Calcolo delle variazioni. — *Functionals with $p(x)$ growth and regularity.* Nota di Emilio Acerbi e Giuseppe Mingione, presentata (*) dal Socio A. Ambrosetti.

Abstract. — We consider the integral functional $\int f(x, Du)\,dx$ under non standard growth assumptions of (p,q)-type: namely, we assume that $|z|^{p(x)} \le f(x,z) \le L(1+|z|^{p(x)})$, a relevant model case being the functional $\int |Du|^{p(x)}\,dx$. Under sharp assumptions on the continuous function $p(x) > 1$ we prove regularity of minimizers both in the scalar and in the vectorial case, in which we allow for quasiconvex energy densities. Energies exhibiting this growth appear in several models from mathematical physics.

Key words: Integral functionals; Minimizers; Nonstandard growth; Partial regularity.

Riassunto. — *Funzionali a crescita $p(x)$ e regolarità.* Consideriamo il funzionale integrale $\int f(x, Du)\,dx$ sotto ipotesi di crescita non standard di tipo (p,q): precisamente, supponiamo che $|z|^{p(x)} \le f(x,z) \le L(1+|z|^{p(x)})$, ottenendo un funzionale il cui modello è $\int |Du|^{p(x)}\,dx$. In ipotesi ottimali sulla funzione continua $p(x) > 1$, dimostriamo la regolarità dei minimi sia nel caso scalare che vettoriale, nel quale copriamo anche il caso di densità di energia quasiconvesse. Energie con crescite come quelle considerate compaiono in diversi modelli della fisica matematica.

In the last decades great attention was paid to the study of regularity and existence properties of local minimizers of integral functionals of the calculus of variations of the type

$$\mathcal{F}(u,\Omega) := \int_\Omega f(x, Du)\,dx \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $f : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Carathéodory integrand satisfying the growth assumptions of (p,q) type:

$$|z|^p \le f(x,z) \le L(1+|z|^q)\,, \qquad q \ge p > 1\,. \tag{2}$$

The case $p = q$ was intensively studied over the last thirty years and a rich existence and regularity theory is now available, especially for the scalar case $N = 1$. Instead, the "nonstandard growth" case, that is when (2) is not verified with $p = q$, has been investigated only in the last decade. In particular the regularity theory, in which we are interested here, was initiated by Marcellini and rapidly grew through several contributions from many authors. For the case of an autonomous energy depending only on the gradient,

$$\int_\Omega f(Du)\,dx\,,$$

we now have a quite satisfying picture, at least in the scalar case (see [5]). Anyway obtaining sharp regularity results for functionals as (1) still remains a largely open problem.

In this note we report on some recent results concerning the regularity of local minimizers of functionals with nonstandard growth of (p,q) type satisfying the following assumptions, that we called $p(x)$ growth:

$$|z|^{p(x)} \le f(x,z) \le L(1+|z|^{p(x)})\,, \tag{3}$$

where $p : \Omega \to]1, +\infty[$ is a continuous function. A prominent model example for us will be

$$\int_\Omega (\mu^2 + |Du|^2)^{p(x)/2}\, dx \ ,$$

with $\mu \geq 0$. The study of this kind of problem is motivated by many models coming from various branches of mathematical physics and exhibiting this type of growth. In particular we mention a recent differential system developed by Rajagopal and Růžička to describe the dynamics of electrorheological fluids. These are particular non-Newtonian fluids, characterized by their ability of changing their mechanical properties when interacting with an electromagnetic field $\mathbf{E}(x)$. In this case the system modeling the phenomenon, as introduced by Rajagopal and Růžička, is:

$$\begin{cases} \operatorname{div} \mathbf{E} = 0 \qquad \operatorname{curl} \mathbf{E} = 0 \\ v_t - \operatorname{div} S(x, \mathbf{E}, \mathcal{E}(v)) + \langle v, Dv \rangle + D\pi = g(x, \mathbf{E}) \\ \operatorname{div} v = 0 \end{cases} \tag{4}$$

where v is the velocity of the fluid, $\mathcal{E}(v)$ is the symmetric part of the gradient Dv and π is the pressure. For our purpose the interesting fact is that the interaction between $\mathbf{E}$ and the fluid is modeled by assigning a particular growth structure to the "extra stress" tensor S, which turns out to be a monotone vector field in the Leray-Lions sense satisfying the ellipticity condition

$$DS(x, \mathbf{E}, z)\lambda \otimes \lambda \geq \nu(\mathbf{E})(1 + |z|^2)^{(p-2)/2}|\lambda|^2 \ ,$$

where $\nu(|\mathbf{E}|) \geq \overline{\nu} > 0$, for all symmetric matrixes z, λ with null trace and, most important, where the exponent p depends on the modulus of the field $\mathbf{E}$, that is $p \equiv p(|\mathbf{E}|)$. As soon as the boundary conditions are assigned, the structure of the system allows to determine $\mathbf{E}$ so that the exponent actually depends on the point, thus $p \equiv p(x)$. For this kind of system Rajagopal and Růžička give an existence theory which is particularly satisfying in the steady case, when the second equation in (4) reduces to:

$$-\operatorname{div} S(x, \mathcal{E}(v)) + D\pi = g(x, \mathbf{E}) \ .$$

As was pointed out by Zhikov, there are further instances of materials and fluids exhibithing such kind of sensitivity to temperature. In particular Zhikov considers the thermistor problem, in which one wants to determine the temperature-dependent conductivity in a matter; the system is

$$\begin{cases} -\operatorname{div}(a(T)|Du|^{p(T)-2}Du) = f \\ \triangle T = g(T)|Du|^2 \ , \end{cases}$$

where u is the potential and T denotes the temperature; this system shows a strong coupling between the two equations and this eventually leads to looking for minimal regularity assumptions on the exponent p.

In this note, as a starting point for the analysis of models like those described above, we treat regularity issues for functionals with $p(x)$ growth as in (3). Indeed though all our results are formulated in a variational context, our methods and techniques can be applied to systems as well. Further results for (steady) Stokes type systems as (4) are in preparation.

The main feature of our results is that we are able to construct a complete regularity theory for local minimizers of these functionals, drawing a striking parallel to the theory of regularity for minimizers in the case p is constant. Due to the nonstandard growth condition, we shall adopt the following notion of local minimizer:

Definition . *We say that a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ if $|Du|^{p(x)} \in L^1_{\mathrm{loc}}(\Omega)$ and $\mathcal{F}(u, \operatorname{spt}\varphi) \le \mathcal{F}(u+\varphi, \operatorname{spt}\varphi)$ for any $\varphi \in W^{1,1}_0(\Omega;\mathbb{R}^N)$ with compact support in Ω.*

On the Carathéodory function $f : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ we require the following growth, quasiconvexity and continuity assumptions:

$$L^{-1}(\mu^2 + |z|^2)^{p(x)/2} \le f(x,z) \le L(\mu^2 + |z|^2)^{p(x)/2} \tag{5}$$

$$\begin{aligned}\int_{Q_1} &[f(x_0, z_0 + D\varphi) - f(x_0, z_0)]\,dx \\ &\ge L^{-1}\int_{Q_1}(\mu^2 + |z_0|^2 + |D\varphi|^2)^{(p(x_0)-2)/2}|D\varphi|^2\,dx\end{aligned} \tag{6}$$

for each $z_0 \in \mathbb{R}^n,\ x_0 \in \Omega$ and each $\varphi \in C^\infty_0(Q_1)$ where $0 \le \mu \le 1$, $Q_1 = (0,1)^n$, and

$$\begin{aligned}&|f(x,z) - f(x_0,z)| \\ &\le L\omega(|x-x_0|)\left((\mu^2+|z|^2)^{p(x)/2} + (\mu^2+|z|^2)^{p(x_0)/2}\right)[1 + |\log(\mu^2+|z|^2)|]\end{aligned} \tag{7}$$

for any $z_0 \in \mathbb{R}^n,\ x, x_0 \in \Omega$ and where $L \ge 1$; here $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function vanishing at zero which represents the modulus of continuity of $p(x)$:

$$|p(x) - p(y)| \le \omega(|x-y|)\ .$$

Let us observe that despite its apparently involved formulation, condition (6) is quite general: indeed in the scalar case $N = 1$ it provides a qualified form of convexity which allows to cover all integrands of the form

$$f(x,z) = (\mu^2 + |z|^2)^{p(x)/2} + h(x,z)$$

where h is a generic (thus not necessarily differentiable) convex function of z satisfying (7) and such that $0 \le h(x,z) \le L(\mu^2 + |z|^2)^{p(x)/2}$. Moreover when the function f is of class C^2 with respect to the variable z, condition (6) allows to recover the classical ellipticity of the matrix $D^2 f(x,z)$ in the sense that

$$D^2 f(x,z)\lambda \otimes \lambda \ge \overline{\nu}(\mu^2 + |z|^2)^{(p(x)-2)/2}|\lambda|^2.$$

The first regularity results obtained for functionals with $p(x)$ growth in the scalar case are summarized in the following theorem (see [2,4,6]):

Theorem 1 . (Zhikov, Fan-Zhao, Alkhutov) *Let* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *be a local minimizer of the functional* $\mathcal{F}(\cdot,\Omega)$, *where* f *is a Carathéodory function satisfying* (5) *with* $N = 1$*; if*

$$\limsup_{R\to 0} \omega(R)\log\Big(\frac{1}{R}\Big) < +\infty \tag{8}$$

then there exist two constants $\delta > 0$ *and* $0 < \alpha < 1$, *both depending on* $(n, p(x), L)$, *such that* $Du \in L^{p(x)+\delta}_{\mathrm{loc}}(\Omega)$ *and* $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$.

It is worth remarking that condition (8) is sharp since in general, as recently shown by Zhikov, dropping it causes the loss of any type of regularity of minimizers, like Hölder continuity and even higher integrability. Moreover condition (8) seems to play a central role in the theory of functionals with $p(x)$-growth since again Zhikov proved that such functionals exhibit the Lavrentiev phenomenon iff (8) is violated, while very recently Acerbi, Bouchittè and Fonseca proved that the singular part of the measure representation of relaxed integrals with this growth disappears iff (6) holds true. Theorem 1 represents the "$p(x)$" counterpart of the classical theorems by Giaquinta and Giusti valid for the case $p(x) = p =$ constant, asserting the Hölder continuity of u without any continuity property of the function f.

The first higher regularity result in the vectorial case is the following (see [3]):

Theorem 2 . (Coscia-Mingione) *Let* $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *be a local minimizer of the functional*

$$\int_\Omega |Du|^{p(x)}\,dx\ ;$$

then there exists $\alpha \equiv \alpha(n, N, p(x))$ *such that* $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *provided* $p(x)$ *is locally Hölder continuous.*

The previous theorem generalizes to our case the classical theorem of K. Uhlenbeck for solutions of p-Laplacian type systems and it is also optimal.

We present here the results concerning higher regularity of minimizers for general structures as (3), obtained in [1]; we first deal with the scalar case.

Theorem 3 . *Let* $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ *be a local minimizer of the functional* $\mathcal{F}(\cdot,\Omega)$ *where* f *is a continuous function satisfying* (5)–(7). *Suppose moreover that*

$$\limsup_{R\to 0} \omega(R)\log\Big(\frac{1}{R}\Big) = 0\ . \tag{9}$$

Then $u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, *for any* $0 < \alpha < 1$.

In view of theorem 1, this result is the natural generalization to the case of $p(x)$ growth of the theorems valid in the standard case when a continuous dependence on the variable x must be assumed to reach any exponent $\alpha < 1$; indeed, in our situation this reflects on the fact that (8) must be naturally reinforced into (9). Moreover it is worth noting that in the previous theorem we do not require any differentiability property of the integrand f with respect to z, recovering also in this case some recent results for the standard case by Cupini, Fusco and Petti.

The proof of theorem 3 is based on a very careful freezing argument, that turns out to be delicate since the perturbation is performed in the exponent. This requires a suitable use of the higher integrability estimates of theorem 1. Moreover an interesting technical feature of the proof is that, in order estimate some quantities appearing in the comparison procedures, we apply a recent result due to T. Iwaniec concerning the $L\log L$ spaces. More precisely we use the fact that the usual Luxemburg norm in $L\log L$ is equivalent to the integral functional

$$[h]_{L\log L} := \int_\Omega |h| \log\Big(e + \frac{|h|}{\|h\|_1}\Big)\, dx$$

for any $h \in L\log L(\Omega)$: hence this functional, quite surprisingly, also defines a norm, equivalent to the usual one.

In order to have local $C^{1,\alpha}$ regularity we have to require a Hölder dependence of f with respect to the variable x. Indeed the arguments employed to prove theorem 3 also lead to the following:

THEOREM 4. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathcal{F}(\cdot,\Omega)$ where $\omega(R) \le LR^\alpha$ for some $0 < \alpha \le 1$ and all $R \le 1$, and f is a function of class C^2 with respect to the variable z in $\Omega \times (\mathbb{R}^n \setminus \{0\})$ satisfying (5)–(7) and with $D^2 f$ satisfying*

$$L^{-1}(\mu^2 + |z|^2)^{(p(x)-2)/2}|\lambda|^2 \le D^2 f(x,z)\lambda \otimes \lambda \le L(\mu^2 + |z|^2)^{(p(x)-2)/2}|\lambda|^2$$

for all $\lambda \in \mathbb{R}^n$. Then Du is locally Hölder continuous in Ω.

Now we turn our attention to the vectorial case $N > 1$; in this case, we cannot expect everywhere regularity as in the scalar case, as suggested by well known counterexamples valid even for convex functionals with quadratic growth. We focus on the concept of quasiconvexity, a central definition in the vectorial case, and we prove partial regularity of minimizers:

THEOREM 5. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ where f is a function of class C^2 with respect to the variable z satisfying hypotheses (5)–(7) with $\mu > 0$ and with $\omega(R) \le cR^\alpha$ for some $c < +\infty$, $0 < \alpha \le 1$ and for any $R \le 1$. Then there exists an open subset $\Omega_0 \subset \Omega$ such that $|\Omega \setminus \Omega_0| = 0$ and Du is locally Hölder continuous in Ω_0.*

The previous result is the analogous, in our context, of the partial regularity results valid for convex and quasiconvex integrals with polynomial growth, that is $p =$ constant. The proof in our case is pretty intricate and it is based on an appropriate localization of the blow-up arguments leading to partial regularity. In particular, the central idea is to blow up minimizers, not in the whole Ω, but in small open subsets where the function $p(x)$ has suitably small variations. At this stage we have to carefully combine some higher integrability inequalities from theorem 1 and the Hölder estimates employed in the proof of theorem 4. The final result is then obtained passing through a covering argument involving small subsets depending both on u and $p(x)$.

Acknowledgements

This work has been performed as a part of the Research Project "Modelli Variazionali sotto Ipotesi non Standard," supported by GNAFA-CNR

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