Curvature-dependent energies

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Abstract. We report our recent results from [1, 2] on the total curvature of graphs of curves in high codimension Euclidean space. We introduce the corresponding relaxed energy functional and provide an explicit representation formula. In the case of continuous Cartesian curves, i.e. of graphs c_u of continuous functions u on an interval, the relaxed energy is finite if and only if the curve c_u has bounded variation and finite total curvature. In this case, moreover, the total curvature does not depend on the Cantor part of the derivative of u. We also deal with the "elastic" case, corresponding to a superlinear dependence on the pointwise curvature. Different phenomena w.r.t. the"plastic" case are observed. A p-curvature functional is well-defined on continuous curves with finite relaxed energy, and the relaxed energy is given by the length plus the p-curvature. We treat the wider class of graphs of one-dimensional BV-functions.

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1. Introduction

Energies of the second order are frequent in mathematical physics, say in the study of thin structures such as wires and plates. In the mathematical literature, functionals depending on second order derivatives have recently been applied e.g. in image restoration processes, in order to overcome some drawbacks typical of approaches based on first order functionals, as the total variation. One instance is the approach by Chan-Marquina-Mulet [5] who proposed to consider regularizing terms given by second order functionals of the type

$$\int_\Omega |\nabla u|\, dx + \int_\Omega \psi(|\nabla u|)\, h(\Delta u)\, dx$$

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for scalar-valued functions u defined in two-dimensional domains, where the function ψ satisfies suitable conditions at infinity in order to allow jumps.

The downscaled one-dimensional version of the above functional is given by

$$\int_a^b |\dot u|\, dt + \int_a^b \psi(|\dot u|)\, |\ddot u|^p\, dt\,, \qquad p \geq 1\,, \qquad u : [a,b] \to \mathbb{R} \tag{1.1}$$

and it has been thoroughly studied in [8], where Dal Maso-Fonseca-Leoni-Morini proved an explicit formula for the relaxed energy, under suitable assumptions on the function ψ.

The prototypical example is the *p-curvature energy functional*, obtained by choosing

$$\psi_p(t) := \frac{1}{(1+t^2)^{(3p-1)/2}}\,, \quad p \geq 1\,.$$

In this case, in fact, the above functional takes the form

$$\mathcal{E}_p(u) := \int_a^b |\dot u|\, dt + \int_a^b \sqrt{1+\dot u(t)^2} \cdot k_u(t)^p\, dt\,, \qquad k_u(t) := \frac{|\ddot u(t)|}{(1+\dot u(t)^2)^{3/2}}\,.$$

Therefore, in the smooth case, considering the *Cartesian curve* $c_u(t) := (t, u(t))$, and replacing the first term with the integral of $\sqrt{1+\dot u^2}$, since $k_u(t)$ is the curvature at the point $c_u(t)$, by the area formula one obtains an intrinsic formulation on the graph curve c_u as

$$\mathcal{E}_p(u) = \mathcal{L}(c_u) + \int_{c_u} k_u^p\, d\mathcal{H}^1\,, \tag{1.2}$$

where $\mathcal{L}$ is the length.

Formula (1.2) can be taken in higher codimension $N \geq 2$ as the definition of our energy functional. In fact, the curvature of a smooth Cartesian curve $c_u(t) = (t, u^1(t), \ldots, u^N(t))$ is given at the point $c_u(t)$ by the formula

$$k_u = \mathbf{k}_{c_u} = \frac{|\dot c_u \wedge \ddot c_u|}{|\dot c_u|^3} = \frac{|\dot u \wedge \ddot u|}{|\dot c_u|^3} = \frac{\big((1+|\dot u|^2)|\ddot u|^2 - (\dot u \bullet \ddot u)^2\big)^{1/2}}{(1+|\dot u|^2)^{3/2}}\,. \tag{1.3}$$

The relaxed energy. In the papers [1, 2] we have provided in high codimension a complete explicit formula for the relaxed energy. More precisely, for any $p \geq 1$ we define

$$\overline{\mathcal{E}}_p(u) := \inf\big\{\liminf_{h\to\infty} \mathcal{E}_p(u_h) \mid \{u_h\} \subset C^2(I, \mathbb{R}^N)\,,\ u_h \to u \text{ in } L^1\big\} \tag{1.4}$$

for any summable function $u \in L^1(I, \mathbb{R}^N)$, where $I = [a,b]$. We correspondingly denote:

$$\mathcal{E}_p(I, \mathbb{R}^N) := \{u \in L^1(I, \mathbb{R}^N) \mid \overline{\mathcal{E}}_p(u) < \infty\}\,, \quad p \geq 1\,. \tag{1.5}$$

Any function u with finite relaxed energy (1.4) has bounded variation, with distributional derivative decomposed as usual by $Du = \dot u\, \mathcal{L}^1 + D^C u +$

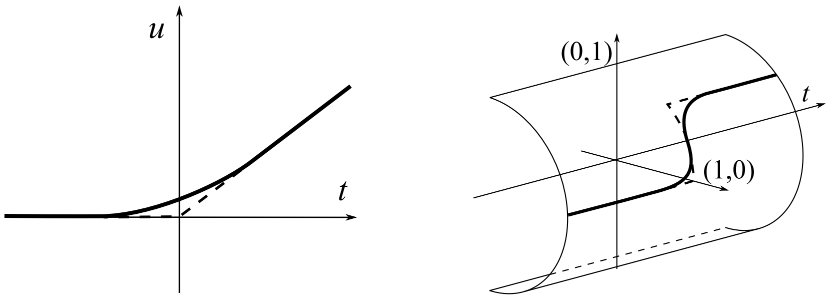

FIGURE 1. The curve c_α (dashed) and the smooth approximation of $c_{\alpha,\varepsilon}$.
On the right: the graph $(t, \mathbf{k}_c(t))$ of the corresponding unit tangent vector.

$D^J u$, compare [3]. A crucial role is played here by the *Gauss map* τ_u that is defined a.e. in I by means of the *approximate gradient* $\dot{u}$, namely

$$\tau_u = \frac{\dot{c}_u}{|\dot{c}_u|}, \qquad \dot{c}_u = (1, \dot{u}^1, \ldots, \dot{u}^N). \tag{1.6}$$

1.1. The plastic case

We first deal with the "plastic" case $p = 1$, and we see how the relaxed formula of the curvature functional must in general contain an angle term reminding of the regular version.

Example 1.1. Consider a curvature functional with "plastic" property:

$$\mathcal{F}(c) := \int_{-1}^{1} |\dot{c}(t)| \, (\lambda_1 + \lambda_2 \, g(\mathbf{k}_c(t))) \, dt, \qquad \lambda_1, \lambda_2 \in \mathbb{R}^+ \tag{1.7}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a non-negative Lipschitz-continuous function such that $g(0) = 0$ and the limit

$$g^\infty := \lim_{t \to +\infty} \frac{g(t)}{t} \tag{1.8}$$

exists and is finite. The *recession* g^∞ of g comes into play in case of occurrence of angles.

In fact, let $0 < \alpha < \pi/4$ and $c_\alpha : [-1, 1] \to \mathbb{R}^2$ be the piecewise affine curve with a turning angle of width 2α at the origin, given by

$$c_\alpha(t) := \begin{cases} (t, 0) & \text{if } t < 0 \\ \dfrac{(t, m_\alpha t)}{\sqrt{1 + m_\alpha^2}} & \text{if } t \geq 0 \end{cases} \qquad m_\alpha := \tan(2\alpha).$$

This function is not of class C^2, so the functional $\mathcal{F}$ is not defined on c. But we now see that, in a relaxed sense,

$$\mathcal{F}(c_\alpha) = \lambda_1 \cdot \mathcal{L}(c_\alpha) + \lambda_2 \cdot 2\alpha \cdot g^\infty.$$

In fact, outside the origin the curvature of c_α is zero, and we can approximate c_α near the origin by small regular arcs, as e.g. $c_{\alpha,\varepsilon} : [-\pi/2, -\pi/2 + 2\alpha] \to \mathbb{R}^2$ defined by

$$c_{\alpha,\varepsilon}(t) := \varepsilon\,(-\sin\alpha + \cos\alpha\cos t, \cos\alpha + \cos\alpha\sin t)\,.$$

We thus have $|\dot c_{\alpha,\varepsilon}(t)| = \varepsilon\,\cos\alpha$ and $\det[\dot c_{\alpha,\varepsilon}(t)|\ddot c_{\alpha,\varepsilon}(t)] = \varepsilon^2\cos^2\alpha$, so that

$$\int_{-\pi/2}^{-\pi/2+2\alpha} |\dot c_{\alpha,\varepsilon}(t)|\, g\Big(\frac{|\det[\dot c_{\alpha,\varepsilon}(t)|\ddot c_{\alpha,\varepsilon}(t)]|}{|\dot c_{\alpha,\varepsilon}(t)|^3}\Big)\, dt = 2\alpha\,\varepsilon\cos\alpha\, g((\varepsilon\cos\alpha)^{-1})$$

and hence $\mathcal{F}(c_{\alpha,\varepsilon}) \to \lambda_1 \cdot \mathcal{L}(c_\alpha) + \lambda_2 \cdot 2\alpha \cdot g^\infty$ as $\varepsilon \to 0^+$.

In Figure 1 we have divided the graph curve $(c(t), \mathbf{k}_c(t))$ by drawing on the left side the image of $c(t)$ and on the right the graph $(t, \mathbf{k}_c(t))$ of the unit tangent vector.

Remark 1.2. Take a sequence $\{u_h\}$ of smooth functions, and assume that their graphs c_{u_h} converge weakly as currents to a rectifiable (not necessarily Cartesian) curve c; for each h the tantrix τ_{u_h} has positive first component $\tau^0_{u_h}$, so also for the limit curve c the tantrix has non-negative first component, i.e. it takes values into the half-sphere

$$\mathbb{S}^N_+ := \{y = (y_0, y_1, \dots, y_N) \in \mathbb{R}^{N+1}_y \ :\ |y| = 1\,,\ y_0 \geq 0\}\,.$$

We thus correspondingly define

$$\mathbb{S}^{N-1}_0 := \{y \in \mathbb{S}^N_+ \mid y_0 = 0\}\,. \tag{1.9}$$

Total curvature. We recall that the *total curvature* $\mathrm{TC}(c)$ of a curve c has been defined by Milnor [11] as the supremum of the total curvature (i.e. the sum of the turning angles) of the polygons inscribed in c. A curve with finite total curvature is rectifiable, and hence it admits a Lipschitz parameterization. Therefore, it is well defined the *tantrix* (or tangent indicatrix), that assigns to a.e. point the oriented unit tangent vector $\mathfrak{t}_c$. Moreover, the total curvature agrees with the *essential total variation* of the tantrix.

For smooth Cartesian curves c_u the tantrix $\mathfrak{t}_{c_u}$ agrees with the Gauss map τ_u, whence the total curvature $\mathrm{TC}(c_u)$ is equal to the total variation of τ_u. Therefore, for C^2-functions u one has

$$\mathrm{TC}(c_u) = \int_{c_u} k_u\, d\mathcal{H}^1 = \int_I |\dot\tau_u|\, dt\,.$$

Actually, in the relaxation process the role of the tantrix is played by the Gauss map (1.6).

Example 1.3. Let $c_{u_h} : [-1,1] \to \mathbb{R}^2$ be the piecewise affine Cartesian curve $c_{u_h}(t) = (t, u_h(t))$, where

$$u_h(t) := \begin{cases} 0 & \text{if } t < -\pi/h \\ ht + \pi & \text{if } -\pi/h \leq t \leq \pi/h \\ 2\pi & \text{if } t > \pi/h \end{cases} \qquad h \in \mathbb{N}^+ \quad \text{large}\,, \tag{1.10}$$

so that c_{u_h} has two corners with two turning angles both of width $\arctan h$ at the points $(-\pi/h, 0)$ and $(\pi/h, 2\pi)$. With the same notation as in Exam-

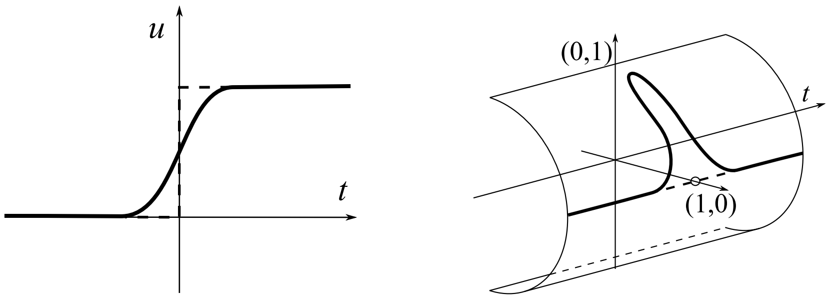

FIGURE 2. The curve c (dashed) and the smooth approximation of c_{u_h}.
On the right: the corresponding curves in the (t,τ)-space.

ple 1.1, and in the same relaxed sense since $u_h \notin C^2$,

$$\mathcal{F}(c_{u_h}) := \int_{-1}^{1} |\dot{c}_{u_h}(t)|\,(\lambda_1 + \lambda_2\, g(k_{u_h}(t)))\,dt = \lambda_1 \cdot \mathcal{L}(c_{u_h}) + \lambda_2 \cdot 2\arctan h \cdot g^\infty\,,$$

so that $\mathcal{F}(c_{u_h}) \to \lambda_1 \cdot (2+2\pi) + \lambda_2 \cdot 2 \cdot \dfrac{\pi}{2} \cdot g^\infty$ as $h \to \infty$. Although the functions u_h converge to a jump function u, the graphs c_{u_h} converge to a curve c which closes the jump of u with a vertical segment of length 2π, see Figure 2, and we get in a further relaxed sense

$$\mathcal{F}(c) = \lambda_1 \cdot \mathcal{L}(c) + \lambda_2 \cdot g^\infty \cdot \mathrm{TC}(c)\,.$$

Gauss graphs. We make use of some features from the theory of *Gauss graphs* re-written in the context of Cartesian curves. Some ideas are therefore taken from [4], see also [6].

We thus recall that for a smooth rectifiable 1-1 curve with support $\mathcal{C} \subset \mathbb{R}^{N+1}_x$, the Gauss graph can be viewed as the graph in $\mathbb{R}^{N+1}_x \times \mathbb{S}^N$ of the unit tangent vector $\mathfrak{t}_\mathcal{C}(x) \in \mathbb{S}^N \subset \mathbb{R}^{N+1}_y$ at $x \in \mathcal{C}$, i.e.,

$$\mathcal{M}_\mathcal{C} := \{(x, \mathfrak{t}_\mathcal{C}(x)) \mid x \in \mathcal{C}\}\,,$$

and an i.m. rectifiable current is naturally associated to $\mathcal{M}_\mathcal{C}$. In the sequel we shall then denote by $(e_0, e_1, \ldots, e_N)$ and $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N)$ the canonical basis in $\mathbb{R}^{N+1}_x$ and $\mathbb{R}^{N+1}_y$, respectively.

Assume that c_u is a smooth Cartesian curve defined as the graph of a C^2-function $u : I \to \mathbb{R}^N$, so that $\mathcal{C} = \{(t, u(t)) \mid t \in I\}$ and $\mathfrak{t}_\mathcal{C}(x) = \tau_u(t)$ if $x = c_u(t)$. We thus introduce the map $\Phi_u : I \to \overline{U} \times \mathbb{S}^N$

$$\Phi_u(t) := (c_u(t), \tau_u(t))\,, \qquad t \in I$$

where

$$
\begin{aligned}
c_u(t) &:= (t,u(t)) = t\,e_0 + \sum_{j=1}^N u^j(t)\,e_j\,, \\
\tau_u(t) &:= \frac{1}{\sqrt{1+|\dot u(t)|^2}}\Bigl(\varepsilon_0 + \sum_{j=1}^N \dot u^j(t)\,\varepsilon_j\Bigr)\,.
\end{aligned}
\tag{1.11}
$$

Therefore, the Gauss graph associated to c_u is identified by

$$\mathcal{GG}_u := \{\Phi_u(t) \mid t\in I\}\,, \qquad \mathcal{GG}_u \subset \overline{U}\times\mathbb{S}^N.$$

Moreover, the set $\mathcal{GG}_u$ is the support of the curve Φ_u, it is 1-rectifiable and naturally oriented by the unit vector

$$\xi_u(t) := \frac{\dot\Phi_u(t)}{|\dot\Phi_u(t)|}\,. \tag{1.12}$$

Denoting by $\bullet$ the scalar product in $\mathbb{R}^N$, we compute for each $t\in I$

$$\dot\Phi_u(t) = e_0 + \sum_{j=1}^N \dot u^j\,e_j + \frac{-(\dot u\bullet\ddot u)}{(1+|\dot u|^2)^{3/2}}\,\varepsilon_0 + \sum_{j=1}^N \frac{\ddot u^j(1+|\dot u|^2)-\dot u^j\,(\dot u\bullet\ddot u)}{(1+|\dot u|^2)^{3/2}}\,\varepsilon_j\,. \tag{1.13}$$

In any codimension N we thus have

$$|\dot\Phi_u(t)| = |\dot c_u(t)|\,\sqrt{1+k_u(t)^2}\,, \qquad |\dot c_u(t)| = \sqrt{1+|\dot u(t)|^2}\,, \qquad |\dot c_u|k_u = |\dot\tau_u|\,. \tag{1.14}$$

where the curvature k_u of the Cartesian curve c_u is given by (1.3).

Notice that $2^{-1/2}(1+k_u)\le\sqrt{1+k_u^2}\le(1+k_u)$. This gives that

$$2^{-1/2}|\dot c_u|\,(1+k_u)\le|\dot\Phi_u|\le|\dot c_u|\,(1+k_u)\,. \tag{1.15}$$

In particular $\dot\Phi_u$ is summable in I if and only if both the functions $|\dot u|$ and $|\dot c_u|\,k_u$ are summable. We thus have for every $u\in C^2(I,\mathbb{R}^N)$

$$|\dot\Phi_u|\in L^1(I) \qquad\Longleftrightarrow\qquad \mathcal{E}_1(u)<\infty\,.$$

Example 1.4. The length of the Gauss graph is some sort of an average between the length and the total curvature. Let e.g. $N=2$ and $u_h:[0,2\pi/h]\to\mathbb{R}^2$ be given by $u_h(t)=R(\cos(ht),\sin(ht))$, so that the curve c_{u_h} parameterizes one turn of the helix of radius $R>0$ and step $2\pi/h$. The tantrix τ_{u_h} describes a circle in $\mathbb{S}^2_+$ of radius $R(h)=Rh/\sqrt{1+R^2h^2}$ that converges to one as $h\to\infty$. Moreover, the limit curve c_R is a circle of radius R and total curvature 2π. In fact, we have $\dot u_h\bullet\ddot u_h=0$ and

$$|\dot\Phi_{u_h}(t)| = \sqrt{1+R^2h^2}\,\sqrt{1+\frac{R^2h^4}{(1+R^2h^2)^2}} = \sqrt{\frac{1+2R^2h^2+(R^2+R^4)h^4}{1+R^2h^2}}$$

for every $t\in[0,2\pi/h]$, so that the limit

$$\lim_{h\to\infty}\int_0^{2\pi/h}|\dot\Phi_{u_h}(t)|\,dt = 2\pi\sqrt{1+R^2}$$

is equal to the length of the Gauss graph of the curve c_R. Since moreover $|\dot{c}_{u_h}|\, k_{u_h} = Rh^2/\sqrt{1+R^2h^2}$, then the limit of the total curvature functional gives

$$\lim_{h\to\infty} \mathrm{TC}(c_{u_h}) = \lim_{h\to\infty} \int_0^{2\pi/h} |\dot{c}_{u_h}|\, k_{u_h}\, dt = 2\pi$$

i.e. the total curvature of the limit curve c_R.

The codimension one case. In codimension $N = 1$, the relaxed functional (1.4) has been studied in [8], where the authors introduce the class $X(I)$ of real valued functions u in $\mathrm{BV}(I) = \mathrm{BV}(I, \mathbb{R})$ satisfying the following properties:

- (a) the function $t \mapsto \arctan(\dot{u}(t))$ belongs to $BV(I)$;
- (b) the positive and negative parts $(D^C u)^\pm$ of the Cantor-type component are respectively concentrated on the sets

$$Z^\pm[\dot{u}] := \left\{ t \in I \,:\, \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \dot{u}(s)\, ds = \pm\infty \right\}.$$

The class $X(I)$ trivially contains the Sobolev space $W^{2,1}(I)$. Therefore, a function $u \in SBV(I)$ with a finite Jump set, $\mathcal{H}^0(J_u) < \infty$, belongs to $X(I)$ if it is a pure Jump-function, i.e. $\dot{u} = 0$ a.e., or more generally if $\arctan(\dot{u}) \in BV(I)$. However, in [8] it is shown the existence of functions u in $X(I)$ with non-trivial Cantor component, $D^C u \neq 0$.

In [8] it is proved that $\mathcal{E}_1(I, \mathbb{R}) \subset X(I)$. Moreover, the explicit representation of the relaxed functional is given for $u \in X(I)$ by

$$\overline{\mathcal{E}}_1(u) = |Dc_u|(I) + \mathcal{G}(u)\,, \tag{1.16}$$

where the second term is:

$$\mathcal{G}(u) := |D\arctan(\dot{u})|(I \setminus J_u) + \sum_{t\in J_u} \Phi(\nu_u(t), \dot{u}(t_-), \dot{u}(t_+))\,. \tag{1.17}$$

For Sobolev functions $u \in W^{2,1}(I, \mathbb{R})$, the functional $\mathcal{G}(u)$ agrees with the total curvature functional

$$\int_{c_u} \mathbf{k}_{c_u}\, d\mathcal{H}^1 = \int_I \frac{|\ddot{u}(t)|}{1 + \dot{u}(t)^2}\, dt\,, \qquad c_u(t) = (t, u(t))\,.$$

In general, the second addendum in the definition of $\mathcal{G}(u)$ depends on the sign $\nu_u(t)$ of the jump $[u(t)] := u(t_+) - u(t_-)$ and on the left and right limits of $\dot{u}$ at the Jump point of u

$$\dot{u}(t_-) := \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} \dot{u}(s)\, ds\,, \qquad \dot{u}(t_+) := \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_{t}^{t+\varepsilon} \dot{u}(s)\, ds\,.$$

Such limits always exist in $\overline{\mathbb{R}}$ at all points $t \in I$ provided that $u \in X(I)$. In fact, compare [8], we have $\dot{u} = \tan v$, where v is a good representative of the BV-function $\arctan \dot{u}$. The general definition of $\Phi(\nu_u(t), \dot{u}(t_-), \dot{u}(t_+))$ from

[8], for the case of the curvature functional as in our context, agrees with the sum of the two turning angles

$$\arccos\Big(\frac{(1,\dot u(t_\pm))\bullet(0,\nu_u(t))}{|(1,\dot u(t_\pm))|}\Big)\,,\quad t\in J_u$$

provided that $\dot u(t_\pm)\in\mathbb{R}$, and with the obvious extensions if $|\dot u(t_\pm)|=\infty$, yielding to the corresponding terms 0 or π according to the sign of the product $\dot u(t_\pm)\,\nu_u(t)$.

Finally, notice that if $u\in X(I)$, so that $v:=\arctan\dot u\in \mathrm{BV}(I)$, setting $\tau_u:=(1,\dot u)/\sqrt{1+\dot u^2}$, then $\tau_u=(\cos v,\sin v)$, whence $\tau_u\in \mathrm{BV}(I,\mathbb{R}^2)$, and by the chain-rule formula one has $|D\tau_u|(A)=|Dv|(A)$ for each Borel set $A\subset I$.

BV-property of the Gauss map. Let now $u\in\mathcal{E}_1(I,\mathbb{R}^N)$ and let $\Phi_u(t):=(c_u(t),\tau_u(t))$ be defined a.e. as in the smooth case, see (1.11), but in terms of the approximate gradient $\dot u$ of the BV-function u. We already know that $c_u\in \mathrm{BV}(I,\overline{U})$. In [1] we proved in any codimension that also *the Gauss map* $\tau_u:I\to\mathbb{S}^N_+$ *is a function with bounded variation.*

Theorem 1.5. *Let $N\geq 1$ and $u\in\mathcal{E}_1(I,\mathbb{R}^N)$. Let $\{u_h\}\subset C^2(I,\mathbb{R}^N)$ such that $u_h\to u$ in L^1 and $\sup_h\mathcal{E}_1(u_h)<\infty$. Then we have:*

1. *the function $t\mapsto\Phi_u(t)$ belongs to $BV(I,\overline{U}\times\mathbb{S}^N)$;*
2. *possibly passing to a subsequence $\{\Phi_{u_h}\}$ converges weakly in the BV-sense to the function $\Phi_u(t)$;*
3. *by lower semicontinuity, $|D\Phi_u|(I)\leq\liminf_h\int_I|\dot\Phi_{u_h}(t)|\,dt$;*
4. *possibly passing to a subsequence $\dot u_h\to\dot u$ a.e. in I.*

Notice that property iv) is false if $\{u_h\}\subset C^2(I,\mathbb{R}^N)$ weakly converges only in the BV-sense, i.e., when the bound $\sup_h\int_I|\dot\tau_{u_h}(t)|\,dt<\infty$ on the total curvature of the Cartesian curves c_{u_h} is not satisfied. Taking e.g. $N=1$, $I=[0,2\pi]$, and $u_h(t):=\sin(ht)/h$, the sequence $\{u_h\}$ converges both weakly in the BV-sense and uniformly to the null function $u\equiv 0$, but we have $\int_0^{2\pi}|\dot\tau_{u_h}(t)|\,dt=h\cdot\pi$, and it is false that $\dot u_h(t)=\cos(ht)\to 0$ for a.e. $t\in[0,2\pi]$.

A geometric formula. In the case of continuous functions, by exploiting the geometric structure of the energy we obtained the following.

Theorem 1.6. *Let $u\in L^1(I,\mathbb{R}^N)$ be a continuous function. Then u has finite relaxed energy if and only if the Cartesian curve c_u has finite length and total curvature. In this case, moreover, the total variation $|D\tau_u|(I)$ agrees with the total curvature $\mathrm{TC}(c_u)$ of the Cartesian curve c_u. Finally, we have:*

$$\overline{\mathcal{E}}_1(u)=\mathcal{L}(C_u)+\mathrm{TC}(c_u)\,. \tag{1.18}$$

More generally, we denote by $\widetilde{c}_u$ the oriented curve obtained by connecting the jumps in the graph of u with oriented line segments from $c_u(t_-)$ to $c_u(t_+)$ at each point $t\in J_u$, so that its length is $\mathcal{L}(\widetilde{c}_u)=|Dc_u|(I)$. This time we deduce that a function $u \in L^1(I, \mathbb{R}^N)$ has finite relaxed energy if and only if the curve $\widetilde{c}_u$ has finite length and total curvature, i.e.

$$u \in \mathcal{E}(I, \mathbb{R}^N) \iff \mathcal{L}(\widetilde{c}_u) + \mathrm{TC}(\widetilde{c}_u) < \infty\,.$$

Finally, for every function $u \in \mathcal{E}_1(I, \mathbb{R}^N)$ we have:

$$\overline{\mathcal{E}}_1(u) = \mathcal{L}(\widetilde{c}_u) + \mathrm{TC}(\widetilde{c}_u)\,. \tag{1.19}$$

1.2. The elastic case

In [2] we analyzed the "elastic" case $p > 1$, where different phenomena appear, as we now briefly illustrate.

The superlinear growth of the curvature term implies that in the relaxation process the energy does not remain bounded if the curvature radius goes to zero at some point, see Example 4.2, as it instead happens in the plastic case. In codimension $N = 1$, one then obtains that a Cartesian curve with finite relaxed energy cannot have creases or edges, compare [8].

However, in the higher codimension case, corner points (only of a special type, say "vertical") are allowed. This phenomenon is illustrated in Example 4.4. Roughly speaking, there is sufficient freedom in the "vertical" directions in order that a sequence of smooth Cartesian curves approaches a curve with a corner point, by producing a "twist" in order to keep the curvature radius greater than some positive threshold, depending on $p > 1$.

Finding the optimal twist at some corner point seems to be a great challenge. This is due to the difficulties in solving the Euler equation satisfied by energy minimizing loops.

Notwithstanding, differently to the case $p = 1$, the set of corner points is always finite. This follows from the fact that the contribution of the relaxed energy at any corner point is at least $\pi/2$, as soon as the exponent $p > 1$, compare Theorem 4.5.

The Gauss map. If $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ for some $p > 1$, then the Gauss map τ_u has no Cantor part, i.e. $D^C \tau_u = 0$, hence it is a *special function of bounded variation.* Therefore, for continuous functions u with finite relaxed p-energy and no corner points, actually τ_u is a Sobolev function in $W^{1,1}(I, \mathbb{S}^N)$.

This fact is false in general in the plastic case $p = 1$, even in codimension $N = 1$. In fact, if $I = [0, 1]$ and $u(t) := \int_0^t v(s)\, ds$, where $v : I \to \mathbb{R}$ is the classical Cantor-Vitali function associated to the "middle thirds" Cantor set, one has $u \in C^1(I, \mathbb{R})$, the Gauss map $\tau_u \in \mathrm{BV}(I, \mathbb{S}^1)$ is continuous, whence $J_{\Phi_u} = \emptyset$, but $D^C \tau_u \neq 0$, as $D^C \tau_u = \dot{f}(v)\, D^C v$ with $f(t) = (1, t)/\sqrt{1 + t^2}$, whereas $\dot{\tau}_u = \dot{f}(v)\, \dot{v} = 0$. We get:

$$\overline{\mathcal{E}}_1(u) = \mathcal{L}^1(c_u) + \mathrm{TC}(c_u) < \infty\,, \quad \mathcal{L}^1(c_u) = \int_I \sqrt{1 + v^2}\, dt\,,$$
$$\mathrm{TC}(c_u) = |D^C \tau_u|(I) = \frac{\pi}{4}$$

whereas $\overline{\mathcal{E}}_p(u) = +\infty$ for every $p > 1$.

The p-curvature functional. We define the *p-curvature functional* of smooth Cartesian curves as

$$\mathrm{TC}_p(c_u) := \int_{c_u} \mathbf{k}_{c_u}^p \, d\mathcal{H}^1 , \quad p > 1 . \tag{1.20}$$

Actually, using that $|\dot{c}_u| \, k_u = |\dot{\tau}_u|$, by the area formula we get

$$\mathrm{TC}_p(c_u) = \int_I |\dot{c}_u|^{1-p} \, |\dot{\tau}_u|^p \, dt , \quad p > 1 . \tag{1.21}$$

We wish to extend our definition to the non-smooth case of continuous functions u with finite relaxed energy and with no corner points. In fact, it turns out that the arc-length parameterization $I_L \ni s \mapsto \gamma(s)$ of the curve c_u is a Sobolev function in $W^{2,p}$, where $I_L := [0, L]$ and $L = \mathcal{L}(c_u)$. We can thus define the p-curvature functional by means of (1.20), where the curvature of c_u at the point $\gamma(s)$ is given a.e. by

$$\mathbf{k}_{c_u}(\gamma(s)) := \frac{|\dot{\gamma}(s) \wedge \ddot{\gamma}(s)|}{|\dot{\gamma}(s)|^3} , \quad s \in I_L .$$

In fact, formula (1.21) continues to hold, and actually the p-curvature functional agrees with the integral of the p-power of the second derivative of the arc length parameterization:

$$\mathrm{TC}_p(c_u) = \int_{I_L} |\ddot{\gamma}(s)|^p \, ds < \infty .$$

Explicit formula. For continuous functions with finite relaxed energy and *no corner points*, we have

$$\overline{\mathcal{E}}_p(u) = \mathcal{L}(c_u) + \mathrm{TC}_p(c_u) . \tag{1.22}$$

The above expression extends to the wider class of continuous functions with finite relaxed energy. In fact, using that *the set* $J_{\dot{u}}$ *of corner points is finite* we introduce the *generalized p-curvature functional* by setting

$$\mathrm{TC}_p(c_u) := \int_I |\dot{c}_u|^{1-p} \, |\dot{\tau}_u|^p \, dt + \sum_{t \in J_{\dot{u}}} \mathcal{E}_p^0(\Gamma_t^p) .$$

In the above formula, roughly speaking, the energy contribution $\mathcal{E}_p^0(\Gamma_t^p)$ is the integral of the p-power of the curvature of the optimal "vertical" curve that allows to smoothly connect the two edges on the graph of u at the corner $c_u(t)$, i.e., the optimal twist that we previously described.

The above results are presented with some more detail in the following sections.

2. Gauss graphs of Cartesian curves

We formalize what we learned from the previous examples.

Cartesian currents. Returning to Example 1.3, we denote by G_{u_h} the 1-current in $(-1,1)\times\mathbb{R}$ carried by the graph of u_h, see (1.10), i.e. $G_{u_h} := c_{u_h\#}[\![-1,1]\!]$. It is easy to check that G_{u_h} weakly converges to the *Cartesian current* $T := c_\#[\![0,2(1+\pi)]\!]$, compare [10], given by the integration of 1-forms in $\mathcal{D}^1((-1,1)\times\mathbb{R})$ over the (oriented) limit curve $c:[0,2(1+\pi)]\to\mathbb{R}^2$

$$c(s) := \begin{cases} (s-1,0) & \text{if } 0\le s\le 1 \\ (0,s-1) & \text{if } 1<s<1+2\pi \\ (s-2\pi-1,2\pi) & \text{if } 1+2\pi\le s\le 2(1+\pi)\,. \end{cases} \tag{2.1}$$

The above computation suggests to extend the functional $\mathcal{F}$ in (1.7) to the corresponding class of 1-dimensional Cartesian currents in such a way that $\mathcal{F}(G_u)=\mathcal{F}(c_u)$ for currents G_u carried by the graph of smooth functions, and in our example

$$\mathcal{F}(T)=\lambda_1\cdot\mathbf{M}(T)+\lambda_2\cdot 2\cdot\frac{\pi}{2}\cdot g^\infty\,,\qquad \mathbf{M}(T)=\mathcal{L}(c)=2(1+\pi)\,. \tag{2.2}$$

This will be shown in Example 3.5 below.

Example 2.1. Let $I=[-\pi,3\pi]$ and consider the sequence of functions from I into $\mathbb{S}^1\subset\mathbb{R}^2$

$$u_h(t) := \begin{cases} (\cos ht,\sin ht) & \text{if } t\in\,[0,2\pi/h] \\ (1,0) & \text{elsewhere} \end{cases}$$

so that we have

$$\int_I |\dot u_h(t)|\,dt = 2\pi\,,\qquad \mathcal{L}(c_{u_h})=\int_I |\dot c_{u_h}(t)|\,dt = 2\pi\left(2-\frac{1}{h}+\frac{\sqrt{1+h^2}}{h}\right).$$

Moreover $u_h\rightharpoonup u_\infty$ weakly in the BV-sense, where $u_\infty(t)\equiv(1,0)$, but the degree satisfies

$$\deg\, u_h = 1\quad\forall\, h\,,\qquad \deg\, u_\infty=0$$

and the following gaps hold:

$$\begin{aligned} \int_I |\dot u_\infty(t)|\,dt = 0 \quad &< \quad 2\pi = \lim_{h\to\infty}\int_I |\dot u_h(t)|\,dt\,, \\ \mathcal{L}(c_{u_\infty})=4\pi \quad &< \quad 6\pi=\lim_{h\to\infty}\mathcal{L}(c_{u_h})\,, \end{aligned}$$

whence the weak BV convergence fails to preserve the geometry and to read the energy concentration.

On the other hand, the graphs G_{u_h} weakly converge to the Cartesian current $T=G_{u_\infty}+T^s$, where $G_{u_\infty}=[\![-\pi,3\pi]\!]\times\delta_{(1,0)}$ and the singular term $T^s=\delta_0\times[\![\mathbb{S}^1]\!]$ is a vertical 1-cycle. The total variation and degree can be defined on T in such a way that

$$\text{total variation } T=2\pi\,,\qquad \deg\, T=1\,,\qquad \mathbf{M}(T)=6\pi\,.$$

Therefore, one recovers concentration and loss of geometry from the limit of graphs.

Cartesian Gauss graphs. If $u : I \to \mathbb{R}^N$ is a C^2-function, recalling that $U = \mathring{I} \times \mathbb{R}^N$, we may associate to the Gauss graph of the Cartesian curve c_u a one-dimensional current $GG_u \in \mathcal{D}_1(U \times \mathbb{S}^N)$ defined by integrating 1-forms on the set $\mathcal{GG}_u$, which is naturally oriented by the unit vector ξ_u defined in (1.12). Then we have

$$\langle GG_u, \omega \rangle := \int_{\mathcal{GG}_u} \omega = \int_{\mathcal{GG}_u} \langle \omega, \xi_u \rangle \, d\mathcal{H}^1 , \qquad \omega \in \mathcal{D}^1(U \times \mathbb{S}^N)$$

and by the Remark 1.2 it turns out that $\operatorname{spt} GG_u \subset \overline{U} \times \mathbb{S}^N_+$.

Moreover, if $|\dot{\Phi}_u| \in L^1(I)$, by means of the area formula we compute

$$\langle GG_u, \omega \rangle = \int_{\mathcal{GG}_u} \langle \omega, \dot{\Phi}_u \rangle \, |\dot{\Phi}_u|^{-1} \, d\mathcal{H}^1 = \int_I \langle \omega(\Phi_u(t)), \dot{\Phi}_u(t) \rangle \, dt \, .$$

Therefore, it turns out that $GG_u = [\![\, \mathcal{GG}_u, 1, \xi_u \,]\!]$ is an i.m. rectifiable current in $\mathcal{R}_1(U \times \mathbb{S}^N)$ with null interior boundary, $\partial GG_u = 0$, and finite mass:

$$\mathbf{M}(GG_u) = \mathcal{H}^1(\mathcal{GG}_u) = \int_I |\dot{\Phi}_u(t)| \, dt < \infty \, , \qquad |\dot{\Phi}_u| = |\dot{c}_u| \sqrt{1 + k_u^2} \, .$$

By (1.15), for smooth functions $u : I \to \mathbb{R}^N$ we have

$$2^{-1/2} \mathcal{E}_1(u) \le \mathbf{M}(GG_u) \le \mathcal{E}_1(u) \tag{2.3}$$

and hence, *u has finite energy $\mathcal{E}_1$ if and only if the corresponding current GG_u has finite mass.*

The Gauss graph of Cartesian curves. We wish to extend the previous notation to the wider class of functions u in $\mathcal{E}_1(I, \mathbb{R}^N)$. Our definition relies on the fact that the Gauss map $\tau_u : I \to \mathbb{S}^N$ is a function of bounded variation, Theorem 1.5.

Recalling that $U := \mathring{I} \times \mathbb{R}^N$, we define a rectifiable 1-current GG_u in $U \times \mathbb{S}^N$ carried by the "Gauss graph" of u that has three components

$$GG_u = GG_u^a + GG_u^C + GG_u^J$$

the absolute continuous, Cantor, and Jump ones, respectively. It turns out that $GG_u^J = 0$ if u has a continuous representative, and that also $GG_u^C = 0$ if $u \in W^{1,1}(I, \mathbb{R}^N)$. The component GG_u^a is well-defined in terms of the approximate gradient of the BV-function Φ_u.

If $u \in \mathcal{E}_1(I, \mathbb{R}^N)$, by Theorem 1.5 it turns out that the *approximate gradient function* $t \mapsto \dot{\Phi}_u(t)$ is well-defined a.e. in I as in (1.13), where this time $\ddot{u}$ denotes the approximate gradient of $\dot{u}$, and $\dot{\Phi}_u \in L^1(I, \overline{U} \times \mathbb{R}^{N+1}_y)$. Moreover, taking good representatives of each component, the right and left limits $\Phi_u(t_\pm)$ exist at each point $t \in \mathring{I}$, with $\Phi_u(t_\pm) = (t, u(t_\pm), \tau_u(t_\pm))$, where $\tau_u = (\tau_u^0, \tau_u^1, \ldots, \tau_u^N)$.

The absolute continuous component. As in the smooth case, since $\dot{\Phi}_u$ is summable, see Theorem 1.5, we define the current $GG_u^a \in \mathcal{D}_1(U \times \mathbb{S}^N)$ by setting for each $\omega \in \mathcal{D}^1(U \times \mathbb{S}^N)$

$$\langle GG_u^a, \omega \rangle := \int_I \langle \omega(\Phi_u(t)), \dot{\Phi}_u(t) \rangle \, dt \, . \tag{2.4}$$

To our purposes, we compute $\langle \partial GG_u^a, f \rangle$ for any $f \in C_c^\infty(U \times \mathbb{S}^N)$. By the definition of boundary current we obtain:

$$\langle \partial GG_u^a, f \rangle := \langle GG_u^a, df \rangle = \int_I \langle df(\Phi_u(t)), \dot{\Phi}_u(t) \, dt \rangle = \int_I \nabla f(\Phi_u) \bullet \dot{\Phi}_u \, dt \, .$$

Moreover, the composition function $f \circ \Phi_u$ belongs to $BV(I)$, and since $f \in C_c^\infty(U \times \mathbb{S}^N)$ by the definition of distributional derivative we deduce that

$$\int_I D(f \circ \Phi_u) = 0 \, .$$

Also, choosing $\Phi_{u+}(t) = \Phi_u(t_+)$ as a precise representative, by the chain-rule formula we get

$$D(f \circ \Phi_u) = \nabla f(\Phi_u) \bullet \dot{\Phi}_u \, dt + \nabla f(\Phi_{u+}) \bullet D^C \Phi_u + \big(f(\Phi_{u+}) - f(\Phi_{u-}) \big) \, \mathcal{H}^0 \llcorner J_{\Phi_u} \, .$$

Therefore, we obtain that for each $f \in C_c^\infty(U \times \mathbb{S}^N)$

$$\langle \partial GG_u^a, f \rangle = - \int_I \nabla f(\Phi_{u+}) \bullet d D^C \Phi_u - \sum_{t \in J_{\Phi_u}} \big(f(\Phi_u(t_+)) - f(\Phi_u(t_-)) \big) \, . \tag{2.5}$$

The Cantor component. We have $D^C \Phi_u \bullet e_0 = 0$, $D^C \Phi_u \bullet e_j = D^C u^j$, and

$$D^C \Phi_u \bullet \varepsilon_0 = D^C \tau_u^0 \, , \qquad D^C \Phi_u \bullet \varepsilon_j = D^C \tau_u^j$$

for $j = 1, \ldots, N$. We define the Cantor component $GG_u^C \in \mathcal{D}_1(U \times \mathbb{S}^N)$ extending by linearity the action on basic forms. Denote by $(dx^0, dx^1, \ldots, dx^N)$ and $(dy^0, dy^1, \ldots, dy^N)$ the canonical dual bases of 1-forms in $\mathbb{R}_x^{N+1}$ and $\mathbb{R}_y^{N+1}$, respectively. For any $g \in C_c^\infty(U \times \mathbb{S}^N)$ we set:

1. $\langle GG_u^C, g(x,y) \, dx^0 \rangle := 0$
2. $\langle GG_u^C, g(x,y) \, dx^j \rangle := \int_I g(\Phi_{u+}) \, d D^C u^j \, , \; j = 1, \ldots, N$
3. $\langle GG_u^C, g(x,y) \, dy^0 \rangle := \int_I g(\Phi_{u+}) \, d D^C \tau_u^0$
4. $\langle GG_u^C, g(x,y) \, dy^j \rangle := \int_I g(\Phi_{u+}) \, d D^C \tau_u^j \, , \; j = 1, \ldots, N \, .$

Therefore, for each $f \in C_c^\infty(U \times \mathbb{S}^N)$ we clearly obtain

$$\langle \partial GG_u^C, f \rangle := \langle GG_u^C, df \rangle = \int_I \nabla f(\Phi_{u+}) \bullet d D^C \Phi_u \, . \tag{2.6}$$

The Jump component. For each Jump point $t \in J_u$, we denote by $\gamma_t(u)$ the oriented line segment in $U = \mathring{I} \times \mathbb{R}^N$ with initial point $c_u(t_-)$ and final point $c_u(t_+)$. Since $\gamma_t(u)$ is oriented by the unit vector

$$\Big(0, \frac{[u(t)]}{|[u(t)]|} \Big) \in \mathbb{S}_+^N \, , \qquad [u(t)] := (u_+(t) - u_-(t)) \in \mathbb{R}^N \setminus \{0\} \, ,$$

we correspondingly denote by $\widetilde{\gamma}_t(u)$ the oriented rectifiable arc in $U \times \mathbb{S}^N_+$ given by

$$\widetilde{\gamma}_t(u) := \Big(\gamma_t(u), \Big(0, \frac{[u(t)]}{|[u(t)]|}\Big)\Big)$$

and we set

$$\langle GG_u^J, \omega\rangle := \sum_{t\in J_u} \int_{\widetilde{\gamma}_t(u)} \omega\,, \qquad \omega \in \mathcal{D}^1(U \times \mathbb{S}^N)\,. \tag{2.7}$$

In particular, for each $f \in C_c^\infty(U \times \mathbb{S}^N)$ we obtain

$$\langle \partial GG_u^J, f\rangle := \langle GG_u^J, df\rangle = \sum_{t\in J_u} \int_{\widetilde{\gamma}_t(u)} df = \sum_{t\in J_u} \big(f(P_+(t)) - f(P_-(t))\big)\,, \tag{2.8}$$

where we have set

$$P_\pm(t) := \Big(t, u(t_\pm), 0, \frac{[u(t)]}{|[u(t)]|}\Big)\,, \qquad t \in J_u\,. \tag{2.9}$$

In conclusion, by the formulas (2.5), (2.6), and (2.8), we deduce that for any $f \in C_c^\infty(U \times \mathbb{S}^N)$

$$\begin{aligned} \langle \partial GG_u, f\rangle &= \langle GG_u^a, df\rangle + \langle GG_u^C, df\rangle + \langle GG_u^J, df\rangle \\ &= \sum_{t\in J_u} \big(f(P_+(t)) - f(P_-(t))\big) \\ &\qquad - \textstyle\sum_{t\in J_{\Phi_u}} \big(f(\Phi_u(t_+)) - f(\Phi_u(t_-))\big)\,. \end{aligned} \tag{2.10}$$

Closing the Gauss graph. There is a natural way to find a "vertical" current $S_u \in \mathcal{D}_1(U \times \mathbb{S}^N)$ such that the current

$$\Sigma_u := GG_u + S_u \tag{2.11}$$

is i.m. rectifiable in $\mathcal{R}_1(U \times \mathbb{S}^N)$ and has no interior boundary. Moreover, in the case $N = 1$ it turns out that the mass of Σ_u essentially agrees with the relaxed energy $\overline{\mathcal{E}}_1(u)$ as it is computed in [8].

The current S_u lives upon the Jump points t in $J_{\Phi_u} = J_u \cup J_{\dot u}$. It is given by two terms:

$$S_u = S_u^{Jc} + S_u^c$$

a "Jump-corner" component S_u^{Jc} that is concentrated upon the discontinuity points in J_u, and a "corner" component S_u^c that is concentrated upon the discontinuity points of the approximate gradient $\dot u$ where u is continuous, the so called "corner" points in $J_{\dot u} \setminus J_u$. Roughly speaking, the first component takes into account of the turning angles that appear when the "graph" of u meets a vertical part of the Cartesian current T_u, possibly giving rise to two corners at the points $(t, u_\pm(t))$, where one side of each corner is "vertical", since it follows the jump. The second component deals with the turning angles where u is continuous but $\dot u$ has a jump point.

In Figures 3 and 4 we illustrate an example in codimension $N = 2$ with occurrence of a jump-corner term. This is due to a jump point of both the graph function $c_u(t) = (t, u(t))$ and of its derivative. A crucial role is

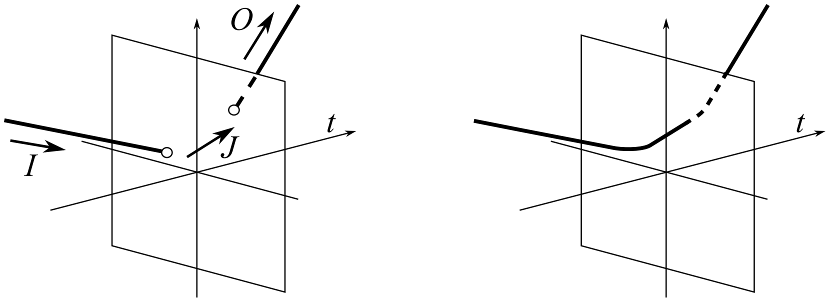

FIGURE 3. The codimension-two "curve" on the left has a jump-corner point at $t = 0$, with incoming, jump, and outgoing directions given by I, J, and O, respectively.
On the right: a codimension-two smooth approximating Cartesian curve.

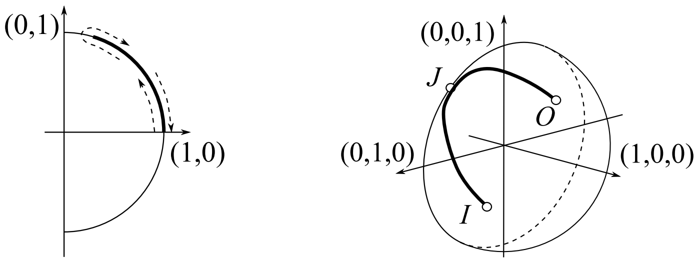

FIGURE 4. On the left: we revise the codimension-one curve in Example 1.3, drawing the image of the tantrix of the smooth approximation of c_{u_h}. It corresponds to the τ-projection of the curve on the right-hand side of Figure 2.
On the right: the image on the 2-sphere of the tantrix of the codimension-two smooth approximating Cartesian curve from Figure 3.

played by the incoming, jump, and outgoing directions, denoted by I, J, and O. The jump direction is determined by the last $N+1$ components in the formula (2.9). The incoming and outgoing directions are determined by the last $N+1$ components of $\Phi_u(t_-)$ and $\Phi_u(t_+)$, respectively, i.e. by the left and right limits $\tau_u(t_\pm)$ of the Gauss map.

A similar example with a corner term is readily obtained by gluing together the two line segments of the graph $c_u(t)$. In this case, only the incoming and outgoing directions $\tau_u(t_\pm)$ come into play.

The Jump-corner component. For each point $t \in J_u$, we denote by $\Gamma_t^\pm(u)$ an oriented geodesic arc in $\{c_u(t_\pm)\} \times \mathbb{S}_+^N$ with initial point $P_\pm(t)$, see (2.9), and final point $\Phi_u(t_\pm)$, and we set

$$\langle S_u^{Jc}, \omega\rangle := \sum_{t\in J_u}\Big(\int_{\Gamma_t^+(u)}\omega - \int_{\Gamma_t^-(u)}\omega\Big)\,, \qquad \omega\in\mathcal{D}^1(U\times\mathbb{S}^N)\,. \tag{2.12}$$

Therefore, for each $f\in C_c^\infty(U\times\mathbb{S}^N)$ we compute

$$\langle \partial S_u^{Jc}, f\rangle := \langle S_u^{Jc}, df\rangle = \sum_{t\in J_u}\big(f(\Phi_u(t_+)) - f(P_+(t)) - f(\Phi_u(t_-)) + f(P_-(t))\big)\,. \tag{2.13}$$

The corner component. Instead, for each point $t\in J_{\dot u}\setminus J_u$, we denote by $\Gamma_t(u)$ an oriented geodesic arc in $\{c_u(t)\}\times\mathbb{S}^N_+$ with initial point $\Phi_u(t_-)$ and final point $\Phi_u(t_+)$, and we set

$$\langle S_u^c, \omega\rangle := \sum_{t\in J_{\dot u}\setminus J_u}\int_{\Gamma_t(u)}\omega\,, \qquad \omega\in\mathcal{D}^1(U\times\mathbb{S}^N)\,.$$

This time for each $f\in C_c^\infty(U\times\mathbb{S}^N)$ we get

$$\langle \partial S_u^c, f\rangle := \langle S_u^c, df\rangle = \sum_{t\in J_{\dot u}\setminus J_u}\big(f(\Phi_u(t_+)) - f(\Phi_u(t_-))\big)\,. \tag{2.14}$$

Properties. The current $\Sigma_u\in\mathcal{D}_1(U\times\mathbb{S}^N)$ is supported in $\overline{U}\times\mathbb{S}^N_+$, and the null-boundary condition $\partial\Sigma_u = 0$ holds. In fact, by (2.10), (2.13), and (2.14), we check

$$\langle\partial\Sigma_u, f\rangle = \langle\partial GG_u, f\rangle + \langle\partial S_u^{Jc}, f\rangle + \langle\partial S_u^c, f\rangle = 0 \qquad \forall\, f\in C_c^\infty(U\times\mathbb{S}^N)\,.$$

Also, the mass of Σ_u decomposes as

$$\mathbf{M}(\Sigma_u) = \mathbf{M}(GG_u^a) + \mathbf{M}(GG_u^C) + \mathbf{M}(GG_u^J) + \mathbf{M}(S_u^{Jc}) + \mathbf{M}(S_u^c)\,. \tag{2.15}$$

More explicitly, the mass of the absolute continuous component is

$$\mathbf{M}(GG_u^a) = \int_I |\dot\Phi_u(t)|\,dt\,, \qquad |\dot\Phi_u| = |\dot c_u|\sqrt{1+k_u^2}\,.$$

As to the Cantor component, we have $\mathbf{M}(GG_u^C) = |D^C\Phi_u|(I)$, and hence

$$2^{-1/2}\big(|D^C u|(I) + |D^C\tau_u|(I)\big) \le \mathbf{M}(GG_u^C) \le |D^C u|(I) + |D^C\tau_u|(I)\,.$$

Moreover, for the other three components we compute

$$\begin{array}{rcl} \mathbf{M}(GG_u^J) &=& \sum_{t\in J_u}\mathcal{H}^1(\widetilde\gamma_t(u)) = \sum_{t\in J_u}|[c_u(t)]| = |D^J u|(I)\\ \mathbf{M}(S_u^{Jc}) &=& \sum_{t\in J_u}\big(\mathcal{H}^1(\Gamma_t^+(u)) + \mathcal{H}^1(\Gamma_t^-(u))\big)\\ \mathbf{M}(S_u^c) &=& \sum_{t\in J_{\dot u}\setminus J_u}\mathcal{H}^1(\Gamma_t(u))\,. \end{array} \tag{2.16}$$

Finally, and most importantly, it turns out that Σ_u *is an i.m. rectifiable current in* $\mathcal{R}_1(U\times\mathbb{S}^N)$, actually an integral 1-cycle in $U\times\mathbb{S}^N$.

3. The plastic case

On account of the bound (2.3), the object of our analysis is the subclass of 1-currents in $\mathcal{D}_1(U \times \mathbb{S}^N)$ that are *weak limits* (in the sense of currents) *of sequences* $\{GG_{u_h}\}$ *of Gauss graphs of smooth Cartesian curves with equi-bounded masses.*

We thus introduce the class $\mathrm{Gcart}(U \times \mathbb{S}^N)$, defined by

$$\begin{aligned}\mathrm{Gcart}(U \times \mathbb{S}^N) := \\ &\{\Sigma \in \mathcal{D}_1(U \times \mathbb{S}^N) \mid \exists\{u_h\} \subset C^2(I, \mathbb{R}^N) \text{ such that} \\ &GG_{u_h} \rightharpoonup \Sigma \text{ in } \mathcal{D}_1(U \times \mathbb{S}^N)\,,\ \sup_h \mathbf{M}(GG_{u_h}) < \infty\}\,.\end{aligned} \tag{3.1}$$

In the following structure theorem, we shall denote by Π_x and Π_y the canonical projections of $\mathbb{R}^{N+1}_x \times \mathbb{R}^{N+1}_y$ onto the first and second factor, respectively.

Theorem 3.1. *Let* $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^N)$ *and let* $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ *satisfy* $\sup_h \mathbf{M}(GG_{u_h}) < \infty$ *and* $GG_{u_h} \rightharpoonup \Sigma$ *weakly in* $\mathcal{D}_1(U \times \mathbb{S}^N)$*. Then we have:*

1. *the current* Σ *is* i.m. rectifiable *in* $\mathcal{R}_1(U \times \mathbb{S}^N)$*, with finite* mass
$$\mathbf{M}(\Sigma) \le \liminf_{h\to\infty} \mathbf{M}(GG_{u_h}) < \infty$$
and it satisfies the null-boundary condition $\partial\Sigma = 0$ *;*
2. *the sequence* $\{u_h\}$ *weakly converges in the* BV*-sense to some function* $u \in \mathrm{BV}(I, \mathbb{R}^N)$*, and actually* $u \in \mathcal{E}_1(I, \mathbb{R}^N)$ *;*
3. *the* projection $T = T(\Sigma) := \Pi_{x\#}\Sigma$ *is a Cartesian current in* $\mathrm{cart}(\mathring{I} \times \mathbb{R}^N)$*, with underlying function* u_T *equal to* u*, compare* [10] *;*
4. *the function* $t \mapsto \Phi_u(t)$ *belongs to* $BV(I, \overline{U} \times \mathbb{S}^N)$ *and it is equal to the weak* BV*-limit of the sequence* $\{\Phi_{u_h}\}$ *;*
5. *the current* Σ *decomposes as*
$$\Sigma = GG_u^a + GG_u^C + \widetilde{\Sigma} \tag{3.2}$$
where GG_u^a *and* GG_u^C *are the* absolute continuous *and the* Cantor *component of the current* GG_u *defined w.r.t. the limit function* u *;*
6. *the component* $\widetilde{\Sigma}$ *has support contained in* $\overline{U} \times \mathbb{S}^N_+$*, and it satisfies the* verticality condition
$$\langle \widetilde{\Sigma}, g(x,y)\, dx^0 \rangle = 0 \qquad \forall\, g \in C^\infty_c(U \times \mathbb{S}^N)$$
and the boundary condition
$$\partial\widetilde{\Sigma} = \sum_{t \in J_{\Phi_u}} \big(\delta_{\Phi_u(t_+)} - \delta_{(\Phi_u(t_-))}\big) \quad on\ C^\infty_c(U \times \mathbb{S}^N)\,;$$
7. *the following* decomposition in mass *holds:*
$$\widetilde{\Sigma} = \widehat{\Sigma} + \sum_{t \in J_{\Phi_u}} \Gamma_{t,\Sigma}\,, \qquad \mathbf{M}(\widetilde{\Sigma}) = \mathbf{M}(\widehat{\Sigma}) + \sum_{t \in J_{\Phi_u}} \mathbf{M}(\Gamma_{t,\Sigma})\,,$$
where the current $\widehat{\Sigma} \in \mathcal{R}_1(U \times \mathbb{S}^N)$ *satisfies the null-boundary condition* $\partial\widehat{\Sigma} = 0$*, and* $\Gamma_{t,\Sigma}$ *is for each* $t \in J_{\Phi_u}$ *an* a-cyclic *i.m. rectifiable current*

in $\mathcal{R}_1(U \times \mathbb{S}^N)$, *supported in* $\{t\} \times \mathbb{R}^N \times \mathbb{S}^N_+$, *and with boundary*

$$\partial \Gamma_{t,\Sigma} = \delta_{\Phi_u(t_+)} - \delta_{\Phi_u(t_-)} \,. \tag{3.3}$$

Example 3.2. We explicitly compute all the current Σ in a simple case. Referring to Example 1.3, we let Σ denote the weak limit of the sequence of Gauss graphs $\{GG_{v_h}\}$ corresponding to a "smoothing" $v_h : [-1,1] \to \mathbb{R}$ of the sequence $\{u_h\}$ from (1.10) at the corner points $(-\pi/h, 0)$ and $(\pi/h, 2\pi)$. The smoothing can be performed as in Example 1.1. By a diagonal argument, we may choose the smooth sequence $\{v_h\}$ in an optimal way, so that the total curvature of the Cartesian curve c_{v_h} is equal to $2 \arctan h$ for each h, and v_h agrees with u_h outside two small intervals centered at the points $\pm\pi/h$, in such a way that $\|v_h - u_h\|_\infty \to 0$ as $h \to \infty$.

The limit curve c has length $L = 2(1+\pi)$ and arc-length parameterization given by (2.1), whence

$$\dot c(s) := \begin{cases} (1,0) & \text{if } 0 \le s < 1 \\ (0,1) & \text{if } 1 < s < 1 + 2\pi \\ (1,0) & \text{if } 1 + 2\pi < s < 2(1+\pi)\,. \end{cases}$$

Therefore, denoting $I_L := [0, L]$, the image current $(c, \dot c)_\# [\![I_L]\!]$ agrees with the Gauss graph current GG_u, where $u : [-1,1] \to \mathbb{R}$ is the weak limit BV-function

$$u(t) := \begin{cases} 0 & \text{if } t < 0 \\ 2\pi & \text{if } t > 0\,. \end{cases} \tag{3.4}$$

More precisely, we have $GG_u = GG_u^a + GG_u^C + GG_u^J$, where the absolute continuous component is given by (2.4) with $I = [-1,1]$ and

$$\Phi_u(t) = \begin{cases} (t, 0, 1, 0) & \text{if } t < 0 \\ (t, 2\pi, 1, 0) & \text{if } t > 0 \end{cases}$$

i.e. $GG_u^a = \Phi_{u\#} [\![I]\!]$; the Cantor component is zero, $GG_u^C = 0$, and the Jump component GG_u^J, according to (2.7), agrees with the integration on the oriented line segment in $\{0\} \times \mathbb{R} \times \mathbb{S}^1$ with initial point $(0,0,0,1)$ and final point $(0, 2\pi, 0, 1)$. In particular, since $J_{\Phi_u} = J_u = \{0\}$, in accordance with (2.10) and (2.9) we have

$$\partial (c, \dot c)_\# [\![I_L]\!] = \partial GG_u = -\big(\delta_{\Phi_u(0_+)} - \delta_{\Phi_u(0_-)}\big) + \big(\delta_{P_+(0)} - \delta_{P_-(0)}\big)$$

on $C^\infty_c(U \times \mathbb{S}^1)$, where

$$\begin{aligned} \Phi_u(0_+) &= (0, 2\pi, 1, 0)\,, \quad \Phi_u(0_-) = (0,0,1,0)\,, \\ P_+(0) &= (0, 2\pi, 0, 1)\,, \quad P_-(0) = (0,0,0,1)\,. \end{aligned}$$

Moreover, by our optimal choice of the sequence $\{v_h\}$, it turns out that the weak limit current Σ of the sequence $\{GG_{v_h}\}$ agrees with the current Σ_u. This means that $\Sigma = \Sigma_u = GG_u + S_u^{Jc} + S_u^c$, where the corner component $S_u^c = 0$, since there are no points in $J_{\dot u} \setminus J_u$, and the Jump-corner component S_u^{Jc}, according to (2.12), is given by

$$S_u^{Jc} = [\![\Gamma_0^+(u)]\!] - [\![\Gamma_0^-(u)]\!]\,, \tag{3.5}$$

where $\Gamma_0^{\pm}(u)$ is the oriented geodesic arc in $\{c_u(0_\pm)\}\times\mathbb{S}^1_+$ with initial point $P_\pm(0)$ and final point $\Phi_u(0_\pm)$. We thus have

$$\mathbf{M}(\Sigma_u)=\mathbf{M}(GG_u^a)+\mathbf{M}(GG_u^J)+\mathbf{M}(S_u^{J_c}),$$

where

$$\mathbf{M}(GG_u^a)=2\,,\quad \mathbf{M}(GG_u^J)=|Du|(I)=2\pi\,,\quad \mathbf{M}(S_u^{J_c})=2\cdot\frac{\pi}{2}\,. \tag{3.6}$$

Geometry of Gauss graphs. A current Σ in $\mathrm{Gcart}(U\times\mathbb{S}^N)$ preserves the geometry of Gauss graphs: indeed, for a regular Gauss graph (c_u,τ_u) the second component τ_u is a normalization of the tangent vector to the first component. Now, when the first component of the tangent vector to Σ at a point $z=(x,y)\in U\times\mathbb{S}^N_+$ is non zero, then it has to be parallel (and with the same verse) to the second component y of the point z, see (3.7) below.

More precisely, there is a Lipschitz-continuous function $\Psi\in\mathrm{Lip}(\widetilde{I},\overline{U}\times\mathbb{S}^N)$ defined on some closed interval $\widetilde{I}$ such that the image current $\Psi_\#[\![\widetilde{I}]\!]$ agrees with Σ. We thus get:

Theorem 3.3. *For a.e. $s\in\widetilde{I}$ such that $|\Pi_x(\dot\Psi(s))|\neq 0$, we have*

$$\frac{\Pi_x(\dot\Psi(s))}{|\Pi_x(\dot\Psi(s))|}=\Pi_y(\Psi(s))\in\mathbb{S}^N_+\,. \tag{3.7}$$

The energy functional on currents. In order to define the energy functional on the class $\mathrm{Gcart}(U\times\mathbb{S}^N)$, we remark that all these currents are of the type $S=[\![\mathcal{M},\theta,\xi]\!]$, i.e.

$$\langle S,\omega\rangle=\int_{\mathcal{M}}\langle\omega,\xi\rangle\,\theta\,d\mathcal{H}^1\qquad\forall\,\omega\in\mathcal{D}^1(U\times\mathbb{S}^N)\,,$$

where $\mathcal{M}$ is a countably 1-rectifiable set, ξ is the orienting unit vector and θ is the integer-valued non-negative multiplicity function, so that $\mathbf{M}(S)=\int_{\mathcal{M}}\theta\,d\mathcal{H}^1$. The unit vector ξ in $\mathbb{R}^{N+1}_x\times\mathbb{R}^{N+1}_y$ orienting $\mathcal{M}$ at $\mathcal{H}^1\llcorner\mathcal{M}$-a.e. point can be decomposed as $\xi=(\xi^{(x)},\xi^{(y)})$, where $\xi^{(x)}:=\Pi_x(\xi)$ and $\xi^{(y)}:=\Pi_y(\xi)$.

Definition 3.4. *For any current $S=[\![\mathcal{M},\theta,\xi]\!]$ we let*

$$\mathcal{E}_1^0(S):=\int_{\mathcal{M}}\theta\left(|\xi^{(x)}|+|\xi^{(y)}|\right)d\mathcal{H}^1\,.$$

The following properties hold:

1. (SMOOTH MAPS) If $S=GG_u$ for some smooth function $u\in C^2(I,\mathbb{R}^N)$, then $\mathcal{E}_1^0(GG_u)=\mathcal{E}_1(u)$.
2. (LOWER SEMICONTINUITY) If $\{u_h\}\subset C^2(I,\mathbb{R}^N)$ is such that $GG_{u_h}\rightharpoonup\Sigma$ weakly in $\mathcal{D}_1(U\times\mathbb{S}^N)$ to some $\Sigma\in\mathrm{Gcart}(U\times\mathbb{S}^N)$, then $\mathcal{E}_1^0(\Sigma)\leq\liminf_h\mathcal{E}_1(u_h)$.
3. (ENERGY DECOMPOSITION) If $\Sigma\in\mathrm{Gcart}(U\times\mathbb{S}^N)$ decomposes as in (3.2), then

$$\mathcal{E}_1^0(\Sigma)=\int_I|\dot c_u|(1+k_u)\,dt+|D^Cu|(I)+|D^C\tau_u|(I)+\mathcal{E}_1^0(\widetilde{\Sigma})\,.$$

If Σ is the Gauss graph Σ_u of some function u in $\mathcal{E}_1(I,\mathbb{R}^N)$, the third term $\widetilde{\Sigma}=\widetilde{\Sigma}_u$ in the formula (3.2) is the sum of the Jump, Jump corner and corner components. We thus have the energy splitting:

$$\widetilde{\Sigma}_u = GG_u^J + S_u^{Jc} + S_u^c\,, \quad \mathcal{E}_1^0(\widetilde{\Sigma}_u) = \mathbf{M}(GG_u^J) + \mathbf{M}(S_u^c) + \mathbf{M}(S_u^{Jc})\,. \tag{3.8}$$

From the mass estimates after (2.15), see also (1.15), we deduce that

$$2^{-1/2}\,\mathcal{E}_1^0(\Sigma_u) \le \mathbf{M}(\Sigma_u) \le \mathcal{E}_1^0(\Sigma_u)$$

and hence for every $u\in\mathcal{E}_1(I,\mathbb{R}^N)$ we have

$$\mathbf{M}(\Sigma_u)<\infty \iff \mathcal{E}_1^0(\Sigma_u)<\infty\,.$$

Finally, the following equivalent formula for the energy (3.8) holds:

$$\mathcal{E}_1^0(\Sigma_u) = |Dc_u|(I) + |D\tau_u|(I\setminus J_u) + \mathbf{M}(S_u^{Jc}) \qquad \forall\, u\in\mathcal{E}_1(I,\mathbb{R}^N)\,,$$

where the *jump-corner* term $\mathbf{M}(S_u^{Jc})$ is given by formula (2.16).

Example 3.5. Returning to Example 3.2, that refers to Example 1.3, we recall that $\Sigma_u = GG_u + S_u^{Jc}$, where $u:[-1,1]\to\mathbb{R}$ is the piecewise constant function in (3.4), and the Jump-corner component S_u^{Jc} is defined by (3.5), i.e. it is the sum of two oriented arcs in $\{c_u(0_\pm)\}\times\mathbb{S}^1_+$ both of length $\pi/2$. By using (3.6) and (3.8), we thus obtain

$$\mathcal{E}_1^0(\Sigma_u) = 2(1+\pi) + 2\cdot\frac{\pi}{2}\,,$$

so that the expected formula from (2.2) holds, as with our choices $\lambda_1=\lambda_2=1$ and $g^\infty=1$.

The case of codimension one. Now, when $N=1$ we have

$$|\dot c_u(t)|\,k_u(t) = |\dot v(t)|\,, \quad |D\tau_u|(I) = |Dv|(I)\,, \tag{3.9}$$

where $v:=\arctan\dot u\in\mathrm{BV}(I)$. Moreover, comparing formula (2.16) for the mass of the jump-corner component S_u^{Jc} with the explicit computation from [8] for the last addendum in (1.17), in the case of the curvature functional, we readily check that

$$\mathbf{M}(S_u^{Jc}) = \sum_{t\in J_u}\Phi(\nu_u(t),\dot u(t_-),\dot u(t_+))\,.$$

Therefore, the representation formula from [8] yields that for every $u\in\mathcal{E}_1(I,\mathbb{R})$

$$\overline{\mathcal{E}}_1(u) = \mathcal{E}_1^0(\Sigma_u)\,. \tag{3.10}$$

Energy bounds. Actually, formula (3.10) holds true for every $u\in\mathcal{E}_1(I,\mathbb{R}^N)$ in higher codimension $N\ge 1$, too. More precisely, for any $\Sigma\in\mathrm{Gcart}(U\times\mathbb{S}^N)$, we shall denote by u_Σ the function $u\in\mathrm{BV}(I,\mathbb{R}^N)$ for which decomposition (3.2) holds. Correspondingly, we define

$$\mathrm{Gcart}_u := \{\Sigma\in\mathrm{Gcart}(U\times\mathbb{S}^N)\mid u_\Sigma=u\}\,, \qquad u\in\mathrm{BV}(I,\mathbb{R}^N) \tag{3.11}$$

so that by lower semicontinuity and closure-compactness we infer:

$$\forall\, u \in \mathrm{BV}(I,\mathbb{R}^N), \quad u \in \mathcal{E}_1(I,\mathbb{R}^N) \iff \mathrm{Gcart}_u \neq \emptyset\,.$$

Since by lower semicontinuity we have

$$\overline{\mathcal{E}}_1(u) \geq \inf\{\mathcal{E}_1^0(\Sigma) \mid \Sigma \in \mathrm{Gcart}_u\}\,,$$

the energy lower bound in (3.10) holds if $\mathcal{E}_1^0(\Sigma) \geq \mathcal{E}_1^0(\Sigma_u)$ for every $\Sigma \in \mathrm{Gcart}_u$. This inequality is proved by exploiting the geometric property from Theorem 3.3, and by using the following average formula proved in [12, Prop. 4.1], that goes back to Fáry [7]:

Proposition 3.6. *Given a curve c in $\mathbb{R}^n$, and some fixed integer $1 \leq k < n$, then*

$$\mathrm{TC}(c) = \int_{G_k\mathbb{R}^n} \mathrm{TC}(\pi_p(c))\, d\mu_k(p)\,.$$

Here we have denoted by $G_k\mathbb{R}^n$ the Grassmannian of k-planes in $\mathbb{R}^n$, by μ_k the corresponding Haar measure, and by π_p the orthogonal projection of $\mathbb{R}^n$ onto $p \in G_k\mathbb{R}^n$,

Finally, the energy upper bound in (3.10) is obtained by means of the following density result.

Theorem 3.7. *For every $u \in \mathcal{E}_1(I,\mathbb{R}^N)$, there exists a sequence of smooth functions $\{u_h\} \subset C^2(I,\mathbb{R}^N)$ such that $u_h \to u$ strongly in L^1, $GG_{u_h} \rightharpoonup \Sigma_u$ weakly in $\mathcal{D}_1(U \times \mathbb{S}^N)$ and $\mathcal{E}_1(u_h) \to \mathcal{E}_1^0(\Sigma_u)$ as $h \to \infty$.*

In conclusion, by (3.10) we thus deduce

$$\forall\, u \in \mathcal{E}_1(I,\mathbb{R}^N)\,, \qquad \overline{\mathcal{E}}_1(u) = |Dc_u|(I) + |D\tau_u|(I \setminus J_u) + \mathbf{M}(S_u^{Jc}) \tag{3.12}$$

where the jump-corner term $\mathbf{M}(S_u^{Jc})$ is given by (2.16). For continuous function, we also get the geometric formula stated in Theorem 1.6. More generally, formula (1.19) follows by observing that

$$|D\tau_u|(I \setminus J_u) + \mathbf{M}(S_u^{Jc}) = \mathrm{TC}(\widetilde{c}_u)\,.$$

4. The elastic case

We now deal with the elastic case $p > 1$.

p-curvature functional on currents. If $u \in C^2(I,\mathbb{R}^N)$, setting

$$\mathcal{E}_p^0(GG_u) := \int_{\mathcal{G}\mathcal{G}_u} |\xi_u^{(x)}|^{1-p}\big(|\xi_u^{(x)}|^p + |\xi_u^{(y)}|^p\big)\, d\mathcal{H}^1\,,$$

where

$$|\xi_u^{(x)}| = \frac{|\dot{c}_u|}{|\dot{\Phi}_u|}\,, \qquad |\xi_u^{(y)}| = \frac{|\dot{\tau}_u|}{|\dot{\Phi}_u|}\,, \qquad |\dot{\tau}_u| = |\dot{c}_u|\, k_u\,,$$

by the area formula we have

$$\mathcal{E}_p^0(GG_u) = \int_I |\dot{\Phi}_u| \Big(\frac{|\dot{c}_u|}{|\dot{\Phi}_u|} + \frac{|\dot{c}_u|\, k_u^p}{|\dot{\Phi}_u|}\Big)\, dt = \int_I |\dot{c}_u|\big(1 + k_u^p\big)\, dt = \mathcal{E}_p(u)\,. \tag{4.1}$$

This suggests to introduce for smooth functions $(u,v) : I \to \mathbb{R}^N \times \mathbb{R}^{N+1}$, with $|v| \equiv 1$, the energy

$$\mathcal{F}_p(u,v) := \int_I f_p(\dot{c}_u, \dot{v})\, dt\,, \qquad f_p(\dot{c}_u, \dot{v}) := |\dot{c}_u|^{1-p} \left(|\dot{c}_u|^p + |\dot{v}|^p \right)$$

so that $\mathcal{F}_p(u, \tau_u) = \mathcal{E}_p(u)$, and to *denote by* $F_p : \mathbb{R}_x^{N+1} \times \mathbb{R}_y^{N+1} \to [0, +\infty]$ *the parametric convex l.s.c. extension of the integrand* f_p in the sense of [10]. Denoting by $\xi^0 \in \mathbb{R}$ the first component of ξ, for every $p > 1$ one has:

$$F_p(\xi) = \begin{cases} |\xi^{(x)}| + |\xi^{(x)}|^{1-p} |\xi^{(y)}|^p & \text{if} \quad |\xi^{(x)}| > 0 \text{ and } \xi^0 \geq 0 \\ +\infty & \text{otherwise}. \end{cases} \tag{4.2}$$

Definition 4.1. The *p-curvature energy functional* is given on currents $\Sigma = [\![\mathcal{M}, \theta, \xi]\!]$ by

$$\mathcal{E}_p^0(\Sigma) := \int_{\mathcal{M}} F_p(\xi)\, \theta\, d\mathcal{H}^1\,.$$

Therefore, in (4.1) we have just shown that $\mathcal{E}_p^0(GG_u) = \mathcal{E}_p(u)$ in the smooth case.

The following *regularity property* holds: every current $\Sigma = [\![\mathcal{M}, \theta, \xi]\!]$ with finite p-energy has finite mass, too. More precisely, if $\Sigma = [\![\mathcal{M}, \theta, \xi]\!]$, then $\mathbf{M}(\Sigma) \leq 2 \cdot \mathcal{E}_p^0(\Sigma)$ for all exponents $p > 1$.

We recall that Σ has finite mass if and only if it has finite plastic energy $\mathcal{E}_1^0(\Sigma)$. This equivalence fails to hold in the elastic case. In fact, as the following example shows, it may happen that a sequence $\{\Sigma_h\}$ has equibounded masses but $\sup_h \mathcal{E}_p^0(\Sigma_h) = \infty$ for every $p > 1$.

Example 4.2. Let $I = [-1, 1]$ and $N = 1$. For $0 < R < 1$, let $\gamma_R : [-1, 1 + \pi/2] \to U \times \mathbb{S}^1$ given by

$$\gamma_R(\theta) := \begin{cases} (\theta, -R, 1, 0) & \text{if } -1 \leq \theta \leq 0 \\ (R\sin\theta, -R\cos\theta, \cos\theta, \sin\theta) & \text{if } 0 \leq \theta \leq \frac{\pi}{2} \\ (R, \theta - \pi/2, 0, 1) & \text{if } \frac{\pi}{2} \leq \theta \leq 1 + \frac{\pi}{2} \end{cases}$$

and define

$$\Sigma_R := \gamma_{R\#} [\![-1, 1 + \pi/2]\!] \in \mathcal{R}_1(U \times \mathbb{S}^1)\,,$$

so that $\Sigma_R = [\![\mathcal{M}_R, 1, \xi_R]\!]$ where $\mathcal{M}_R = \gamma_R([-1, 1 + \pi/2])$ and $\xi_R(z) = \dot{\gamma}_R(\theta)/|\dot{\gamma}_R(\theta)|$ if $z = \gamma_R(\theta)$. Since

$$\dot{\gamma}_R(\theta) = \begin{cases} (1, 0, 0, 0) & \text{if } -1 \leq \theta < 0 \\ (R\cos\theta, R\sin\theta, -\sin\theta, \cos\theta) & \text{if } 0 < \theta < \frac{\pi}{2} \\ (0, 1, 0, 0) & \text{if } \frac{\pi}{2} < \theta \leq 1 + \frac{\pi}{2} \end{cases}$$

we compute

$$|\dot{\gamma}_R(\theta)| = \begin{cases} 1 & \text{if } -1 \leq \theta < 0 \\ \sqrt{1 + R^2} & \text{if } 0 < \theta < \frac{\pi}{2} \\ 1 & \text{if } \frac{\pi}{2} < \theta \leq 1 + \frac{\pi}{2} \end{cases}$$

and hence $|\xi_R^{(x)}| > 0$ on $\mathcal{M}_R$, as

$$\xi_R(\gamma_R(\theta)) = \begin{cases} (1,0,0,0) & \text{if } -1 \le \theta < 0 \\ \dfrac{1}{\sqrt{1+R^2}}(R\cos\theta, R\sin\theta, -\sin\theta, \cos\theta) & \text{if } 0 < \theta < \frac{\pi}{2} \\ (0,1,0,0) & \text{if } \frac{\pi}{2} < \theta \le 1 + \frac{\pi}{2}\,. \end{cases}$$

Now, the mass of Σ_R is equal to the length of the simple curve γ_R, and by the area formula

$$\mathbf{M}(\Sigma_R) = \int_{[-1,1+\pi/2]} |\dot\gamma_R(\theta)|\, d\theta = 2 + \frac{\pi}{2}\sqrt{1+R^2}\,.$$

As before, we let $\gamma_R^{(x)} := \Pi_x \circ \gamma_R$ and $\gamma_R^{(y)} := \Pi_x \circ \gamma_R$, so that $\gamma_R = (\gamma_R^{(x)}, \gamma_R^{(y)})$. Using that

$$|\dot\gamma_R^{(x)}(\theta)| = \begin{cases} 1 & \text{if } -1 \le \theta < 0 \\ R & \text{if } 0 < \theta < \frac{\pi}{2} \\ 1 & \text{if } \frac{\pi}{2} < \theta \le 1 + \frac{\pi}{2} \end{cases} \qquad |\dot\gamma_R^{(y)}(\theta)| = \begin{cases} 0 & \text{if } -1 \le \theta < 0 \\ 1 & \text{if } 0 < \theta < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < \theta \le 1 + \frac{\pi}{2} \end{cases}$$

for every $p > 1$, by (4.2) and by the area formula we compute

$$\begin{aligned} \mathcal{E}_p^0(\Sigma_R) &= \int_{[-1,1+\pi/2]} |\dot\gamma_R(\theta)| \Big(\frac{|\dot\gamma_R^{(x)}(\theta)|}{|\dot\gamma_R(\theta)|} + \frac{|\dot\gamma_R^{(x)}(\theta)|^{1-p}}{|\dot\gamma_R(\theta)|^{1-p}} \cdot \frac{|\dot\gamma_R^{(x)}(\theta)|^p}{|\dot\gamma_R(\theta)|^p} \Big)\, d\theta \\ &= \int_{[-1,1+\pi/2]} \big(|\dot\gamma_R^{(x)}(\theta)| + |\dot\gamma_R^{(x)}(\theta)|^{1-p} |\dot\gamma_R^{(y)}(\theta)|^p \big)\, d\theta \end{aligned}$$

where

$$\int_{[-1,1+\pi/2]} |\dot\gamma_R^{(x)}(\theta)|\, d\theta = 2 + \frac{\pi}{2}\, R$$

and

$$\int_{[-1,1+\pi/2]} |\dot\gamma_R^{(x)}(\theta)|^{1-p} |\dot\gamma_R^{(y)}(\theta)|^p\, d\theta = \frac{\pi}{2}\, R^{1-p}\,,$$

whence

$$\mathcal{E}_p^0(\Sigma_R) = 2 + \frac{\pi}{2}\, R + \frac{\pi}{2}\, R^{1-p}\,.$$

Similarly, one has

$$\mathcal{E}_1^0(\Sigma_R) = 2 + \frac{\pi}{2}\,(R+1)\,,$$

whence we recover the energy estimates

$$2^{-1/2}\, \mathcal{E}_1^0(\Sigma_R) \le \mathbf{M}(\Sigma_R) \le \mathcal{E}_1^0(\Sigma_R)\,, \qquad 2 \cdot \mathcal{E}_p^0(\Sigma_R) \ge \mathcal{E}_1^0(\Sigma_R) \quad \forall\, p > 1\,.$$

In particular, we have as $R \to 0^+$

$$\mathbf{M}(\Sigma_R) \to 2 + \frac{\pi}{2}\,, \qquad \mathcal{E}_1^0(\Sigma_R) \to 2 + \frac{\pi}{2}\,, \qquad \mathcal{E}_p^0(\Sigma_R) \to +\infty \quad \forall\, p > 1\,.$$

The cited example is given by $\Sigma_h = \Sigma_{R_h}$ for a sequence of radii $R_h \searrow 0$.

Minimal currents. In order to obtain a similar formula to (3.12) in the case $p>1$, for every $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ we now introduce an optimal current Σ^p_u of the type :

$$\Sigma^p_u := GG^a_u + GG^C_u + S^p_u\,, \tag{4.3}$$

where the third component $S^p_u\in\mathcal{R}_1(U\times\mathbb{S}^N)$ is given by

$$S^p_u := \sum_{t\in J_{\Phi_u}} \Gamma^p_t$$

for suitable minimal currents Γ^p_t that we now describe.

For any point $t\in J_{\Phi_u}$ we first introduce a suitable class $\mathcal{F}(u,t)$ of a-cyclic i.m. rectifiable currents Γ in $\mathcal{R}_1(U\times\mathbb{S}^N)$, supported in $\{t\}\times\mathbb{R}^N\times\mathbb{S}^N_+$, with boundary

$$\partial\Gamma = \delta_{\Phi_u(t_+)} - \delta_{\Phi_u(t_-)}$$

and such that $\mathcal{E}^0_p(\Gamma)<\infty$. By Federer's structure theorem [9, 4.2.25], for any such Γ we find a Lipschitz function $\gamma=\gamma_\Gamma : [-1/2,1/2]\to U\times\mathbb{S}^N$ with constant velocity $|\dot\gamma(s)| = \mathbf{M}(\Gamma)$ for a.e. s, and $\gamma_\#[\![-1/2,1/2]\!]=\Gamma$. Therefore, we deduce that $|\dot\gamma^{(x)}(s)|>0$ a.e., and as in Example 4.2 we get

$$\mathcal{E}^0_p(\Gamma) = \int_{[-1/2,1/2]} \big(|\dot\gamma^{(x)}(s)| + |\dot\gamma^{(x)}(s)|^{1-p}|\dot\gamma^{(y)}(s)|^p\big)\,ds\,. \tag{4.4}$$

According to the geometric property (3.7), *we say that* $\Gamma\in\mathcal{F}(u,t)$ *if in addition one has*

$$\frac{\dot\gamma^{(x)}(s)}{|\dot\gamma^{(x)}(s)|} = \gamma^{(y)}(s) \qquad \text{for a.e. } s\in[-1/2,1/2]\,, \qquad \gamma=\gamma_\Gamma\,. \tag{4.5}$$

Remark 4.3. Let $\Gamma\in\mathcal{F}(u,t)$ for some $t\in J_{\Phi_u}$. By property (4.5) we deduce that the second component $\gamma^{(y)} := \Pi_y(\gamma_\Gamma)$ is the tantrix of the first component $\gamma^{(x)} := \Pi_x(\gamma_\Gamma)$, and that the curve $\gamma^{(x)}$ has positive length, as $|\dot\gamma^{(x)}(s)|>0$ a.e. Therefore, formula (4.4) yields that the p-energy of Γ is equal to the *p-curvature functional* of the curve $\Pi_x(\gamma_\Gamma)$, i.e.

$$\mathcal{E}^0_p(\Gamma) = \int_c (1+k^p_c)\,d\mathcal{H}^1\,, \qquad c:=\Pi_x(\gamma_\Gamma)\,. \tag{4.6}$$

Finally, notice that the vertical cycle $\widetilde\Sigma$ defined in Example 4.4 below satisfies the above property (4.5), whence $\widetilde\Sigma\in\mathcal{F}(u,t)$, with $t=0\in J_{\dot u}$.

Actually, it turns out that *for every* $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ *and* $t\in J_{\Phi_u}$*, the minimum of the problem*

$$\inf\{\mathcal{E}^0_p(\Gamma)\mid\Gamma\in\mathcal{F}(u,t)\}$$

is attained. For each $t\in J_{\Phi_u}$, we thus denote by Γ^p_t *a minimum point for the p-energy* $\mathcal{E}^0_p$ *in the class* $\mathcal{F}(u,t)$.

In particular, Γ^p_t is an a-cyclic current in $\mathcal{R}_1(U\times\mathbb{S}^N)$. From our definition (4.3) we thus obtain:

$$\mathcal{E}^0_p(\Sigma^p_u) = \mathcal{E}^0_p(GG^a_u)+\mathcal{E}^0_p(GG^C_u)+\mathcal{E}^0_p(S^p_u)\,, \qquad \mathcal{E}^0_p(S^p_u) = \sum_{t\in J_{\Phi_u}} \mathcal{E}^0_p(\Gamma^p_t)\,. \tag{4.7}$$

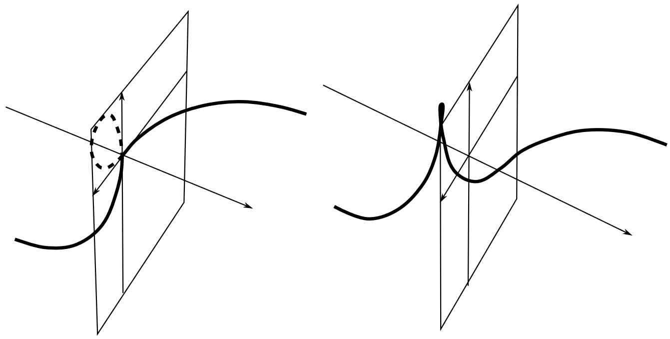

FIGURE 5. To the left, the curve c_u and (dashed) the current $\widetilde{\Sigma}$ which lies in the vertical plane $\{t=0\}$; to the right, one term of the smooth approximating sequence c_{u_h}.

As a consequence, we readily obtain the following *energy lower bound*:

$$\forall\, u \in \mathcal{E}_p(I,\mathbb{R}^N)\,, \quad \overline{\mathcal{E}}_p(u) \geq \mathcal{E}^0_p(GG^a_u) + \mathcal{E}^0_p(GG^C_u) + \mathcal{E}^0_p(S^p_u)\,.$$

Corner points. If $u \in \mathcal{E}_p(I,\mathbb{R}^N)$ for some $p>1$, then $\tau^0_u(t_\pm)=0$ for every $t \in J_{\Phi_u}$. This property of the Gauss map is false in the case $p=1$, compare Remark 4.6 below.

Note that condition $\tau^0_u(t_\pm)=0$ is equivalent to $|\dot u(t_\pm)|=+\infty$. Therefore, in case of codimension $N=1$, one infers that $\dot u(t_+)=\dot u(t_-)\in\{\pm\infty\}$ for each $t\in J_{\Phi_u}$, otherwise one would obtain as above that $\mathcal{E}^0_p(\Gamma_{t,\Sigma})=+\infty$, a contradiction. This implies that $J_{\dot u}=\emptyset$ and hence that the corner component $\Sigma^c=0$ if $N=1$, as already shown in [8].

Now, Example 4.2 may suggest the absence of corner points for Cartesian curves c_u of continuous functions $u\in\mathcal{E}_p(I,\mathbb{R}^N)$, when $p>1$. We now see that this is not the case in high codimension $N\geq 2$.

Example 4.4. For $I=[-1,1]$ and $N=2$, consider the continuous $W^{1,1}$-function $u: I\to\mathbb{R}^2$ given by

$$u(t) := \begin{cases} (0, -\sqrt{-t^2-2t}) & \text{if } t \leq 0 \\ (\sqrt{-t^2+2t}, 0) & \text{if } t \geq 0 \end{cases}$$

so that the graph curve c_u is the union of two quarters of unit circles meeting at the point $0_{\mathbb{R}^3}$, centered at the points $(-1,0,0)$ and $(1,0,0)$ and lying in the hyperplanes $x_1=0$ and $x_2=0$, respectively. Since $\dot c_u(0_-)=(1,0,+\infty)$ and $\dot c_u(0_+)=(1,+\infty,0)$, we have $\tau_u(0_-)=(0,0,1)$ and $\tau_u(0_+)=(0,1,0)$, thus a corner point with turning angle equal to $\pi/2$ appears at the point $0_{\mathbb{R}^3}$, whence $J_{\dot u}\setminus J_u=\{0\}$.

It is not difficult to check that $u\in\mathcal{E}_p(I,\mathbb{R}^2)$ for each $p>1$. In fact, one may approximate u by a smooth sequence $\{u_h\}$ such that the Gauss graphs GG_{u_h} weakly converge to a current Σ in $\mathrm{Gcart}(U\times\mathbb{S}^2)$ given by $\Sigma = GG^a_u + \widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is the 1-current integration of a smooth curve

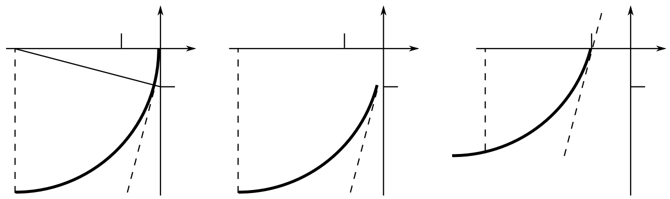

FIGURE 6. The small notches on the t and x_2 axes are $1/h$ away from the origin; the angles lines thus have slopes $1/h$ and h.

in $\{0\} \times \mathbb{R}^2 \times \mathbb{S}^1_0$, see (1.9), with end points $\Phi_u(0_\pm) = (0_{\mathbb{R}^3}, \tau_u(0_\pm))$ and satisfying the geometric property (3.7). Since $\sup_h \mathcal{E}_p(u_h) < \infty$, we deduce that $u \in \mathcal{E}_p(I, \mathbb{R}^2)$.

When $p = 1$, one simply has to smoothen the angle by means of an arc with small curvature radius. For $p > 1$, instead, good approximations (i.e., with small energy contribution) are performed by means of vertical arcs which have curvature radius greater than a positive constant, depending on p, see Figure 5 on the left.

Differently to the case $p = 1$, the vertical current $\widetilde{\Sigma}$ is not supported in $\{c_u(0)\} \times \mathbb{S}^1_0$, compare (1.9). An example is $\widetilde{\Sigma} := \gamma_\# [\![\pi/2, \pi]\!]$, where $\gamma : [\pi/2, \pi] \to \mathbb{R}^3_x \times \mathbb{R}^3_y$ is the regular curve defined in components $\gamma^{(x)} := \Pi_x \circ \gamma$ and $\gamma^{(y)} := \Pi_y \circ \gamma$ by $\gamma^{(x)}(\theta) := (0, \gamma_x(\theta))$ and $\gamma^{(y)}(\theta) := (0, \gamma_y(\theta))$, where

$$\gamma_x(\theta) := \left(-\sin\theta\cos^2\theta, -\cos\theta\sin^2\theta\right), \qquad \gamma_y(\theta) := \frac{\dot\gamma^{(x)}(\theta)}{|\dot\gamma^{(x)}(\theta)|},$$

so that the geometric property (3.7) holds true. More explicitly, one has

$$\gamma^{(y)}(\theta) = \frac{\left(\cos\theta(3\sin^2\theta - 1), \sin\theta(1 - 3\cos^2\theta)\right)}{(3\sin^4\theta - 3\sin^2\theta + 1)^{1/2}} \qquad \forall\, \theta \in [\pi/2, \pi]$$

whence $\gamma(\pi/2) = (0_{\mathbb{R}^3}, \tau_u(0_-)) = \Phi_u(0_-)$ and $\gamma(\pi) = (0_{\mathbb{R}^3}, \tau_u(0_+)) = \Phi_u(0_+)$, and according to (3.3)

$$\partial\widetilde{\Sigma} = \partial\gamma_\# [\![\pi/2, \pi]\!] = \delta_{\gamma(\pi)} - \delta_{\gamma(\pi/2)} = \delta_{\Phi_u(0_+)} - \delta_{\Phi_u(0_-)}.$$

An approximating sequence $u_h : [-1, 1] \to \mathbb{R}^2$ satisfying $\sup_h \mathcal{E}_p(u_h) < \infty$ is defined by widening the base of the vertical arc so that the loop is made on the interval $-1/h \le t \le 1/h$; the resulting arc does no longer begin and end vertically, but with slope h, so we cannot simply glue the two parts of the graph of u to it (conveniently separated); the problem is easily solved by cutting away a suitable (and small) terminal portion of each arc and translating it so that the three pieces fit, see Figure 6. The resulting curve is defined on a very slightly larger interval than $[-1, 1]$ but a reparameterization does the trick, see Figure 5 on the right. Finally, a smoothing and a diagonal argument yield the existence of a sequence $\{\widetilde{u}_h\}_h \subset C^1(I, \mathbb{R}^N)$ with $\sup_h \mathcal{E}_p(\widetilde{u}_h) < \infty$ and $\widetilde{u}_h \to u$ strongly in L^1.

Now, *for every* $p>1$, *the set* $J_{\dot u}\setminus J_u$ *of corner points of a function* $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ *with finite relaxed energy is always finite.* This is a consequence of the following

Theorem 4.5. *Let* $u\in\mathcal{E}_p(I,\mathbb{R}^N)$ *for some* $p>1$. *For every* $t\in J_{\dot u}\setminus J_u$ *we have* $\mathcal{E}^0_p(\Gamma^p_t)\geq\pi/2$.

Remark 4.6. Theorem 4.5 fails to hold in the case $p=1$. It suffices to consider a piecewise affine and continuous function $u:[0,1]\to\mathbb{R}$ with a countable set of corner points $J_{\dot u}=\{t_j:=1-2^{-j}\mid j\in\mathbb{N}^+\}$ such that (setting $t_0=0$) the slope of u at the interval $]t_{j-1},t_j[$ is equal to 2^{-j} for every j. In fact, the length of the Cartesian curve c_u is lower than 2, and its total curvature is equal to $\pi/4$, whence $u\in\mathcal{E}_1(I,\mathbb{R})$. Notice that we have $\overline{\mathcal{E}}_p(u)=+\infty$ for every $p>1$.

Discontinuity points. In general a function with finite relaxed energy may have a non-trivial jump set J_u. This was already observed in [8] when $N=1$, and we give here an example for $N=2$.

Example 4.7. Similarly to Example 4.4, consider the function $u:[-1,1]\to\mathbb{R}^2$ given by

$$u(t):=\begin{cases}(0,-\sqrt{-t^2-2t}) & \text{if } t\leq 0\\ (\sqrt{-t^2+2t},3) & \text{if } t>0\end{cases}$$

so that u has a jump point at the origin, as $u(0_-)=(0,0)$ and $u(0_+)=(0,3)$. This time the graph of u is the union of two quarters of unit circles centered at the points $(-1,0,0)$ and $(1,1,3)$ and lying in the hyperplanes $x_1=0$ and $x_2=3$, respectively. We again have $\dot c_u(0_-)=(1,0,+\infty)$ and $\dot c_u(0_+)=(1,+\infty,0)$, so that $\tau_u(0_-)=(0,0,1)$ and $\tau_u(0_+)=(0,1,0)$. Again, it is not difficult to check that $u\in\mathcal{E}_p(I,\mathbb{R}^2)$ for each $p>1$. In fact, one may approximate u by a smooth sequence $\{u_h\}$ such that the Gauss graphs GG_{u_h} weakly converge to a current Σ in $\mathrm{Gcart}(U\times\mathbb{S}^2)$ given by $\Sigma=GG^a_u+\widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is the 1-current integration of a smooth curve in $\{0\}\times\mathbb{R}^2\times\mathbb{S}^1_0$ with end points $\Phi_u(0_-)=((0,0,0),(0,0,1))$ and $\Phi_u(0_+)=((0,0,3),(0,1,0))$ and satisfying the geometric property (3.7). Since $\sup_h\mathcal{E}_p(u_h)<\infty$, we deduce again that $u\in\mathcal{E}_p(I,\mathbb{R}^2)$.

An explicit formula for $\widetilde{\Sigma}$ can be obtained by slightly modifying the definition of γ from Example 4.4, this time defining a regular curve $\gamma:[\pi/2,\pi]\to\mathbb{R}^3_x\times\mathbb{R}^3_y$ such that

$$\gamma_y(\theta)=\frac{\dot\gamma^{(x)}(\theta)}{|\dot\gamma^{(x)}(\theta)|}\qquad\forall\,\theta\in[\pi/2,\pi]$$

whereas

$$\gamma(\pi/2)=((0,0,0),\tau_u(0_-))=\Phi_u(0_-),\quad \gamma(\pi)=((0,0,3),\tau_u(0_+))=\Phi_u(0_+)\,.$$

Thus the geometric property (3.7) is satisfied and according to (3.3) we again have $\partial\widetilde{\Sigma}=\delta_{\Phi_u(0_+)}-\delta_{\Phi_u(0_-)}$.

Note that the lower bound in Theorem 4.5 is false if $t \in J_u$, i.e. on the Jump component Σ_u^J. As a consequence, in general the Jump set J_u of a function $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ is countable, for any $p > 1$. We sketch here an example in codimension $N = 2$.

Consider $u : [0,1] \to \mathbb{R}^2$ whose components u^j are increasing and bounded functions that are discontinuous on the countable set $J_u = \{t_i = 1 - 2^{-i} \mid i \in \mathbb{N}^+\}$ and smooth outside J_u, in such a way that the integral $\int_I |\dot c_u|^{p-1}(|\dot c_u|^p + |\tau_u|^p)\, dt$ is finite, the series $\sum_i |u(t_i+) - u(t_i-)|$ is convergent, and u is a BV-function with no Cantor-part, $|D^C u|(I) = 0$. We define u with unbounded right and left derivative at the discontinuity set, i.e. $|\dot u(t_i\pm)| = +\infty$ for each i, and so that the Gauss map $\tau_u : I \to \mathbb{S}^1$ is a BV-function with no Cantor part, $D^C \tau_u = 0$, and discontinuity set $J_{\tau_u} = J_u$, i.e. u has no corner points. Then $\tau_u(t_i\pm) \in \mathbb{S}^1_0$ for each i. Denoting by d_i the geodesic distance between $\tau_u(t_i-)$ and $\tau_u(t_i+)$ in $\mathbb{S}^1_0$, we can define u in such a way that the series $\sum_i d_i$ is convergent. Finally, denoting by $\Gamma^p_{t_i}$ a minimum point for $\mathcal{E}^0_p$ in the class $\mathcal{F}(u, t_i)$, we can also require that

$$\mathcal{E}^0_p(\Gamma^p_{t_i}) \le c_p\,(|u(t_i+) - u(t_i-)| + d_i) \qquad \forall\, i$$

for some constant $c_p > 0$ not depending on i. By our relaxation result, we have

$$\overline{\mathcal{E}}_p(u) = \int_I |\dot c_u|^{1-p}\left(|\dot c_u|^p + |\dot\tau_u|^p\right) dt + \sum_{i=1}^{\infty} \mathcal{E}^0_p(\Gamma^p_{t_i}) < \infty$$

and hence $u \in \mathcal{E}_p(I, \mathbb{R}^2)$, but $\mathcal{H}^0(J_u) = +\infty$.

Energy upper bound. It follows from our density result:

Theorem 4.8. *Let $p > 1$ and $N \ge 1$. For every $u \in \mathcal{E}_p(I, \mathbb{R}^N)$, there exists a sequence of smooth functions $\{u_h\} \subset C^2(I, \mathbb{R}^N)$ such that $u_h \to u$ strongly in L^1, $GG_{u_h} \rightharpoonup \Sigma^p_u$ weakly in $\mathcal{D}_1(U \times \mathbb{S}^N)$ and $\mathcal{E}_p(u_h) \to \mathcal{E}^0_p(\Sigma^p_u)$ as $h \to \infty$.*

Recall that Gcart_u denotes the subclass of currents in $\mathrm{Gcart}(U \times \mathbb{S}^N)$ such that $u_\Sigma = u$, see (3.1) and (3.11). For any function $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ we denote by Gcart^p_u the class of currents Σ in Gcart_u such that $\mathcal{E}^0_p(\Sigma) < \infty$. For every $p > 1$ and $u \in L^1(I, \mathbb{R}^N)$, we thus have:

$$u \in \mathcal{E}_p(I, \mathbb{R}^N) \iff \mathrm{Gcart}^p_u \ne \emptyset\,.$$

Moreover, if $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ we obtain

$$\overline{\mathcal{E}}_p(u) = \min\{\mathcal{E}^0_p(\Sigma) \mid \Sigma \in \mathrm{Gcart}^p_u\}$$

where, we recall, the Gauss map τ_u is a function of bounded variation in $\mathrm{BV}(I, \mathbb{S}^N)$ whose derivative has no Cantor part, i.e., $D^C \tau_u = 0$. More explicitly, for every $u \in \mathcal{E}_p(I, \mathbb{R}^N)$ we infer:

$$\overline{\mathcal{E}}_p(u) = \int_I |\dot c_u|^{1-p}\left(|\dot c_u|^p + |\dot\tau_u|^p\right) dt + |D^C u|(I) + \sum_{t \in J_{\Phi_u}} \mathcal{E}^0_p(\Gamma^p_t)$$

where Γ_t^p denotes for every $t \in J_{\Phi_u}$ a minimum point for $\mathcal{E}_p^0$ in the class $\mathcal{F}(u,t)$.

The lack of precise knowledge of the minimal arcs Γ_t^p prevents further explicitation of the relaxed energy $\overline{\mathcal{E}}_p(u)$, which is instead possible when u is continuous if we assume that u has no corner points, so that no such arcs appear, compare (1.22). Finally, recall that the absence of corner points is always true in codimension $N=1$, when $p>1$, as already observed in [8].

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