t) = 0$ for $3\delta/4 \leqslant t \leqslant \delta$, and $\|\eta'\| \leqslant 4/\delta$. Then, to each smooth n-form $\omega \in \mathcal{D}^n(\mathcal{X}\times\mathcal{Y})$ we associate an n-form $\widehat{\omega} \in \mathcal{D}^n(\mathcal{X}\times\mathbb{R}^N)$ such that $(\mathrm{Id}_{\mathcal{X}} \bowtie \mathrm{i}_{\mathcal{Y}})^\#\widehat{\omega} = \omega$, and then the smooth n-form $\widetilde{\omega}$ in $\mathcal{M}_\delta \times \mathcal{Y}$ given by

$$\widetilde{\omega} := (\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})^\#\widehat{\omega} \wedge \eta, \qquad (\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})(z,y) := \big(\Phi(z), y\big). \tag{2.2}$$

Now, since U is smooth out of $\mathcal{X}$, the above formula (2.1) reads as

$$\langle G_u, \omega\rangle = \langle G_u, \widetilde{\omega}\rangle := (-1)^{n-1}\langle G_U, d\widetilde{\omega}\rangle \quad \forall \omega \in \mathcal{D}^{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y}), \tag{2.3}$$

where we can choose $\eta = \eta(\delta)$ independently of $0 < \delta < d$.

Remark 2.3. The above definition, introduced in [13], does not depend on the choice of the extension. In fact, in [12, Sec. 3.2.5] it is shown that *two Sobolev maps $U_1, U_2 \in W^{1,p}(\mathcal{M}, \mathbb{R}^N)$ have the same traces on $\partial\mathcal{M}$, i.e., $U_{1|\mathcal{X}} = U_{2|\mathcal{X}}$, if and only if*

$$(\partial G_{U_1}) \llcorner (\mathcal{X}\times\mathbb{R}^N) = (\partial G_{U_2}) \llcorner (\mathcal{X}\times\mathbb{R}^N) \quad \text{on } \mathcal{D}^{n,\mathfrak{p}-1}(\mathcal{X}\times\mathbb{R}^N).$$

Moreover, see [12, Sec. 3.2.5], it may happen that a Sobolev map $U \in W^{1,p}(\mathcal{M}, \mathbb{R}^N)$ has a smooth trace u on $\mathcal{X} = \partial\mathcal{M}$, but the boundary current $(\partial G_U) \llcorner (\mathcal{X}\times\mathbb{R}^N)$ does not agree (up to the sign) with the graph current G_u carried by the trace u, if $p < \min\{n+1, N\}$.

Also, notice that by Federer's support theorem [10], the semi-current G_u in Definition 2.2 belongs to the class $\mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y})$. Finally, the following null-boundary condition holds true:

Proposition 2.4. *If $\mathfrak{p} \geqslant 2$, for every $u \in W^{1/p}(\mathcal{X}, \mathcal{Y})$ we have*

$$\langle \partial G_u, \xi\rangle = 0 \quad \forall \xi \in \mathcal{D}^{n-1,\mathfrak{p}-2}(\mathcal{X}\times\mathcal{Y}). \tag{2.4}$$

Proof. By Proposition 1.1, choose a smooth sequence $\{u_k\} \subset W^{1/p} \cap C^\infty(\mathcal{X}, \mathbb{R}^N)$ converging to u strongly in $W^{1/p}$. Setting $U_k := \operatorname{Ext}(u_k)$, clearly $U_k \to U := \operatorname{Ext}(u)$ strongly in $W^{1,p}(\mathcal{M}, \mathbb{R}^N)$ and hence, by the dominated convergence theorem, $\langle G_{U_k}, \alpha\rangle \to \langle G_U, \alpha\rangle$ for each $\alpha \in \mathcal{D}^{n+1,\mathfrak{p}}(\mathcal{M}\times\mathbb{R}^N)$. Since u_k is smooth, each current G_{u_k} satisfies the null-boundary condition (2.4). Therefore, by (2.2) and (2.3) we obtain for each $\xi \in \mathcal{D}^{n-1,\mathfrak{p}-2}(\mathcal{X}\times\mathcal{Y})$

$$0 = \langle G_{u_k}, d\xi\rangle = \langle G_{u_k}, \widetilde{d\xi}\rangle = (-1)^{n-1}\langle G_{U_k}, d\widetilde{\omega}\rangle, \quad \widetilde{\omega} := \widetilde{d\xi}.$$

Since $d\widetilde{\omega}$ is a smooth $(n+1)$-form in $\mathcal{M}\times\mathbb{R}^N$ with bounded Lipschitz coefficients, and with at most $\mathfrak{p}$ "vertical" differentials, again by dominated convergence we have $\langle G_{U_k}, d\widetilde{\omega}\rangle \to \langle G_U, d\widetilde{\omega}\rangle$, whence

$$0 = (-1)^{n-1}\langle G_U, d\widetilde{\omega}\rangle = \langle G_u, d\xi\rangle = \langle \partial G_u, \xi\rangle,$$

as required. □

Maps in VMO. We now see how the above properties can be improved for maps u in $W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y})$. In fact, by Remark 2.3, in this case we deduce that

$$\langle G_u, \omega\rangle = (-1)^{n-1}\big\langle G_U, d\big[(\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})^{\#}\omega \wedge \eta_\delta\big]\big\rangle \quad \forall \omega \in \mathcal{D}^{n-1,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y}), \tag{2.5}$$

where $U \in W^{1,p}(\mathcal{M}_\delta, \mathcal{Y})$ is given by Proposition 1.2 and d this time denotes the tangential differential in $\mathcal{M}\times\mathcal{Y}$. Denoting by

$$\mathcal{Z}^{n-1,r}(\mathcal{X}\times\mathcal{Y}) := \big\{\xi \in \mathcal{D}^{n-1,r}(\mathcal{X}\times\mathcal{Y}) \;\big|\; d_y\xi = 0\big\},$$

where d_y is the tangential differential in the vertical y-directions, we thus extend (2.4) as follows:

Proposition 2.5. *Let $u \in W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y})$ and G_u given by Definition 2.2. Then*

$$\langle \partial G_u, \xi\rangle = 0 \quad \forall \xi \in \mathcal{Z}^{n-1,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y}). \tag{2.6}$$

Proof. Let $\{u_k\} \subset W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y})$ denote the smooth approximating sequence given by Theorem 1.5. Using the argument from Proposition 1.2, we may and do assume that

$$\langle G_{u_k}, \omega\rangle = (-1)^{n-1}\big\langle G_{U_k}, d\big[(\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})^{\#}\omega \wedge \eta_\delta\big]\big\rangle \quad \forall \omega \in \mathcal{D}^{n-1,\mathfrak{p}-1}(\mathcal{X}\times\mathcal{Y}),$$

where $U_k \in W^{1,p}(\mathcal{M}_\delta, \mathcal{Y})$ is smooth and $U_k \to U$ strongly in $W^{1,p}$. Let $\xi \in \mathcal{Z}^{n-1,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})$. Since $d[(\Phi \bowtie \mathrm{Id}_\mathcal{Y})^\# d\xi \wedge \eta_\delta]$ contains at most $\mathfrak{p} - 1$ "vertical" differentials, arguing as in the proof of Proposition 2.4 we get

$$0 = \langle G_{u_k}, d\xi \rangle = (-1)^{n-1} \langle G_{U_k}, d\big[(\Phi \bowtie \mathrm{Id}_\mathcal{Y})^\# d\xi \wedge \eta_\delta\big]\rangle,$$
$$\lim_{k\to\infty} (-1)^{n-1} \langle G_{U_k}, d\big[(\Phi \bowtie \mathrm{Id}_\mathcal{Y})^\# d\xi \wedge \eta_\delta\big]\rangle = \langle G_u, d\xi \rangle,$$

as required. □

Remark 2.6. The null-boundary condition (2.6) yields that u is a Cartesian map in $\mathrm{cart}^{1/\mathfrak{p}}(\mathcal{X}, \mathcal{Y})$ in the sense of [13]. Notice that such a condition is violated in general if $u \notin \mathrm{VMO}$, and only the weaker property (2.4) is satisfied. In fact, the map u from the example (1.4) belongs to $W^{1/2}(\mathbb{S}^2, \mathbb{S}^1)$ but one has

$$\partial G_u = (\delta_{P_-} - \delta_{P_+}) \times [\![\mathbb{S}^1]\!] \quad \text{on } \mathcal{D}^1\big(\mathbb{S}^2 \times \mathbb{S}^1\big),$$

where $\delta_{P_\pm}$ denotes the unit Dirac mass at the point $P_\pm := (\pm 1, 0, 0)$.

BV-maps in VMO. The distributional derivative of a BV-map decomposes as $Du = (Du)^a + (Du)^J + (Du)^C$, where

$$(Du)^a = \nabla u\, dx, \qquad (Du)^J = (u_+ - u_-) \otimes \nu \mathcal{H}^{n-1} \llcorner J_u,$$

compare [5]. The Cantor–Vitali function gives an example of BV-map in VMO such that the distributional derivative is concentrated on the Cantor set, hence Du only contains a non-zero Cantor part, $Du = (Du)^C$. However, in general a BV-map in VMO does not have a Jump part.

Proposition 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If $u \in \mathrm{BV} \cap \mathrm{VMO}(\Omega, \mathbb{R}^N)$ then $(Du)^J = 0$.*

Proof. Following [5, Sec. 3.6], let $x \in J_u$ and denote $a, b \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$ so that

$$\lim_{\varepsilon\to 0} \fint_{B_\varepsilon^+(x,\nu)} \big|u(y) - a\big|\, dy = 0, \qquad \lim_{\varepsilon\to 0} \fint_{B_\varepsilon^-(x,\nu)} \big|u(y) - b\big|\, dy = 0,$$

where $B_\varepsilon^\pm(x, \nu) := \{y \in B_\varepsilon(x) \mid \pm(y - x) \cdot \nu > 0\}$. Setting $\widetilde{u}(x) := \frac{a+b}{2}$, for each $\varepsilon > 0$ small we have

$$\fint_{B_\varepsilon^+(x,\nu)} \big|u(y) - \widetilde{u}(x)\big|\, dy + \fint_{B_\varepsilon^-(x,\nu)} \big|u(y) - \widetilde{u}(x)\big|\, dy \leqslant 2J_\varepsilon^u(x) + \big|u_\varepsilon(x) - \widetilde{u}(x)\big|$$

and hence, by the characterization of the class VMO, using that

$$2\big(u_\varepsilon(x) - \widetilde{u}(x)\big) = \fint_{B_\varepsilon^+(x,\nu)} \big(u(y) - a\big)\, dy + \fint_{B_\varepsilon^-(x,\nu)} \big(u(y) - b\big)\, dy,$$

we get

$$\lim_{\varepsilon\to 0} \fint_{B_\varepsilon^+(x,\nu)} \big|u(y) - \widetilde{u}(x)\big|\, dy = 0, \qquad \lim_{\varepsilon\to 0} \fint_{B_\varepsilon^-(x,\nu)} \big|u(y) - \widetilde{u}(x)\big|\, dy = 0,$$

that clearly gives $a = \widetilde{u}(x) = b$. This yields $(Du)^J = 0$, as required. □

3. Currents carried by graphs in $W^{1/p} \cap \mathrm{VMO}$

Assume that the target manifold $\mathcal{Y}$ has dimension not greater than $\mathfrak{p}$, the integer part of p. In this section we show that for maps $u \in W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y})$, the action of the semi-current G_u from Definition 2.2 can be extended to a current T_u in $\mathcal{D}_n(\mathcal{X} \times \mathcal{Y})$, actually an *integral flat chain*. We shall then discuss some structure properties, and provide some relevant examples.

We first recall that a current $T \in \mathcal{D}_n(\mathcal{M}_\delta \times \mathcal{Y})$ is an integral flat chain in $\mathcal{F}_n(\mathcal{M}_\delta \times \mathcal{Y})$ if there exist two i.m. rectifiable currents $T \in \mathcal{R}_{n+1}(\mathcal{M}_\delta \times \mathcal{Y})$ and $S \in \mathcal{R}_n(\mathcal{M}_\delta \times \mathcal{Y})$ such that $T = \partial R + S$, and the flat norm of any such T is

$$\mathcal{F}(T) := \inf\bigl\{\mathbf{M}(R) + \mathbf{M}(S) \bigm| T \in \mathcal{R}_{n+1}(\mathcal{M}_\delta \times \mathcal{Y}),\ S \in \mathcal{R}_n(\mathcal{M}_\delta \times \mathcal{Y}),\ T = \partial R + S\bigr\}.$$

One correspondingly obtains a metric space. Furthermore, the flat convergence $\mathcal{F}(T_k - T) \to 0$ in the class $\mathcal{F}_n(\mathcal{M}_\delta \times \mathcal{Y})$ implies the weak convergence $T_k \rightharpoonup T$ as currents in $\mathcal{D}_n(\mathcal{M}_\delta \times \mathcal{Y})$.

Theorem 3.1. *Assume that* $\dim \mathcal{Y} \leqslant \mathfrak{p}$, *the integer part of* p. *Let* $u \in W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y})$ *and let* $U \in W^{1,p}(\mathcal{M}_\delta, \mathcal{Y})$ *be given by Proposition 1.2. Then the boundary current*

$$T_u := (-1)^{n-1}(\partial G_U) \llcorner (\mathcal{X} \times \mathcal{Y}) \tag{3.1}$$

is well-defined as an integral flat chain in $\mathcal{F}_n(\mathcal{X} \times \mathcal{Y})$. *Moreover, we can decompose*

$$T_u = G_u + S_u$$

so that the following properties hold:

1. *for each smooth form* $\omega \in \mathcal{D}^n(\mathcal{X} \times \mathcal{Y})$ *we have*
$$\langle T_u, \omega\rangle = \langle G_u, \omega\rangle + \langle S_u, \omega\rangle = (-1)^{n-1}\bigl\langle G_U, d\bigl[(\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})^{\#}\omega \wedge \eta_\delta\bigr]\bigr\rangle;$$
2. $\langle T_u, d\xi\rangle = 0$ *for each* $\xi \in \mathcal{D}^{n-1}(\mathcal{X} \times \mathcal{Y})$;
3. G_u *is the semi-current in* $\mathcal{D}_{n,\mathfrak{p}-1}(\Omega \times \mathcal{Y})$ *satisfying the formula* (2.5);
4. G_u *satisfies the null-boundary condition* (2.6);
5. S_u *is "completely vertical", i.e.,* $\langle S_u, \omega\rangle = 0$ *if* $\omega \in \mathcal{D}^{n,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})$.

Proof. Since $U \in W^{1,p}(\mathcal{M}_\delta, \mathcal{Y})$ and $\mathfrak{p} \geqslant \dim \mathcal{Y}$, the current G_U carried by the graph of U is i.m. rectifiable in $\mathcal{R}_{n+1}(\mathcal{M}_\delta \times \mathcal{Y})$, with finite mass $\mathbf{M}(G_U) < \infty$, see Example 2.1. Moreover, the function U being smooth inside $\mathcal{M}_\delta$, by a slicing argument it turns out that for a.e. $0 < \delta_2 < \delta_1 < \delta$ the current

$$R_{\delta_2}^{\delta_1} := G_U \llcorner \bigl((\mathcal{M}_{\delta_1} \setminus \mathcal{M}_{\delta_2}) \times \mathcal{Y}\bigr)$$

satisfies the following properties:

(a) the boundary $\partial R_{\delta_2}^{\delta_1}$ has finite mass;

(b) by the *boundary rectifiability theorem*, see e.g. [12, Sec. 2.2.7], the current $\partial R_{\delta_2}^{\delta_1}$ is i.m. rectifiable in $\mathcal{R}_n(\mathcal{M}_\delta \times \mathcal{Y})$;

(c) $\partial R_{\delta_2}^{\delta_1} = (-1)^n(G_{u_{\delta_1}} - G_{u_{\delta_2}})$, where $G_{u_{\delta_i}}$ is the i.m. rectifiable current carried by the graph of the smooth Sobolev $W^{1,p}$-map $u_{\delta_i} := v \circ \Phi_{|\Phi^{-1}(\mathcal{X} \times \{\delta_i\})}$, where $v(x,h)$ is given by (1.2).

Choosing now a decreasing sequence $\{\delta_i\}$ of positive numbers with $\delta_i \to 0^+$ that are good in the above sense, since for each k

$$\mathcal{F}(G_{u_{\delta_k}} - G_{u_{\delta_{k+1}}}) \leqslant \mathbf{M}\bigl(R_{\delta_{k+1}}^{\delta_k}\bigr), \qquad \sum_{i=k}^{\infty} \mathbf{M}\bigl(R_{\delta_{i+1}}^{\delta_i}\bigr) \leqslant \mathbf{M}\bigl(G_U \llcorner (\mathcal{M}_{\delta_k} \times \mathcal{Y})\bigr),$$

whereas by absolute continuity

$$\mathbf{M}\bigl(G_U \llcorner (\mathcal{M}_{\delta_k} \times \mathcal{Y})\bigr) \leqslant c \int_{\mathcal{M}_{\delta_k}} \bigl(1 + |DU|^p\bigr)\, d\mathcal{H}^{n+1} \to 0$$

as $k \to \infty$, we deduce that $\{G_{u_{\delta_k}}\} \subset \mathcal{F}_n(\mathcal{M}_\delta \times \mathcal{Y})$ is a Cauchy sequence. Hence, the $G_{u_{\delta_k}}$'s weakly converge in $\mathcal{D}_n(\mathcal{M}_\delta \times \mathcal{Y})$ to some integral flat chain $T_u \in \mathcal{F}_n(\mathcal{M}_\delta \times \mathcal{Y})$. Moreover, the above computation yields that

$$(-1)^n T_u = \sum_{i=1}^{\infty} \partial R^{\delta_i}_{\delta_{i+1}} + G_{u_{\delta_1}}$$

and hence, choosing $\delta_1 > 3\delta/4$ and $\eta = \eta(\delta)$ in (2.2), we deduce that

$$\langle T_u, \omega\rangle = (-1)^{n-1}\big\langle G_U, d\big[(\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})^{\#}\omega \wedge \eta_\delta\big]\big\rangle \quad \forall \omega \in \mathcal{D}^n(\mathcal{X} \times \mathcal{Y}).$$

In particular, the current T_u does not depend on the choice of the sequence $\{\delta_i\}$. Also, by Remark 2.3, since $U_{|\mathcal{X}} = u$, we infer that

$$T_u = G_u \quad \text{on } \mathcal{D}^{n,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y}),$$

where $G_u \in \mathcal{D}_{n,\mathfrak{p}-1}(\mathcal{X} \times \mathcal{Y})$ is given by the formula (2.5), hence by Definition 2.2. The claim follows from Proposition 2.5. □

Remark 3.2. If $\dim \mathcal{Y} > \mathfrak{p}$, with a similar proof one obtains that the formula (3.1) and the property 1 hold true on forms in $\mathcal{D}^{n,\mathfrak{p}}(\mathcal{X} \times \mathcal{Y})$, hence the boundary current T_u is well-defined as a semi-current in $\mathcal{D}_{n,\mathfrak{p}}(\mathcal{X} \times \mathcal{Y})$. This time, the null-boundary condition 2 is satisfied on forms $\xi \in \mathcal{Z}^{n-1,\mathfrak{p}-1}$, whereas the properties 3–5 continue to hold.

Some structure properties. We now write explicitly the “lower” components of T_u in terms of the $W^{1,p}$-extension map U from Proposition 1.2.

Assume for simplicity $\mathcal{X} = \Omega$, a bounded domain in $\mathbb{R}^n$, and $\mathcal{M}_\delta = \Omega \times [0,\delta]$. For $0 < \varepsilon \ll \delta$, set $\eta_\varepsilon(t) := 1 - t/\varepsilon$ for $0 \leqslant t \leqslant \varepsilon$ and $\eta_\varepsilon(t) \equiv 0$ for $t \geqslant \varepsilon$.

Under the hypotheses of Theorem 3.1, we decompose

$$T_u = \sum_{j=0}^{\mathfrak{p}} T_{u(j)},$$

where $T_{u(j)}$ is the component of T_u acting on forms in $\mathcal{D}^n(\Omega \times \mathcal{Y})$ with exactly j vertical differentials:

$$\langle T_{u(j)}, \omega\rangle := \big\langle T_u, \omega^{(j)}\big\rangle, \quad \omega \in \mathcal{D}^n(\Omega \times \mathcal{Y}).$$

Remark 3.3. Theorem 3.1 yields that $T_{u(j)} = G_{u(j)}$, for $j = 0, \dots, \mathfrak{p}-1$, and $T_{u(\mathfrak{p})} = S_u$.

For each $\omega \in \mathcal{D}^n(\Omega \times \mathcal{Y})$ we have

$$(-1)^{n-1}\langle T_u, \omega\rangle = \big\langle G_U, \eta_\varepsilon'(t)\omega \wedge dt + \eta_\varepsilon(t) \wedge d\omega\big\rangle. \tag{3.2}$$

Setting $U = (U^1, \dots, U^N)$, for $j = 1, \dots, N$ we shall denote

$$D_t U^j(x,t) := \frac{\partial U^j}{\partial t}(x,t), \qquad D_i U^j(x,t) := \frac{\partial U^j}{\partial x_i}(x,t), \quad i = 1, \dots, n.$$

Case 1: the component $T_{u(0)}$. If $\omega = \phi(x)\psi(y)\,dx$, where $\phi \in C_c^\infty(\Omega)$ and $\psi \in C_c^\infty(\mathbb{R}^N)$, the above formula gives

$$\big\langle T_u, \phi(x)\psi(y)\,dx\big\rangle = \int_\Omega \phi(x) \fint_{[0,\varepsilon]} \psi\big(U(x,t)\big)\,dt\,dx - \sum_{j=1}^N \int_\Omega \phi(x) \int_0^\varepsilon \eta_\varepsilon(t) \frac{\partial \psi}{\partial y_j}\big(U(x,t)\big) D_t U^j(x,t)\,dt\,dx.$$

Since $U \in W^{1,1}(\Omega \times (0,\delta))$, passing to the limit as $\varepsilon \to 0$ we get

$$\big\langle T_u, \phi(x)\psi(y)\,dx\big\rangle = \int_\Omega \phi(x)\psi\big(u(x)\big)\,dx.$$

By a density argument, this yields that

$$\langle T_u, \phi(x,y)\,dx\rangle = \int_\Omega \phi\big(x,u(x)\big)\,dx \quad \forall \phi \in C_c^\infty(\mathcal{X}\times\mathcal{Y}).$$

In particular, $\mathbf{M}(T_{u(0)}) < \infty$.

Case 2: the component $T_{u(1)}$. Assume for a moment that $p \geqslant 2$. If $\omega = \phi(x)\psi(y)\,\widehat{dx^i} \wedge dy^j$, where $\widehat{dx^i} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$, this time we get

$$\begin{aligned}\langle T_u, \phi(x)\psi(y)\,\widehat{dx^i} \wedge dy^j\rangle = (-1)^{i-1} \int_\Omega \phi(x) \fint_{[0,\varepsilon]} \psi\big(U(x,t)\big) D_i U^j(x,t)\,dt\,dx \\ + (-1)^{i-1} \int_\Omega \frac{\partial\phi}{\partial x_i}(x) \int_0^\varepsilon \eta_\varepsilon(t)\psi\big(U(x,t)\big) D_t U^j(x,t)\,dt\,dx \\ + (-1)^{n+i-1} \sum_{k\neq j} \int_\Omega \phi(x) \int_0^\varepsilon \eta_\varepsilon(t) \frac{\partial\psi}{\partial y_k}\big(U(x,t)\big) \frac{\partial(U^j,U^k)}{\partial(x_i,t)}(x,t)\,dt\,dx.\end{aligned} \tag{3.3}$$

Since $U \in W^{1,2}(\Omega\times(0,\delta))$, both $\frac{\partial(U^j,U^k)}{\partial(x_i,t)}$ and $D_t U$ are summable in $\Omega\times(0,\delta)$, and hence the last two integrals go to zero as $\varepsilon \to 0$. We thus deduce that $T_{u(1)}$ *has finite mass provided that we can find a sequence* $\varepsilon_h \searrow 0$ *along which*

$$\lim_{h\to\infty} \int_\Omega \fint_{[0,\varepsilon_h]} \big|DU(x,t)\big|\,dt\,dx < \infty.$$

Arguing as in Theorem 3 from [12, Sec. 4.2.3], for any choice of $p > 1$ we obtain:

Proposition 3.4. *The component $T_{u(1)}$ has finite mass if and only if u has bounded variation, $u \in \mathrm{BV}(\Omega,\mathcal{Y})$. In this case, moreover, we have $\mathbf{M}(T_{u(1)}) = |Du|(\Omega)$ and*

$$\langle T_u, \phi(x,y)\,\widehat{dx^i} \wedge dy^j\rangle = (-1)^{n-i} \int_\Omega \phi\big(x,u(x)\big)\,dD_i u^j(x) \tag{3.4}$$

for every $\varphi \in C_c^\infty(\Omega\times\mathbb{R}^N)$.

Proof. Assume first that $T_{u(1)}$ has finite mass, and choose the form $\xi = y^j\varphi(x)\,\widehat{dx^i}$, where $\varphi \in C^1(\Omega)$ and $|D\varphi| \in L^\infty$, so that

$$d\xi = (-1)^{i-1} D_i\varphi\, y^j\,dx + \varphi(x)\,dy^j \wedge \widehat{dx^i}.$$

Since the coefficient of $(d\xi)^{(0)}$ grows linearly in y and the coefficient of $(d\xi)^{(1)}$ is bounded, using that $\mathbf{M}(T_{u(0)}) + \mathbf{M}(T_{u(1)}) < \infty$, the action of T_u on $d\xi$ can be computed, by approximation, as limit of $\langle T_u, d\alpha_h\rangle$, the α_h being smooth $(n-1)$-forms in $\Omega\times\mathbb{R}^N$ with compact support and $\alpha_h = \alpha_h^{(0)}$. Property (2.6) gives $\langle T_u, d\alpha_h\rangle = 0$, and passing to the limit

$$0 = \langle T_u, d\xi\rangle = \langle T_u, (-1)^{i-1} D_i\varphi\, y^j\,dx\rangle + \langle T_u, \varphi(x)\,dy^j \wedge \widehat{dx^i}\rangle,$$

whence (by using the case 1)

$$\int_\Omega D_i\varphi\, u^j\,dx = (-1)^i \langle T_u, \varphi(x)\,dy^j \wedge \widehat{dx^i}\rangle.$$

Setting for every $\phi = (\phi^1, \dots, \phi^N) \in C_c^\infty(\Omega, \mathbb{R}^{Nn})$

$$\omega_\phi := \sum_{j=1}^{N} \sum_{i=1}^{n} (-1)^i \phi_i^j \, \widehat{dx^i} \wedge dy^j, \quad \phi^j = \big(\phi_1^j, \dots, \phi_n^j\big),$$

by linearity this gives

$$\sum_{j=1}^{N} \int_\Omega \operatorname{div} \phi^j u^j \, dx = \langle T_u, \omega_\phi \rangle,$$

and hence the estimate $|Du|(\Omega) \leqslant \mathbf{M}(T_{u(1)})$, by the definition of variation.

Conversely, assume that u belongs to $\in \mathrm{BV}(\Omega, \mathcal{Y})$. The averaged integral $⨍_{B_\varepsilon(x)} u(y)\,dy$ agrees (up to an absolute constant) with the convolution product $(u * \rho_\varepsilon)(x)$, where $\rho_\varepsilon(z) := \varepsilon^{-n} \rho(z/\varepsilon)$ for some symmetric kernel $\rho \in L^1(\mathbb{R}^n)$, with $\operatorname{spt} \rho = \overline{B}^n$, $\rho \geqslant 0$, and $\int \rho(z)\,dz = 1$. By [5, Prop. 3.2], then $\nabla(u * \rho_\varepsilon) = Du * \rho_\varepsilon$ in $\Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$, and $\int_U |\nabla(u * \rho_\varepsilon)|\,dx \to |Du|(U)$ for every $U \Subset \Omega$ such that $|Du|(\partial U) = 0$, see [5, Prop. 3.7]. By the proof of Theorem 3.1, we deduce that the sequence of graph currents G_{u_ε} weakly converges in $\mathcal{D}_n(\mathcal{M} \times \mathcal{Y})$ to T_u. Since $\mathbf{M}(G_{u_{\varepsilon(1)}}) \leqslant \int_\Omega |\nabla(u * \rho_\varepsilon)|\,dx$, and by lower semicontinuity $\mathbf{M}(T_{u(1)}) \leqslant \liminf_{\varepsilon \to 0} \mathbf{M}(G_{u_{\varepsilon(1)}})$, the first claim is proved. Moreover, the weak BV-convergence with the mass, $\mathbf{M}(G_{u_{\varepsilon(1)}}) = \int_\Omega |\nabla(u * \rho_\varepsilon)|\,dx \to |Du|(U)$, yield the structure property (3.4), as required. □

Remark 3.5. Recall that in general the distributional derivative of a BV-map in VMO does not have a Jump part, Proposition 2.7.

Case 3: the higher order components $T_{u(j)}$. Assume in addition that $u \in W^{1,q}(\Omega, \mathbb{R}^N)$ for some $q \geqslant 1$. Then the averaged integral $u_\varepsilon(x) := ⨍_{B_\varepsilon(x)} u(y)\,dy$ converges to u strongly in $W^{1,q}$ as $\varepsilon \to 0$. To prove this, arguing as in the proof of Proposition 3.4, recall that $u_\varepsilon(x) = (u * \rho_\varepsilon)(x)$, and that the distributional derivatives $D_i u_\varepsilon$ agree with the convolutions $(D_i u * \rho_\varepsilon)$, see e.g. [5, Sec. 2.2]. As a consequence, by dominated convergence, for $j \leqslant [q]$, the integer part of q, we get

$$\langle T_{u(j)}, \omega \rangle = \lim_{\varepsilon \to 0} \big\langle G_{u_\varepsilon}, \omega^{(j)} \big\rangle = \lim_{\varepsilon \to 0} \int_\Omega (\mathrm{Id} \bowtie u_\varepsilon)^\# \omega^{(j)} = \int_\Omega (\mathrm{Id} \bowtie u)^\# \omega^{(j)}$$

for each $\omega \in \mathcal{D}^n(\Omega \times \mathcal{Y})$. We thus have obtained:

Proposition 3.6. *If in addition $u \in W^{1,q}(\Omega, \mathbb{R}^N)$ for some $q \geqslant 1$, then for each $j = 1, \dots, [q]$ the component $T_{u(j)}$ agrees with the restriction of the current carried by the rectifiable graph of u, i.e.,*

$$\langle T_{u(j)}, \omega \rangle = \int_\Omega (\mathrm{Id} \bowtie u)^\# \omega^{(j)} \quad \forall \omega \in \mathcal{D}^n(\Omega \times \mathcal{Y}).$$

If u is not a Sobolev function, similarly to (3.3) one infers that in order to write the action of T_u on forms with several vertical differentials dy, by (3.2) the integral of minors of higher order of the Jacobian matrix of $DU(x,t)$ comes into play. In this direction, we shall see in Appendix A that if $U(x,t) = ⨍_{B_t(x)} u(y)\,dy$ for some function $u \in L^1(B^n)$, then for each $\varepsilon > 0$ the average $x \mapsto ⨍_{[0,\varepsilon]} U(x,t)\,dt$ agrees with the convolution product of u with $\rho_\varepsilon(z) := \varepsilon^{-n} \rho(z/\varepsilon)$, for a suitable non-negative and symmetric convolution kernel $\rho \in L^1(\mathbb{R}^n)$.

Integration by parts. By using the null-boundary condition 2, one may write the action of $T_{u(j+1)}$ on differentials in terms of the lower component $T_{u(j)}$. More precisely, if $\xi \in \mathcal{D}^{n-1}(\Omega \times \mathcal{Y})$, and $\xi = \xi^{(j)}$, we have

$$d\xi = (d\xi)^{(j)} + (d\xi)^{(j+1)}, \quad (d\xi)^{(j)} = d_x \xi, \ (d\xi)^{(j+1)} = d_y \xi.$$

Using that $\langle T_u, d\xi \rangle = 0$, one obtains a formula of integration by parts:

$$\langle T_{u(j+1)}, d_y \xi \rangle = -\langle T_{u(j)}, d_x \xi \rangle \quad \forall \xi = \xi^{(j)} \in \mathcal{D}^{n-1}(\Omega \times \mathcal{Y}). \tag{3.5}$$

For example, if $N=n$, and $\omega_n := y_1\, dy^2 \wedge\cdots\wedge dy^n$, for each $\varphi\in C_c^\infty(\Omega)$ we have

$$d(\omega_n\wedge\varphi)=d\omega_n\wedge\varphi+(-1)^{n-1}\omega_n\wedge d\varphi,\quad d\omega_n=dy^1\wedge\cdots\wedge dy^n.$$

Therefore, if $p\geqslant n$, by (3.5) we get

$$\langle T_{u(n)},\varphi\, dy^1\wedge\cdots\wedge dy^n\rangle=(-1)^n\langle T_{u(n-1)},\omega_n\wedge d\varphi\rangle.$$

If in addition $u\in W^{1,n-1}(\Omega,\mathbb{R}^n)\cap L^\infty$, by Proposition 3.6 the above formula reads as

$$\langle T_{u(n)},\varphi\, dy^1\wedge\cdots\wedge dy^n\rangle=(-1)^n\int_\Omega u^\#\omega_n\wedge d\varphi,$$

and actually the right-hand side agrees with the action of the *distributional determinant* of ∇u,

$$\langle \operatorname{Det}\nabla u,\varphi\rangle := \sum_{i=1}^n\langle D_i\big(u^1(\operatorname{adj}\nabla u)_i^1\big),\varphi\rangle=(-1)^n\int_\Omega u^\#\omega_n\wedge d\varphi.$$

A graph with concentration on a Cantor-type set. We now show the existence of a map $u\in W^{1/2}\cap \mathrm{VMO}(\Omega,\mathbb{S}^2)$, where $\Omega:=(0,1)^2$, such that *the action of the vertical component $S_u=T_{u(2)}$ of the integral flat chain T_u on a non-trivial class of forms $\omega=\omega^{(2)}$ in $\mathcal{D}^2(\Omega\times\mathbb{S}^2)$ is concentrated on $(C\times C)\times\mathbb{S}^2$, where C is the Cantor set.*

In fact, S. Müller [14] shows that for $n=N=2$ the singular part of the distributional determinant may in general concentrate on a set of Hausdorff dimension α, for any prescribed $0<\alpha<1$. More precisely, there exist bounded Hölder continuous Sobolev functions u in $W^{1,q}(\Omega,\mathbb{R}^2)$ for every $q<2$ such that $\det\nabla u=0$ and $|\nabla u^1||\nabla u^2|=0$ a.e. in Ω, but $\operatorname{Det}\nabla u=V'\otimes V'$, where V is the Cantor–Vitali function. Therefore, the distributional determinant has a “Cantor-type” part and the role played by V' in the Cantor set C is here played by $\operatorname{Det}\nabla u$ in $C\times C$. The “graph” of u is very similar to the graph of the Cantor–Vitali function V and, actually, has infinitely many holes, namely $\mathbf{M}(\partial G_u)=\infty$, compare [12, Sec. 4.2.5].

In the classical case of Cantor's middle thirds set, we have $u\in C^{0,s}(\Omega,\mathbb{R}^2)$, where $s:=\dim_{\mathcal{H}}(C)=\log 2/\log 3$. Since $u(\Omega)\subset[0,1]^2$, one has $u\in W^{1/p}(\Omega,\mathbb{R}^2)\cap L^\infty$ for each $p<1/(1-s)$. In particular, $u\in W^{1/2}\cap\mathrm{VMO}\cap L^\infty$. If $T_u\in\mathcal{D}_2(\Omega\times\mathbb{R}^2)$ is the integral flat chain defined as in Theorem 3.1, in this case we get

$$\langle T_u,\varphi\, dy^1\wedge dy^2\rangle=\langle\operatorname{Det}\nabla u,\varphi\rangle=\langle V'\otimes V',\varphi\rangle\quad\forall\varphi\in C_c^\infty(\Omega). \tag{3.6}$$

Finally, by composing u on the left with a bi-Lipschitz map from $[0,1]^2$ into $\mathbb{S}^2$, we readily obtain our cited example. Notice that such map u belongs to $W^{1,q}(\Omega,\mathbb{S}^2)$ for $q<2$, but $u\notin W^{1,2}$.

Remark 3.7. By Proposition 3.6, we clearly have $\mathbf{M}(T_{u(0)})+\mathbf{M}(T_{u(1)})<\infty$. On account of the computation from [12, Sec. 4.2.5], we expect that the current T_u has finite mass, i.e., $\mathbf{M}(T_{u(2)})<\infty$, compare Example 4.6 below.

Remark 3.8. In the “smooth” case of maps u in $W^{1/2}\cap\mathrm{VMO}(\Omega,\mathbb{S}^2)$ that belong to the Sobolev class $W^{1,2}(\Omega,\mathbb{S}^2)$, Proposition 3.6 yields that T_u agrees with the current carried by the rectifiable graph of u:

$$\langle T_u,\omega\rangle=\int_\Omega(\mathrm{Id}\bowtie u)^\#\omega\quad\forall\omega\in\mathcal{D}^2\big(\Omega\times\mathbb{S}^2\big),$$

and hence it only contains an “absolute continuous” part. This has to be compared with the “non-smooth” example previously described, in which property (3.6) shows the existence of a “Cantor-type” component in the action of the current T_u.

A graph with unbounded mass. We show the existence of maps $\bar u\in W^{1/2}\cap\mathrm{VMO}(B^2,\mathcal{Y})$, where $\dim\mathcal{Y}=2$, that do not have bounded variation, $\bar u\notin\mathrm{BV}(B^2,\mathbb{R}^N)$. By Proposition 3.4, this yields that the corresponding current $T_{\bar u}$ does not have finite mass, namely $\mathbf{M}(T_{\bar u(1)})=\infty$.

Example 3.9. Let $f(x) = |\log|x||^\alpha$, where $x \in \mathbb{R}^n$. In [8, Ex. 5] it is shown that the function f belongs to VMO for each $0 < \alpha < 1$. Moreover, $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$ provided that $n > 1/(1-\alpha)$. As a consequence, setting $\Omega = (-1,1)^2$ and $x = (x_1, x_2) \in \Omega$, the function $v : \Omega \to \mathbb{R}^2$ given by

$$v(x_1, x_2) := \left(\left|\log|x_1|\right|^\alpha, \left|\log|x_2|\right|^\alpha\right), \quad 0 < \alpha < \frac{1}{2}$$

belongs to the class $W^{1/2} \cap \mathrm{VMO}(\Omega, \mathbb{R}^2)$. Notice that, denoting $v = (v^1, v^2)$,

$$\left|Dv^i\right| = \frac{\alpha}{|x_i|}\left|\log|x_1|\right|^{\alpha-1}, \qquad |\det Dv| = \left|Dv^1\right| \cdot \left|Dv^2\right|,$$

whence $|Dv| \notin L^1(\Omega)$ and $\det Dv \notin L^1(\Omega)$. In particular, $v \notin \mathrm{BV}(\Omega, \mathbb{R}^2)$.

We now modify the function v to obtain a function $u = (u^1, u^2) \in W^{1/2} \cap \mathrm{VMO}(\Omega, \mathbb{R}^2)$ such that $0 \leqslant u^i(x) \leqslant 1$ for each i, so that u takes values into the unit square $[0,1]^2$. To this purpose, define $t_0 = 1$ and $t_n := \mathbf{e}^{-n^{1/\alpha}}$, so that $0 < t_n < t_{n-1}$ and $|\log|t_n||^\alpha = n$ for each $n \in \mathbb{N}^+$, and set, for $i = 1, 2$,

$$u^i(x) := \begin{cases} |\log|x_i||^\alpha - n & \text{if } t_{n+1} \leqslant |x_i| \leqslant t_n \text{ and } n \in \mathbb{N} \text{ is even,} \\ n - |\log|x_i||^\alpha & \text{if } t_{n+1} \leqslant |x_i| \leqslant t_n \text{ and } n \in \mathbb{N} \text{ is odd.} \end{cases}$$

Moreover, the function u can be easily extended to a function u from $(-2,2)^2$ onto $[0,1]^2$ that belongs to the class $W^{1/2} \cap \mathrm{VMO}$ and such that $u \equiv (0,0)$ at the boundary of $(-2,2)^2$.

Setting now $u \equiv (0,0)$ outside $(-2,2)^2$ and $U : (-2,2)^2 \times (0,1) \to \mathbb{R}^2$ by

$$U(x,h) := \fint_{B(x,h)} u(x)\,dx, \quad x = (x_1, x_2),\ h \in (0,1)$$

it turns out that $U \in W^{1,2}((-2,2)^2 \times]0,1], \mathbb{R}^2)$, with image contained in the unit square $[0,1]^2$. As in the proof of Theorem 3.1, we correspondingly obtain that the compactly supported integral flat chain

$$T_u := (-1)^{n-1}(\partial G_U) \llcorner \left(\left((-2,2)^2 \times \{0\}\right) \times \mathbb{R}^2\right)$$

satisfies for every $\omega \in \mathcal{D}^2((-2,2)^2 \times \mathbb{R}^2)$

$$\langle T_u, \omega\rangle = \langle T_u, \eta \wedge \omega\rangle := (-1)^{n-1}\langle G_U, d\eta \wedge \omega + \eta \wedge d\omega\rangle,$$

where we can choose the cut-off function η in correspondence to $\delta = 1$.

The above mentioned map $\bar{u} \in W^{1/2} \cap \mathrm{VMO}(B^2, \mathcal{Y}) \setminus \mathrm{BV}(B^2, \mathbb{R}^N)$ is readily obtained by composing u on the left with a bi-Lipschitz map from $[0,1]^2$ into $\mathcal{Y}$.

4. Approximate differentiability and Cartesian currents

In this section we deal with approximate differentiability of maps in our framework. We already mentioned in the introduction the possible existence of maps in $W^{1/p}$ that are not a.e. approximately differentiable. We then show that if the current T_u from Theorem 3.1 has finite mass, then it is a Cartesian current in the sense of Giaquinta, Modica and Souček [12]. We finally deal with some related examples built up in a way similar to the argument from Theorem 3.1, i.e., by means of weak limits of averages.

For the sake of simplicity, we shall assume that $\mathcal{X} = \Omega$, a bounded domain in $\mathbb{R}^n$, and deal with maps $u : \Omega \to \mathcal{Y}$. The general case of mappings $u : \mathcal{X} \to \mathcal{Y}$ is recovered by means of local coordinates and a partition of unity argument.

Approximate differentiability. If $u \in W^{1/p}(\Omega, \mathcal{Y})$ and $\mathbf{M}((G_u)_{(1)}) < \infty$, Proposition 3.4 yields that $u \in \mathrm{BV}(\Omega, \mathcal{Y})$. In particular, u is approximately differentiable a.e. in Ω. Therefore, following [12, Sec. 3.1.5], the *rectifiable graph* of u is well-defined by

$$\mathcal{G}_u := \left\{\left(x, u(x)\right) \,\middle|\, x \in R_u\right\},$$

where R_u is the set of Lebesgue points of u where u is approximately differentiable, and $u(x) \in \mathcal{Y}$ is the Lebesgue value. It turns out that $\mathcal{G}_u$ is $\mathcal{H}^n$-measurable and countably n-rectifiable in $\Omega \times \mathcal{Y}$, hence for $\mathcal{H}^n$-a.e. $z \in \mathcal{G}_u$ the approximate tangent n-space to $\mathcal{G}_u$ at z exists and its linear projection onto the first n coordinates has maximum rank n. However, in general one has $\mathcal{H}^n(\mathcal{G}_u)=+\infty$, even if u is a bounded map in $W^{1/p}\cap\mathrm{VMO}$, see Example 3.9.

The n-current $[\![\mathcal{G}_u]\!]$ integration of n-forms ω in $\mathcal{D}^n(\Omega\times\mathcal{Y})$ over the rectifiable graph $\mathcal{G}_u$ is well-defined, provided that we equip a.e. point $z=(x,u(x))$ with the naturally induced orientation from the one of the domain Ω.

Denote by $\mathcal{A}^1(\Omega,\mathbb{R}^N)$ the subclass of a.e. approximately differentiable L^1-maps such that *each minor of the Jacobian matrix* ∇u *of the approximate gradient is summable in* Ω. Also, set

$$\mathcal{A}^1(\Omega,\mathcal{Y}):=\left\{u\in\mathcal{A}^1\left(\Omega,\mathbb{R}^N\right)\;\middle|\;u(x)\in\mathcal{Y}\text{ for a.e. }x\in\Omega\right\}.$$

Since $\dim\mathcal{Y}=\mathfrak{p}$, if $u\in\mathcal{A}^1(\Omega,\mathcal{Y})$, by the area formula all the minors of ∇u of order greater than $\mathfrak{p}$ are zero. If e.g. $u\in W^{1,\mathfrak{p}}(\Omega,\mathcal{Y})$, then clearly $u\in\mathcal{A}^1(\Omega,\mathcal{Y})$. Moreover, if $u\in\mathcal{A}^1(\Omega,\mathcal{Y})$, again by the area formula it turns out that (in the approximate sense)

$$\left\langle[\![\mathcal{G}_u]\!],\omega\right\rangle=\int_\Omega(\mathrm{Id}\bowtie u)^{\#}\omega\quad\forall\omega\in\mathcal{D}^n(\Omega\times\mathcal{Y}).$$

In particular, we get

$$\mathbf{M}\left([\![\mathcal{G}_u]\!]\right)=\mathcal{H}^n(\mathcal{G}_u)=\int_\Omega J_n(\mathrm{Id}\bowtie u)(x)\,dx<\infty,$$

where $J_n(\mathrm{Id}\bowtie u)\in L^1(\Omega)$ is the n-dimensional tangential Jacobian of $(\mathrm{Id}\bowtie u)$.

Cartesian currents. Following the notation from [12, Sec. 4.2.2], we denote by $\mathrm{cart}(\Omega\times\mathcal{Y})$ the class of Cartesian currents in $\mathrm{cart}(\Omega\times\mathbb{R}^N)$ such that $\mathrm{spt}\,T\subset\overline{\Omega}\times\mathcal{Y}$. Any Cartesian current is i.m. rectifiable, hence it "lives" on an n-rectifiable set. More precisely, we recall that if T is an i.m. rectifiable n-current in $\mathcal{R}_n(\Omega\times\mathcal{Y})$, its action on n-forms $\omega\in\mathcal{D}^n(\Omega\times\mathcal{Y})$ is given by

$$\langle T,\omega\rangle=\int_{\mathcal{M}}\left\langle\xi(x),\omega(x)\right\rangle\theta(x)\,d\mathcal{H}^n(x)\tag{4.1}$$

where $\mathcal{M}\subset\Omega\times\mathcal{Y}$ is $\mathcal{H}^n$-measurable and countably n-rectifiable, with $\mathcal{H}^n(\mathcal{M})<\infty$, the *multiplicity function* $\theta:\mathcal{M}\to[0,+\infty]$ is $(\mathcal{H}^n\llcorner\mathcal{M})$-summable, and $\xi:\mathcal{M}\to\Lambda_n\mathbb{R}^{n+N}$ is $\mathcal{H}^n$-measurable with $|\xi|=1$ for $(\mathcal{H}^n\llcorner\mathcal{M})$-a.e. z. Also, for $\mathcal{H}^n$-a.e. $z\in\mathcal{M}$ the unit n-vector $\xi(z)$ provides an orientation to the approximate tangent space $T_z\mathcal{M}$.

In particular, if $T\in\mathrm{cart}(\Omega\times\mathcal{Y})$, by the structure properties from [12, Sec. 4.2.2], there exists a map $u\in\mathcal{A}^1(\Omega,\mathcal{Y})$ such that the following decomposition into the so-called "graph" and "vertical" parts holds true:

1. $\mathcal{M}=\mathcal{G}_u+\mathcal{M}^v$, with $\mathcal{H}^n(\mathcal{M})=\mathcal{H}^n(\mathcal{G}_u)+\mathcal{H}^n(\mathcal{M}^v)$;
2. at points z in the "vertical" part $\mathcal{M}^v$, the projection of the approximate tangent space $T_z\mathcal{M}^v$ onto the first n coordinates has dimension strictly lower than n;
3. the restriction $T\llcorner\mathcal{G}_u$ agrees with the n-current $[\![\mathcal{G}_u]\!]$, i.e.,

$$\langle T\llcorner\mathcal{G}_u,\omega\rangle=\left\langle[\![\mathcal{G}_u]\!],\omega\right\rangle=\int_\Omega(\mathrm{Id}\bowtie u)^{\#}\omega\quad\forall\omega\in\mathcal{D}^n(\Omega\times\mathcal{Y}).\tag{4.2}$$

Furthermore, for currents $T\in\mathrm{cart}(\Omega\times\mathcal{Y})$, in general the vertical part $T\llcorner\mathcal{M}^v$ is non-trivial, i.e., $\mathcal{H}^n(\mathcal{M}^v)>0$. This is due to the *null-boundary condition* $(\partial T)\llcorner\Omega\times\mathcal{Y}=0$, i.e., roughly speaking, to the necessity of "filling the holes" of the current $[\![\mathcal{G}_u]\!]$, see e.g. the examples below.

The case of finite mass. Let u satisfy the hypotheses of Theorem 3.1, and let T_u be the corresponding integral flat chain. As we have seen in the previous section, in general *the current* T_u *does not have finite mass*. In terms of component, T_u is a vector-valued distribution of order one, as for every $\omega\in\mathcal{D}^n(\Omega\times\mathcal{Y})$

$$\left|\langle T_u,\omega\rangle\right|\leqslant c\left\|\eta'\right\|_\infty\left(\|\omega\|+\|d\omega\|\right),$$

where $c > 0$ is an absolute constant and $\|\eta'\|_\infty \leqslant 4/\delta$, with $\delta > 0$ chosen in terms of the uniform limit $J_h^u(x) \to 0$ as $h \to 0^+$, and hence depending on $\mathcal{Y}$ and u, as shown in the proof of Proposition 1.2.

It is well known that in general, an integral flat chain T with unbounded mass does not "live" in a countably rectifiable set. For example, choose $T = \partial[\![A]\!] \in \mathcal{D}_1(\mathbb{R}^2)$, where A is a von Koch snowflake. In fact, $[\![A]\!] \in \mathcal{R}_2(\mathbb{R}^2)$, and ∂A is $\mathcal{L}^2$-negligible but *purely $\mathcal{H}^1$-unrectifiable*. However, in our context the following property is readily obtained:

Proposition 4.1. *Under the hypotheses of Theorem 3.1, assume that T_u has finite mass. Then $T_u \in \mathrm{cart}(\Omega \times \mathcal{Y})$.*

Proof. Since $\dim \mathcal{Y} \leqslant \mathfrak{p}$, by the boundary rectifiability theorem T_u is i.m. rectifiable in $\mathcal{R}_n(\Omega \times \mathcal{Y})$. In this case, moreover, we obtain a decomposition in mass

$$T_u = G_u + S_u, \qquad \mathbf{M}(T_u) = \mathbf{M}(G_u) + \mathbf{M}(S_u) < \infty.$$

Since $\mathcal{Y}$ is compact, we have $\|u\|_\infty < \infty$. Therefore, from the proof of Theorem 3.1 we deduce that T_u is the weak limit in $\mathcal{D}_n(\Omega \times \mathcal{Y})$ of a sequence of graphs G_{u_k} of equibounded smooth maps $u_k : \Omega \to \mathcal{Y}$. This clearly yields that T_u satisfies all the other properties that give the membership to the class $\mathrm{cart}(\Omega \times \mathcal{Y})$, compare [12, Sec. 4.2.2]. We omit any further detail. □

Remark 4.2. Under the hypotheses of Theorem 3.1, if T_u has finite mass, by Proposition 4.1 we have $T_u \in \mathrm{cart}(\Omega \times \mathcal{Y})$, and hence $u \in \mathcal{A}^1(\Omega, \mathcal{Y})$. In particular, formula (4.2) holds true. However, the map u from Example 3.9, where $n = N = 2$, is such that $\nabla u \notin L^1$ and $\det \nabla u \notin L^1$, hence the above formula (4.2) does not make sense, even if the current $[\![\mathcal{G}_u]\!]$ is well-defined.

Examples. We briefly sketch some related examples built up in a way similar to the argument from Theorem 3.1, i.e., by means of weak limits of averages.

Example 4.3. Let $H : \mathbb{R} \to \mathbb{R}$ denote the Heaviside map $H(x) = 1$ for $x \geqslant 0$ and $H(x) = 0$ for $x < 0$, and set $u := H_{|B^1}$, where $B^1 = (-1, 1)$, so that $u \in \mathrm{BV}(B^1)$, but clearly $u \notin \mathrm{VMO}(B^1)$. Let $U : B^1 \times (0, 1) \to \mathbb{R}$ given by $U(x, h) := ⨍_{B_h(x)} H(y)\, dy$, where $B_h(x) = (x - h, x + h)$, and set $u_h(x) := U(x, h)$. For each $0 < h < 1$, we have

$$u_h(x) = \begin{cases} 0 & \text{if } x < -h, \\ \frac{x+h}{2h} & \text{if } -h \leqslant x \leqslant h, \\ 1 & \text{if } x > h \end{cases}$$

whence $U \in W^{1,p}(B^1 \times (0, 1))$ and hence $u \in W^{1/p}(B^1)$ for each $p < 2$. Moreover, G_{u_h} weakly converges in $\mathcal{D}_1(B^1 \times \mathbb{R})$ as $h \to 0$ to the Cartesian current $T_u = [\![\mathcal{G}_u]\!] + S_u \in \mathrm{cart}(B^1 \times \mathbb{R})$, where

$$[\![\mathcal{G}_u]\!] = [\![-1, 0]\!] \times \delta_0 + [\![0, 1]\!] \times \delta_1, \qquad S_u = \delta_0 \times [\![0, 1]\!],$$

so that the vertical part $T_u \llcorner \mathcal{M}^v$ is non-trivial.

Example 4.4. Let $n \geqslant 2$ and $u : B^n \to \mathbb{R}^{n+1}$ given by $u(x) := \frac{(x,0)}{|x|}$, so that $u \in W^{1,q}(B^n, \mathbb{R}^{n+1}) \cap L^\infty$ for each $q < n$, and of course $u \notin \mathrm{VMO}(B^n, \mathbb{R}^{n+1})$. The image of u being the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1} := \{y \in \mathbb{R}^n \times \{0\}\colon |y| = 1\}$, by the area formula we deduce that $u \in \mathcal{A}^1(B^n, \mathbb{R}^{n+1})$. Since moreover u is the trace of the map $V : B^n \times (0, 1) \to \mathbb{R}^{n+1}$ given by $V(x, h) := \frac{(x,h)}{|(x,h)|}$, we have $V \in W^{1,p}(B^n \times (0, 1), \mathbb{R}^{n+1})$ and hence $u \in W^{1/p}(B^n, \mathbb{R}^{n+1})$ for each $p < n + 1$. Setting then $\mathfrak{p} = n$, on account of Remark 2.3, we deduce that the semi-current $G_u \in \mathcal{D}_{n,n-1}(B^n \times \mathbb{R}^{n+1})$ from Definition 2.2 agrees with the restriction to forms in $\mathcal{D}^{n,n-1}(B^n \times \mathbb{R}^{n+1})$ of the graph current $[\![\mathcal{G}_u]\!]$. By [12, Sec. 3.2.2], we have

$$\partial[\![\mathcal{G}_u]\!] \llcorner B^n \times \mathbb{R}^{n+1} = -\delta_0 \times [\![\mathbb{S}^{n-1}]\!],$$

where $\mathbb{S}^{n-1}$ is equipped with the natural orientation.

Setting as before $U : B^n \times (0, 1) \to \mathbb{R}^{n+1}$ by $U(x, h) := ⨍_{B_h(x)} u(y)\, dy$, and $u_h(x) := U(x, h)$, it turns out that G_{u_h} weakly converges in $\mathcal{D}_n(B^n \times \mathbb{R}^{n+1})$ as $h \to 0$ to the Cartesian current $T = [\![\mathcal{G}_u]\!] + S_u \in \mathrm{cart}(B^n \times \mathbb{R}^{n+1})$, where this time

$$S_u = \delta_0 \times [\![D^n]\!],$$

D^n being the naturally oriented unit n-disk given by the convex envelop of $\mathbb{S}^{n-1}$, so that $\partial[\![D^n]\!] = [\![\mathbb{S}^{n-1}]\!]$. In particular, the vertical part $T \llcorner \mathcal{M}^v$ is non-trivial.

Example 4.5. Let $V : [0,1] \to [0,1]$ denote the Cantor–Vitali function, so that $u \in BV \cap C^{0,\alpha}$ for $\alpha = \log 2/\log 3$, hence $u \in W^{1/2}$. Similarly as before, one deduces that $T_V = \partial[\![SG_V]\!]$, where $SG_V := \{(x,y) \mid x \in [0,1],\ y < V(x)\}$ is the subgraph of V. This can be directly checked by means of the characterization of the class of Cartesian currents in codimension one [12, Sec. 4.2.4], on account of Proposition 3.4. In particular, we have $T_V = [\![\mathcal{G}_V]\!] + S_V$, where $\mathcal{G}_V$ agrees with the set of points in the *reduced boundary* $\partial^- SG_V$ on which the tangent unit vector is "horizontal", and $S_V = T_V \llcorner \mathcal{M}^v$, where $\mathcal{M}^v$ is the set of points in $\partial^- SG_V$ on which the tangent unit vector is "completely vertical".

Example 4.6. As we have seen in the previous section, the example by S. Müller [14] shows the existence of a Hölder-continuous map $u \in W^{1/2} \cap \mathrm{VMO}(\Omega, \mathbb{S}^2)$, where $\Omega := (0,1)^2$, such that the action of the vertical component $S_u = T_{u(2)}$ of the integral flat chain T_u on a non-trivial class of forms $\omega = \omega^{(2)}$ in $\mathcal{D}^2(\Omega \times \mathbb{S}^2)$ is concentrated on $(C \times C) \times \mathbb{S}^2$, where C is the Cantor set, compare (3.6). Moreover, the computation from [12, Sec. 4.2.5] yields the existence of a Cartesian current $T \in \mathrm{cart}(\Omega \times \mathbb{S}^2)$ that "fills the holes" of the graph of u. More precisely, formulas (4.1) and (4.2) hold true, where $\mathcal{G}_u$ is the rectifiable graph of u. For this reason, we expect that our current T_u has finite mass and hence, by Proposition 4.1, that it is a Cartesian current in $\mathrm{cart}(\Omega \times \mathbb{S}^2)$.

5. Degree and a continuity property

In this final section we recover the property discovered in [7] that the degree of $W^{1/\mathfrak{p}}$-maps u from the $\mathfrak{p}$-sphere onto itself is well-defined and integer-valued.

We then show (Theorem 5.2) that the strong convergence in $W^{1/\mathfrak{p}} \cap \mathrm{BMO}$ yields the weak convergence in the sense of the currents of the corresponding integral flat chains given by Theorem 3.1. Our result extends the following continuity property proved in [7]:

Proposition 5.1 *(Brezis–Nguyen). Let $u_k, u \in C^1(\mathbb{S}^{\mathfrak{p}}_x, \mathbb{S}^{\mathfrak{p}}_y)$ be such that $\|u_k - u\|_{\mathrm{BMO}} \to 0$ and $\|u_k - u\|_{W^{1/\mathfrak{p}}} \to 0$ as $k \to \infty$. Then for every $F \in C^{0,\alpha}(\mathbb{S}^{\mathfrak{p}}_y)$, where $0 < \alpha < 1$, and $\psi \in C^1(\mathbb{S}^{\mathfrak{p}}_x)$ we have*

$$\lim_{k\to\infty} \int_{\mathbb{S}^p} F\big(u_k(x)\big)\psi(x) \det \nabla u_k(x)\, ds = \int_{\mathbb{S}^p} F\big(u(x)\big)\psi(x) \det \nabla u(x)\, ds.$$

Degree. Assume $n = \mathfrak{p}$ and $\mathcal{X} = \mathcal{Y} = \mathbb{S}^{\mathfrak{p}}$. For each $f \in C^1(\mathbb{S}^{\mathfrak{p}})$, if $v : \mathbb{S}^{\mathfrak{p}} \to \mathbb{S}^{\mathfrak{p}}$ is smooth we let

$$\big|\mathbb{S}^{\mathfrak{p}}\big| \cdot J(v,f) := \int_{\mathbb{S}^{\mathfrak{p}}} f(x) \det \nabla v(x)\, d\sigma,$$

where $\det \nabla v = \det(\nabla v, v)$, viewing $(\nabla v, v)$ as a square matrix of order $\mathfrak{p}+1$. We thus have

$$\det \nabla v(x) = v^\# \omega_{\mathfrak{p}}(x), \quad \omega_{\mathfrak{p}} := \sum_{j=1}^{\mathfrak{p}+1} (-1)^{j-1} y^j\, \widehat{dy^j},$$

where $\widehat{dy^j} := dy^1 \wedge \cdots \wedge dy^{j-1} \wedge dy^{j+1} \wedge \cdots \wedge dy^{\mathfrak{p}+1}$. Therefore, according to Example 2.1 we get

$$\big|\mathbb{S}^{\mathfrak{p}}\big| \cdot J(v,f) = \int_{\mathbb{S}^{\mathfrak{p}}} (\mathrm{Id} \bowtie v)^\#(f \wedge \omega_{\mathfrak{p}}) =: \langle G_v, f \wedge \omega_{\mathfrak{p}} \rangle.$$

If $u \in W^{1/\mathfrak{p}}(\mathbb{S}^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$, hence $u \in \mathrm{VMO}$, using Corollary 1.4, Theorem 1.5, and Theorem 3.1, it turns out that the Jacobian of u is well-defined by

$$\big|\mathbb{S}^{\mathfrak{p}}\big| \cdot J(u,f) := \lim_{k\to\infty} \langle G_{v_k}, f \wedge \omega_{\mathfrak{p}} \rangle = \langle T_u, f \wedge \omega_{\mathfrak{p}} \rangle = \langle S_u, f \wedge \omega_{\mathfrak{p}} \rangle,$$

where $\{v_k\} \subset C^1(\mathbb{S}^{\mathfrak{p}}, \mathbb{S}^{\mathfrak{p}})$ is the smooth sequence given in the proof of Theorem 3.1 by $v_k := u_{\delta_k}$ for a good sequence $\delta_k \searrow 0$. Since $J(v_k, 1) =: \deg(v_k)$ is integer for each k, and by the weak convergence $|\langle G_{v_k}, f \wedge \omega_{\mathfrak{p}}\rangle - \langle T_u, f \wedge \omega_{\mathfrak{p}}\rangle| \to 0$, one recovers the property discovered in [7] that the degree $\deg(u) := J(u, 1)$ is integer.

More generally, it turns out that under the hypotheses of Theorem 3.1, in any dimension $n \geqslant \mathfrak{p}$, and for each integer $j = 0, \dots, \mathfrak{p}$, the limit as $k \to \infty$ of

$$\langle G_{v_k}, \varphi \wedge \omega\rangle := \int_{\mathcal{X}} \varphi \wedge {v_k}^{\#}\omega, \quad \varphi \in \mathcal{D}^{n-j}(\mathcal{X}),\ \omega \in \mathcal{D}^j(\mathcal{Y})$$

is well-defined and agrees with $T_u(\varphi \wedge \omega)$. Moreover, $\langle T_u, \varphi \wedge \omega\rangle = \langle G_u, \varphi \wedge \omega\rangle$ if $j < \mathfrak{p}$, and $\langle T_u, \varphi \wedge \omega\rangle = \langle S_u, \varphi \wedge \omega\rangle$ if $j = \mathfrak{p}$.

A continuity property. As a consequence of Theorem 3.1, we finally prove:

Theorem 5.2. *Assume that* $\dim \mathcal{Y} \leqslant \mathfrak{p}$*, the integer part of* p*. Let* $u_k, u \in W^{1/p} \cap \mathrm{VMO}(\mathcal{X}, \mathcal{Y})$ *and let* T_{u_k}*,* T_u *denote the corresponding integral flat chains in* $\mathcal{F}_n(\mathcal{X} \times \mathcal{Y})$ *given by* (3.1)*. If* $\|u_k - u\|_{\mathrm{BMO}} \to 0$ *and* $\|u_k - u\|_{W^{1/\mathfrak{p}}} \to 0$ *as* $k \to \infty$*, then* $T_{u_k} \rightharpoonup T_u$ *weakly in* $\mathcal{D}_n(\mathcal{X} \times \mathcal{Y})$*, i.e.,*

$$\lim_{k\to\infty} \langle T_{u_k}, \omega\rangle = \langle T_u, \omega\rangle \quad \forall \omega \in \mathcal{D}^n(\mathcal{X} \times \mathcal{Y}).$$

Proof. Since $\|u_k - u\|_{\mathrm{BMO}} \to 0$, by Corollary 1.4, in Proposition 1.2 we may choose $\delta > 0$ uniformly with respect to $\{u_k\}$. Therefore, by (3.1), we can choose a cut-off function $\eta = \eta(\delta)$ in (2.2) in such a way that for each k we have

$$\langle T_{u_k}, \omega\rangle = (-1)^{n-1}\big\langle G_{U_k}, d\big[(\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})^{\#}\omega \wedge \eta_\delta\big]\big\rangle \quad \forall \omega \in \mathcal{D}^n(\mathcal{X} \times \mathcal{Y}).$$

We can thus write for each k and every $\omega \in \mathcal{D}^n(\mathcal{X} \times \mathcal{Y})$

$$\big|\langle T_{u_k}, \omega\rangle - \langle T_u, \omega\rangle\big| = \big|\big\langle G_{U_k}, d\big[(\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})^{\#}\omega \wedge \eta_\delta\big]\big\rangle - \big\langle G_U, d\big[(\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})^{\#}\omega \wedge \eta_\delta\big]\big\rangle\big|.$$

Moreover, using that $\dim \mathcal{Y} \leqslant \mathfrak{p}$, the $(n+1)$-form $d[(\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})^{\#}\omega \wedge \eta_\delta]$ contains at most $\mathfrak{p}$ vertical differentials in the $\mathcal{Y}$-directions. On the other hand, the strong convergence $\|u_k - u\|_{W^{1/\mathfrak{p}}} \to 0$ yields that $\|U_k - U\|_{W^{1,\mathfrak{p}}} \to 0$. Therefore, by the dominated convergence, we get

$$\lim_{k\to\infty} \big|\big\langle G_{U_k}, d\big[(\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})^{\#}\omega \wedge \eta_\delta\big]\big\rangle - \big\langle G_U, d\big[(\Phi \bowtie \mathrm{Id}_{\mathcal{Y}})^{\#}\omega \wedge \eta_\delta\big]\big\rangle\big| = 0,$$

as required. □

When $\mathcal{X} = \mathcal{Y} = \mathbb{S}^{\mathfrak{p}}$, since in the smooth case

$$\big\langle T_u, F(y)\psi(x) \wedge \omega_{\mathfrak{p}}\big\rangle = \int_{\mathbb{S}^p} F\big(u(x)\big)\psi(x) \det \nabla u(x)\, d\sigma,$$

we readily deduce Proposition 5.1 from the more general Theorem 5.2, by taking $\omega = F(y)\psi(x) \wedge \omega_{\mathfrak{p}}$.

Remark 5.3. On account of Remark 3.2, if $\dim \mathcal{Y} > \mathfrak{p}$, similarly to Theorem 5.2 this time we deduce that $T_{u_k} \rightharpoonup T_u$ weakly in $\mathcal{D}_{n,\mathfrak{p}}(\mathcal{X} \times \mathcal{Y})$, i.e.,

$$\lim_{k\to\infty} \langle T_{u_k}, \omega\rangle = \langle T_u, \omega\rangle \quad \forall \omega \in \mathcal{D}^{n,\mathfrak{p}}(\mathcal{X} \times \mathcal{Y}).$$

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Appendix A

We prove the following

Proposition A.1. *Let $\rho : \mathbb{R}^n \to \mathbb{R}$ denote the summable symmetric convolution kernel given by*

$$\rho(z) := \begin{cases} -\frac{1}{2}\log|z| & \text{if } 0 < |z| < 1, \\ 0 & \text{elsewhere} \end{cases}$$

for $n = 1$, and for $n > 1$ by

$$\rho(z) := \begin{cases} \frac{1}{(n-1)\alpha_n}(|z|^{1-n} - 1) & \text{if } \quad 0 < |z| < 1, \\ 0 & \text{elsewhere}, \end{cases} \qquad \alpha_n := \left|B^n\right|$$

so that $\rho \in L^1(\mathbb{R}^n)$, $\operatorname{spt}\rho = \overline{B}^n$, $\rho \geqslant 0$, and $\int \rho(z)\,dz = 1$. Let $u \in L^1(B^n)$ and $U(x,t) = ⨍_{B_t(x)} u(y)\,dy$. Then for each $\varepsilon > 0$ and $x \in B^n_{1-\varepsilon}$ we have

$$(u * \rho_\varepsilon)(x) = ⨍_{[0,\varepsilon]} U(x,t)\,dt, \qquad \rho_\varepsilon(z) := \varepsilon^{-n}\rho(z/\varepsilon).$$

Therefore, if $u \in \mathrm{BV}(B^n)$ we have that

$$\lim_{\varepsilon\to 0} \int_{B^n} \left| ⨍_{[0,\varepsilon]} D_i U(x,t)\,dt \right| dx = \lim_{\varepsilon \to 0} \int_{B^n} \left|D_i(u * \rho_\varepsilon)(x)\right| dx = |D_i u|\big(B^n\big)$$

and if $u \in W^{1,p}(B^n)$, the map $x \mapsto ⨍_{[0,\varepsilon]} U(x,t)\,dt$ converges to u strongly in $W^{1,p}$, as $\varepsilon \to 0$.

Proof. For $n = 1$ and $|x| < 1 - \varepsilon$, we have

$$\begin{aligned}
⨍_{[0,\varepsilon]} U(x,\lambda)\,d\lambda &= \int_0^1 U(x,\varepsilon t)\,dt \\
&= \frac{1}{2\varepsilon}\int_0^1 \frac{1}{t}\int u(y)\chi_{(x-\varepsilon t, x+\varepsilon t)}(y)\,dy\,dt \\
&= \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} u(y)\int_0^1 \frac{1}{t}\chi_{(x-\varepsilon t, x+\varepsilon t)}(y)\,dt\,dy \\
&= \frac{1}{2\varepsilon}\left\{\int_{x-\varepsilon}^{x} u(y)\int_{(x-y)/\varepsilon}^{1} \frac{1}{t}\,dt\,dy + \int_x^{x+\varepsilon} u(y)\int_{(y-x)/\varepsilon}^{1}\frac{1}{t}\,dt\,dy\right\} \\
&= \int u(y)\rho_\varepsilon(x-y)\,dy =: (u*\rho_\varepsilon)(x).
\end{aligned}$$

Similarly, for $n > 1$ we get

$$\begin{aligned}
⨍_{[0,\varepsilon]} U(x,\lambda)\,d\lambda &= \frac{1}{\alpha_n\varepsilon^n}\int_0^1 \frac{1}{t^n}\int u(y)\chi_{B_{\varepsilon t}(x)}(y)\,dy\,dt \\
&= \frac{1}{\alpha_n\varepsilon^n}\int_{B_\varepsilon(x)} u(y)\int_0^1 \frac{1}{t^n}\chi_{B_{\varepsilon t}(x)}(y)\,dt\,dy
\end{aligned}$$

$$= \frac{1}{\alpha_n \varepsilon^n} \int\limits_{B_\varepsilon(x)} u(y) \int\limits_{|x-y|/\varepsilon}^{1} \frac{1}{t^n}\, dt\, dy$$

$$= \frac{1}{\varepsilon^n} \int\limits_{B_\varepsilon(x)} u(y) \frac{1}{(n-1)\alpha_n} \left(\left| \frac{x-y}{\varepsilon} \right|^{1-n} - 1 \right) dy$$

$$= \int u(y) \rho_\varepsilon(x-y)\, dy =: (u * \rho_\varepsilon)(x).$$

The last assertion follows from standard arguments, compare [5]. □

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