REGULARITY RESULTS FOR A CLASS OF QUASICONVEX FUNCTIONALS WITH NONSTANDARD GROWTH

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ABSTRACT. We consider the integral functional $\int f(x, Du)\, dx$ under non standard growth assumptions of (p, q)-type: namely, we assume that

$$|z|^{p(x)} \leq f(x, z) \leq L(1 + |z|^{p(x)})$$

for some function $p(x) > 1$, a condition appearing in several models from mathematical physics. Under sharp assumptions on the continuous function $p(x)$ we prove partial regularity of minimizers in the vector-valued case $u : \mathbb{R}^n \to \mathbb{R}^N$, allowing for quasiconvex energy densities. This is, to our knowledge, the first regularity theorem for quasiconvex functionals under non standard growth conditions.

1. INTRODUCTION.

Over the last thirty years great attention was reserved to the study of regularity of minimizers of integral functionals of the Calculus of Variations of the type

$$\mathcal{F}(u, \Omega) := \int_\Omega f(x, Du)\ dx\ , \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $f : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Carathéodory integrand and $u \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^N)$. The interest of the topic also rises from the fact that functionals of the type (1.1) naturally represent energies in the context of many problems coming from mathematical physics. Until some years ago it was customary to study existence and regularity of minimizers of functionals as (1.1) under the assumption of p-growth:

$$|z|^p \leq f(x, z) \leq L(1 + |z|^p)\ , \quad p > 1\ . \tag{1.2}$$

A regularity theory (quite satisfying in the scalar case $N = 1$) has been developed for such integrals through various contributions from many authors (see for instance [Gia],[Giu]).
Anyway recent models arising in various branches of mathematical phsysics suggest that (1.2) could to be too restrictive. This is the case, for instance, of the energy associated with the electrorheological fluids: in a recent work (see [RR]) Rajagopal and Růžička propose a model for this class of fluids of non-Newtonian type which are characterized by the property of drastically changing their mechanical properties when in presence of an electromagnetic field $\mathbf{E}(x)$. According to [RR], their stationary flow is characterized by a set of equations which are similar to the modified

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Navier-Stokes system introduced in the sixties by O. Ladyzhenskaya:

$$-\operatorname{div} S(x, \mathcal{E}(v)) = g(x, v, Dv) \;, \qquad \operatorname{div} v = 0$$

where v is the velocity of the fluid, $\mathcal{E}(v)$ is the symmetric part of the gradient Dv and the "extra stress" tensor S satisfies standard monotonicity conditions in the Leray-Lions fashion but with non constant growth in the sense that

$$D^2 S(x, z) \geq \nu(1 + |z|^2)^{(p(x)-2)/2} \mathrm{Id} \;,$$

where $p(x) \equiv p(|\mathbf{E}|^2)$ is a function of $\mathbf{E}$ which is given (see [RR], [R1], [R2]). Moreover models of this type arise for fluids whose viscosity is influenced in a similar way by temperature (see [Z3]). In another context, the differential system modelling the thermistor problem (see [Z1-3]) includes equations like

$$-\operatorname{div} (p(x)|Du|^{p(x)-2} Du) = 0 \;.$$

The natural common energy associated to these problems is modeled by

$$\int_\Omega (\mu^2 + |Du|^2)^{p(x)/2} \, dx \;, \tag{1.3}$$

$\mu \geq 0$, which clearly violates (1.2) but that, instead, meets the more general assumption

$$|z|^p \leq f(x, z) \leq L(1 + |z|^q) \;, \qquad q > p > 1 \;. \tag{1.4}$$

This condition, which is commonly referred to as (p, q)-growth, was introduced in [M1] by Marcellini, who subsequently gave important regularity results for minimizers of functionals with nonstandard growth (see [M1-5]). The regularity theory for such functionals was later studied by many authors (see the references in [M4-5] and [ELM]).

The aim of this paper is to study regularity of minimizers for a class of functionals which lie in an intermediate position between (1.2) and (1.4) and whose model case is (1.3), that is we consider energy densities satisfying the following assumption of $p(x)$ growth:

$$|z|^{p(x)} \leq f(x, z) \leq L(1 + |z|^{p(x)}) \;, \tag{1.5}$$

where $p(x) > 1$ is a continuous function.

The regularity theory for functionals satisfying (1.5) was started by Zhikov, and in the scalar case $N = 1$ has been carried out in the papers [Z1-3],[FZ],[A],[AM1],[CC] under suitable convexity hypotheses on f and mild regularity assumptions on $p(x)$. In this paper we concentrate on the vector-valued case $N > 1$ and we prove partial regularity of minimizers.

As it is well known, in the scalar case a natural assumption in order to guarantee existence and regularity of minimizers is convexity of f. In the vector-valued case this condition is not necessary anymore since, as shown by Morrey (see also [AF1]), in order to ensure lower semicontinuity of an integral functional the weaker assumption of quasiconvexity suffices. Subsequently the hypothesis of quasiconvexity was shown to be the appropriate substitute for convexity in order to obtain regularity of minimizers (see [E], [AF2], [CFM]). Here we consider quasiconvex functionals with $p(x)$ growth and we prove partial $C^{1,\alpha}$-regularity of minimizers, that is Hölder continuity of the gradient in an open subset of full measure, provided the function $p(x)$

is Hölder continuous. Our result covers functionals such as (with $p(x) \geq n = N$)

$$\int_{\Omega} [(1+|Du|^2)^{p(x)/2} + (1+|\det Du|^2)^{p(x)/2n}]\, dx \, .$$

We hope that our methods will be helpful in order to treat also more concrete and precise models arising from mathematical physics; we would like to stress that the difficulties to be overcome are not only due to the nonstandard growth (1.5) in itself but also to the presence in the integrand f of the variable x, which, in contrast to the standard case (1.2), is not easy to handle under nonstandard growth conditions, especially when trying to obtain sharp results. Moreover (see remark 2.2 below) our methods allow to cover more general integrals of the type

$$\int_{\Omega} f(x, u, Du)\, dx \, .$$

Most of these difficulties are present already in the convex case, which happens more frequently in the applications (we note that all the convex vector-valued cases under non standard growth considered in the literature, such as [AF3],[FS],[EM],[BF], are concerned with an integrand depending only on the gradient and the methods used do not seem to extend immediately to the case in which f is allowed to contain also (x, u) or even only x). Moreover we remark that our motivations do not come only from the applicative aspects of the problem but are also theoretical: indeed the regularity theory of functionals with (p, q)-growth has not developed enough to cover quasiconvex integrals and, although we set ourselves in a particular situation, that is the one of $p(x)$-growth, this is, up to our knowledge, the first partial regularity result valid for quasiconvex integrals with (p, q)-growth. We recall that the problem of regularity of minimizers of quasiconvex functionals with (p, q)-growth was raised by Marcellini in [M5].

Finally we spend some words about our techniques. In order to treat condition (1.5), we have to push hard all the technical tools developed (see [AF2],[CFM]) to treat quasiconvexity under assumption (1.2), proceeding through a strong revisitation of the blow-up arguments leading to partial regularity in the traditional p-growth case. We employ a suitable localization of the usual blow-up procedure (see [EG]) in the sense that we blow-up a minimizer u not in the whole Ω but in small open subsets constructed in such a way that both Du satisfies certain average bounds and $p(x)$ has suitably small variations. Nevertheless we present some technical simplifications, even with respect to the p-growth case, that allow us to avoid the use of the selection and extension (via maximal functions) lemmas employed for instance in [AF2] and [CFM] (see the proof of theorem 2.1, step 4).

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2. Notation and statement

In the sequel Ω will denote an open bounded domain in $\mathbb{R}^n$, and $B(x, R)$ the open ball $\{y \in \mathbb{R}^n : \ |x - y| < R\}$. If u is an integrable function defined on $B(x, R)$

we will set

$$(u)_{x,R} = -\!\!\!\!\!\!\int_{B(x,R)} u(x)\, dx = \frac{1}{\omega_n R^n} \int_{B(x,R)} u(x)\, dx \ ,$$

where ω_n is the Lebesgue measure of $B(0,1)$. We shall also adopt the convention of writing B_R and $(u)_R$ instead of $B(x,R)$ and $(u)_{x,R}$ respectively, when the center will not be relevant, or it is clear from the context; moreover, unless otherwise stated, all balls considered will have the same center. Finally, the letter c will freely denote a constant, not necessarily the same in any two occurencies, while only the relevant dependeces will be highlighted.

We are going to deal with the integral functional

$$\mathcal{F}(u,\Omega) = \int_\Omega f(x, Du)\, dx \ , \tag{2.1}$$

defined on $W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$. The Carathéodory function $f : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ will be supposed to satisfy a growth condition of the following type:

$$L^{-1}|z|^{p(x)} \le f(x,z) \le L(1+|z|^{p(x)}) \tag{2.2}$$

for any $z \in \mathbb{R}^{nN}, x \in \Omega$, where $p : \Omega \to (1,+\infty)$ is a continuous function and $L \ge 1$. With this type of nonstandard growth condition we adopt the following notion of (local) minimizer:

Definition 2.1. *We say that a function $u \in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ if $|Du|^{p(x)} \in L^1_{\rm loc}(\Omega)$ and*

$$\int_{{\rm spt}\varphi} f(x,Du)\, dx \le \int_{{\rm spt}\varphi} f(x, Du + D\varphi)\, dx$$

for any $\varphi \in W^{1,1}_0(\Omega;\mathbb{R}^N)$ with compact support in Ω.

Besides (2.2), we shall consider the following ellipticity and continuity conditions:

$$\begin{aligned} &\int_{Q_1} [f(x_0, z_0 + D\varphi) - f(x_0,z_0)]\, dx \\ &\qquad \ge L^{-1} \int_{Q_1} (1+|z_0|^2 + |D\varphi|^2)^{(p(x_0)-2)/2} |D\varphi|^2\, dx \end{aligned} \tag{2.3}$$

for each $z_0 \in \mathbb{R}^n,\ x_0 \in \Omega$ and each $\varphi \in C^\infty_0(Q_1)$ where $Q_1 = (0,1)^n$, and

$$\begin{aligned} &|f(x,z) - f(x_0,z)| \\ &\le L\omega(|x-x_0|)\big((1+|z|^2)^{p(x)/2} + (1+|z|^2)^{p(x_0)/2}\big)\big[1+\log(1+|z|^2)\big] \end{aligned} \tag{2.4}$$

for any $z_0 \in \mathbb{R}^n,\ x, x_0 \in \Omega$ and where $L \ge 1$; here $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function vanishing at zero which represents the modulus of continuity of $p(x)$:

$$|p(x) - p(y)| \le \omega(|x-y|) \ ,$$

thus (2.4) is modeled on the functional (1.3) for $\mu > 0$.

Remark 2.1. We observe that our regularity result needs no other growth assumptions, in particular on the second derivatives of the function f. We recall that if (2.2) holds then (2.3) implies (see e.g. [AF2]) the following growth property for Df (when it exists):

$$|Df(x_0,z)| \le c(1+|z|)^{p(x_0)-1}$$

with $c \equiv c(L,\gamma_1,\gamma_2)$, for any $z \in \mathbb{R}^{nN}$ and $x_0 \in \Omega$.

Our main result is the following:

Theorem 2.1. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ where f is a function of class C^2 with respect to the variable z satisfying* (2.2)–(2.4) *with*

$$\omega(R) \leq LR^{\alpha} \tag{2.5}$$

for some $0 < \alpha \leq 1$ and for any $R \leq 1$. Then there exists an open subset $\Omega_0 \subset \Omega$ such that $|\Omega \setminus \Omega_0| = 0$ and that Du is locally Hölder continuous in Ω_0.

Remark 2.2. In a matter already burdened with technicalities we preferred to avoid the full generality in order to highlight only the main ideas. Anyway our results can be carried out for more general functionals of the type

$$\int_{\Omega} f(x,u,Du)\,dx$$

with f satisfying (2.2)–(2.5) and a continuity assumption with respect to u such as

$$|f(x,u,z) - f(x,u_0,z)| \leq L\omega(|u-u_0|)(1+|z|^2)^{p(x)/2}.$$

3. Preliminary results

In this section and in the following one, since all our results are local in nature, without loss of generality we shall suppose that

$$1 < \gamma_1 \leq p(x) \leq \gamma_2 \quad \forall x \in \Omega\ , \qquad \int_{\Omega} |Du|^{p(x)}\,dx < +\infty\ .$$

We start with a higher integrability result due to Zhikov, which in the following statement appears in [AM1] (see also [Z2], [CM]):

Theorem 3.1. *Let $u \in W^{1,1}(\Omega,\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$, with $f : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ satisfying* (2.2) *and with*

$$\omega(R) \leq L|\log R|^{-1}$$

for all $R < 1$; also let

$$\int_{\Omega} |Du|^{p(x)}\,dx \leq M\ .$$

Then there exist two positive constants $c_0, \delta \equiv c_0, \delta(\gamma_1,\gamma_2,L,M)$ such that for every $B_R \subset\subset \Omega$

$$\left(\fint_{B_{R/2}} |Du|^{p(x)(1+\delta)}\,dx\right)^{1/(1+\delta)} \leq c_0 \fint_{B_R} |Du|^{p(x)}\,dx + c_0\ . \tag{3.1}$$

Remark 3.1. The condition on ω in the previous theorem, which is weaker than (2.5), was first considered by Zhikov (see [Z1-3]).
The next lemma is an up-to-the-boundary higher integrability result, and the version we give here is a vector-valued restatement of lemma 2.7 from [CFP].

Lemma 3.1. *Let $g(x,z) : \Omega \times \mathbb{R}^{nN} \to \mathbb{R}$ be a continuous function satisfying*

$$L^{-1}|z|^p \leq g(x,z) \leq L(|z|^p + a(x))\ ,$$

where $L \geq 1$, $0 \leq a(x) \in L^{\gamma}(B_R)$, $\gamma > 1$, $\gamma_1 \leq p \leq \gamma_2$. Let $\bar{u} \in W^{1,q}(B_{2R};\mathbb{R}^N)$, $p < q$, $B_{2R} \subset\subset \Omega$ and $v \in \bar{u} + W^{1,p}_0(B_R;\mathbb{R}^N)$ be a minimizer of the functional $w \mapsto \int_{B_R} g(x,Dw)\,dx$ in the Dirichlet class $\bar{u} + W^{1,p}_0(B_R;\mathbb{R}^N)$. Then there exist $c, \varepsilon \equiv$

$c, \varepsilon(\gamma_1, \gamma_2, L)$ with $0 < \varepsilon < \min\{\frac{q-p}{p}, \gamma - 1\} := m$, but independent of $R, v, a(x)$ and $\bar{u}$, such that

$$\left(\fint_{B_R} |Dv|^{p(1+\varepsilon)}\, dx\right)^{1/p(1+\varepsilon)}$$
$$\leq c\left(\fint_{B_R} |Dv|^p\, dx\right)^{1/p} + c\left(\fint_{B_{2R}} [|D\bar{u}|^p + a(x)\mathbf{1}_{B_R}]^{(1+m)}\, dx\right)^{1/p(1+m)} .$$

Proof. We only give a sketch, since this result is just a restatement of similar theorems appearing in the literature. When $\gamma = +\infty$ then the proof is contained in lemma 2.7 from [CFP], except for the fact that in our case $N \geq 1$, which is completely unrelevant for the proof. When $\gamma < \infty$ we extend $a(x)$ to the whole B_{2R} just setting $\tilde{a}(x) := a(x)\mathbf{1}_{B_R}$, then once again we follow the proof of lemma 2.7 from [CFP] adding to the right hand side of each inequality the term $\fint \tilde{a}(x)\, dx$ and applying Gehring lemma in the version of Giaquinta and Modica (see [Giu], theorem 6.6) to get our statement. We remark that although from [CFP] it seems that the exponent ε depends on p, the proof of [CFP] itself and an accurate inspection of the statements of the various versions of the Gehring lemma appearing in the literature (again, see for instance [S], or [I], proposition 6.1), reveal that ε can be chosen to be bounded uniformly away from zero as p varies in a compact subset of $]1, +\infty[$ such as our $[\gamma_1, \gamma_2]$. □

We shall widely use the function $V_p : \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$V_p(z) = (1 + |z|^2)^{(p-2)/4} z \tag{3.2}$$

for each $z \in \mathbb{R}^k$ and for any $p > 1$. The properties of V_p may be found in [CFM], lemma 2.1, which we restate here in a way that suits our needs.

Lemma 3.2. *Let $p > 1$, and let $V \equiv V_p : \mathbb{R}^k \to \mathbb{R}^k$ be as in (3.2); then for any $z, \eta \in \mathbb{R}^k$, $t > 0$*

$$
\begin{array}{ll}
(a) & |V(tz)| \leq \max\{t, t^{p/2}\}|V(z)| \\
(b) & |V(z+\eta)| \leq c[|V(z)| + |V(\eta)|] \\
(c) & |V(z) - V(\eta)| \leq c(M)|V(z-\eta)| \quad \text{if } |\eta| \leq M \text{ and } z \in \mathbb{R}^k \\
(d) & \max\{|z|, |z|^{p/2}\} \leq |V(z)| \leq c\max\{|z|, |z|^{p/2}\} \quad \text{if } p \geq 2 \\
 & c^{-1}\min\{|z|, |z|^{p/2}\} \leq |V(z)| \leq \min\{|z|, |z|^{p/2}\} \quad \text{if } 1 < p < 2 \\
(e) & c^{-1}|z-\eta| \leq \dfrac{|V(z) - V(\eta)|}{(1 + |z|^2 + |\eta|^2)^{(p-2)/4}} \leq c|z-\eta| \\
(f) & |V(z-\eta)| \leq c(M)|V(z) - V(\eta)| \quad \text{if } |\eta| \leq M \text{ and } z \in \mathbb{R}^k
\end{array}
$$

with $c \equiv c(k, p) > 0$. Moreover if $1 < \gamma_1 \leq p \leq \gamma_2$ all constants may be replaced by a single constant $c \equiv c(k, \gamma_1, \gamma_2)$.

The next lemma is a further higher integrability result concerning local minimizers of a certain class of functionals that will be needed later. The proof, including the precise dependences of the constants, will be omitted for the sake of brevity and can be easily adapted from [AF2], lemma II.4 for the case $p \geq 2$ and from [CFM], lemma 2.8, for the case $1 < p < 2$.

Lemma 3.3. *Let $\bar{L} > 0$, $0 < \lambda < 1$, $1 < \gamma_1 \leq p \leq \gamma_2$ and let $g : \mathbb{R}^{nN} \to \mathbb{R}$ be a continuous function such that*

$$|g(z)| \leq \bar{L}(1 + \lambda^2|z|^2)^{(p-2)/2}|z|^2 \tag{3.3}$$

$$\int_{B_1} g(D\phi)\,dx \geq \bar{L}^{-1}\int_{B_1} \lambda^{-2}|V_p(\lambda D\phi)|^2\,dx \tag{3.4}$$

for any $\phi \in W_0^{1,p}(B_1;\mathbb{R}^N)$. Let $\mu > 0$ and let $\bar{u} \in W^{1,p}(B_1;\mathbb{R}^N)$ be such that

$$\int_{B_1} g(D\bar{u})\,dx \leq \int_{B_1} g(D\bar{u} + D\phi) + \mu|D\phi|\,dx \tag{3.5}$$

for all $\phi \in W_0^{1,p}(B_1;\mathbb{R}^N)$. Then there exist $c, \delta_2 \equiv c, \delta_2(\gamma_1, \gamma_2, \bar{L})$ but independent of $R, \lambda, \bar{u}, g$ and μ, such that for any $B_{3R} \subset\subset B_1$:

$$\left(\fint_{B_{R/2}} |V_p(\lambda D\bar{u})|^{2(1+\delta_2)}\,dx\right)^{1/(1+\delta_2)} \leq c\fint_{B_{3R}} (\lambda^2\mu + |V_p(\lambda D\bar{u})|^2)\,dx\,.$$

The next is a consequence of the mean value theorem applied to suitable rescalings of the function f (see [AF2], lemma II.3 and [CFM], lemma 3.3):

Lemma 3.4. *Let $M > 1$, and $x_0 \in \Omega$. If $A \in \mathbb{R}^{nN}$, $|A| \leq M$, $\lambda > 0$ set*

$$f_{A,\lambda}(z) := \lambda^{-2}[f(x_0, A + \lambda z) - f(x_0, A) - \lambda Df(x_0, A)z]\,.$$

Then there exists a constant $\bar{L} \equiv \bar{L}(\gamma_1, \gamma_2, L, M) > 0$ such that

$$f_{A,\lambda}(z) \leq \bar{L}(1 + \lambda^2|z|^2)^{(p(x_0)-2)/2}|z|^2 = \bar{L}\lambda^{-2}|V_{p(x_0)}(\lambda z)|^2 \tag{3.6}$$

$$\int_{B_1} f_{A,\lambda}(D\phi)\,dx \geq \bar{L}^{-1}\int_{B_1} \lambda^{-2}|V_{p(x_0)}(\lambda D\phi)|^2\,dx \tag{3.7}$$

for any $\phi \in W_0^{1,p(x_0)}(B_1;\mathbb{R}^N)$.

Now we state a Poincaré-type inequality on increasing spheres, proved in [CFM], and involving the function V_p.

Theorem 3.2. *If $\gamma_1 < p(x_0) < 2$, there exist $\frac{2}{p(x_0)} < \alpha < 2$ and $\sigma > 0$ depending on (n, N, γ_1), such that if $u \in W^{1,p(x_0)}(B(x_0, 3R);\mathbb{R}^N)$ then*

$$\begin{aligned}\left(\fint_{B(x_0,R)} \left|V_{p(x_0)}\Big(\frac{u - (u)_{x_0,R}}{R}\Big)\right|^{2(1+\sigma)} dx\right)^{1/2(1+\sigma)} \\ \leq c\left(\fint_{B(x_0,3R)} |V_{p(x_0)}(Du)|^{\alpha} dx\right)^{1/\alpha},\end{aligned} \tag{3.8}$$

where $c \equiv c(n, N, \gamma_1)$ is independent of R and u.

We conclude the section by stating a well known variational principle due to Ekeland (see [Ek]).

Lemma 3.5. *Let (X, d) be a complete metric space and $\mathcal{G} : X \to (-\infty, +\infty]$ a lower semicontinuous functional such that $\inf_X \mathcal{G}$ is finite. Given $\varepsilon > 0$ let $v \in X$ be such that $\mathcal{G}(v) \leq \inf_X \mathcal{G} + \varepsilon$. Then there exists $\bar{\bar{u}} \in X$ such that*

$$\begin{aligned}\mathrm{d}(\bar{\bar{u}}, v) &\leq 1\\ \mathcal{G}(\bar{\bar{u}}) &\leq \mathcal{G}(v)\\ \mathcal{G}(\bar{\bar{u}}) &\leq \mathcal{G}(w) + \varepsilon\mathrm{d}(w, \bar{\bar{u}}) \quad \text{for any } w \in X\,.\end{aligned}$$

4. Blow-up and Partial Regularity

In this section we prove theorem 2.1. Before starting, we fix some quantities and make some preliminary reductions, that will be crucial for our proof; from now on, $u \in W^{1,1}(\Omega;\mathbb{R}^N)$ will always denote the local minimizer of theorem 2.1.

Choice of some relevant quantities. We start observing that if $\Omega' \subset\subset \Omega$ is open then, via a standard covering argument, by theorem 3.1 we get an exponent δ_1 such that $|Du|^{p(x)} \in L^{1+\delta_1}(\Omega')$. Again, since all our results are local, we shall rightaway suppose that

$$\int_\Omega |Du|^{p(x)(1+\delta_1)}\,dx < +\infty\ .$$

Without loss of generality, we may pick δ_1 as small as we want in order to have

$$0 < \delta_1 \le \min\{\gamma_1 - 1, 1\}\ . \tag{4.1}$$

Now, let $1 < M < \infty$ and denote by $\bar{L} \equiv \bar{L}(M)$ the constant given by lemma 3.4; applying lemma 3.3 to $g(z) \equiv f_{A,\lambda}(z)$ we come up with a further higher integrability exponent $\delta_2 \equiv \delta_2(M)$. We now turn our attention to lemma 3.1 and we apply it when $q = p(1+\delta_1/4)$ and L is replaced by $2L$. In this way we again have another (up-to-the-boundary) higher integrability exponent $0 < \varepsilon < \delta_1/4$, $\varepsilon \equiv \varepsilon(\gamma_1, \gamma_2, L, \delta_1)$, independent of any local minimizer we are going to consider. Finally we set

$$\delta_3 := \min\{\varepsilon, \delta_2\} \equiv \delta_3(M)\ .$$

Select a radius $R_M > 0$ in such a way that $\omega(R_M) \le \delta_3/4$: from now on $\mathcal{O} \subset\subset \Omega$ will always denote an open subset whose diameter does not exceed R_M. In this way if we set

$$p_1 := \inf_{\mathcal{O}} p(x) \qquad p_2 := \sup_{\mathcal{O}} p(x)$$

then it follows that

$$p_2(1+\delta_1/4) \le p_1(1+\delta_1) \le p(x)(1+\delta_1) \tag{4.2}$$

$$p_2(1+\delta_2/4) \le p_1(1+\delta_2) \le p(x)(1+\delta_2) \tag{4.3}$$

$$p_2(1+\varepsilon/4) \le p_1(1+\varepsilon) \le p(x)(1+\varepsilon) \tag{4.4}$$

whenever $x \in \mathcal{O}$. We remark that, while in general δ_3 depends on M in the sense that $\delta_3 \to 0$ if $M \to +\infty$ (since $\delta_2 \to 0$), the exponent ε is independent of M and stays bounded away from zero when M moves.

By C_M we will denote any constant depending on M, possibly varying from line to line, that for simplicity will be assumed to be such that $C_M \ge 1$. We will occasionally denote by $\hat{C}_M$ (or the like) any peculiar occurrence of a constant that we need later.

In the following technical proposition we exploit a variational freezing argument that will be used later:

Proposition 4.1. *Let $B(x_0, 4R) \subset\subset \mathcal{O}$, $(|Du|^{p_2})_{x_0,4R} \le C_M < +\infty$. Then there exist $\beta_1, \beta_2 \equiv \beta_1, \beta_2(\gamma_1, \gamma_2, L, \alpha)$ but independent of M, R and $x_0 \in \mathcal{O}$, a constant $\check{C}_M$ depending also on M, and a function $\bar{\bar{u}} \in u + W_0^{1,p_2}(B(x_0,R);\mathbb{R}^N)$ such that*

$$⨍_{B(x_0,R)} |Du - D\bar{\bar{u}}|^{p_2}\,dx \le \check{C}_M R^{\beta_1} \tag{4.5}$$

$$⨍_{B(x_0,R)} f(x_0, D\bar{\bar{u}})\,dx \tag{4.6}$$

$$\leq \fint_{B(x_0,R)} f(x_0, Dw)\, dx + R^{\beta_2} \fint_{B(x_0,R)} |Dw - D\bar{u}|\, dx$$

for any $w \in u + W_0^{1,p_2}(B(x_0,R);\mathbb{R}^N)$.

Proof. We consider $g(z) := f(x_0, z)$ and we define $v \in u + W_0^{1,p(x_0)}(B_R;\mathbb{R}^N)$ as the unique solution to the Dirichlet problem

$$\min\{\int_{B_R} g(Dw)\, dx : w \in u + W_0^{1,p(x_0)}(B_R;\mathbb{R}^N)\}\ .$$

We remark that the existence of v is ensured since f is quasiconvex. Applying theorem 3.1 to u we find:

$$\begin{aligned}\fint_{B_{2R}} |Du|^{p_2(1+\delta_1/4)}\, dx &\overset{(4.2)}{\leq} c \fint_{B_{2R}} (|Du|^{p(x)(1+\delta_1)} + 1)\, dx \\ &\overset{(3.1)}{\leq} c\left(\fint_{B_{4R}} |Du|^{p(x)}\, dx + 1\right)^{1+\delta_1} \\ &\leq c\left(\fint_{B_{4R}} |Du|^{p_2}\, dx + 1\right)^{1+\delta_1} :\leq C_M\ ;\end{aligned} \tag{4.7}$$

then, applying lemma 3.1 with $\bar{u} \equiv v$, $p \equiv p(x_0)$, $q \equiv p(x_0)(1+\delta_1/4)$, and $a(x) \equiv 1 \in L^\infty$ we have

$$\begin{aligned}\fint_{B_R} |Dv|^{p_2(1+\varepsilon/4)}\, dx &\overset{(4.4)}{\leq} c \fint_{B_R} (|Dv|^{p(x_0)(1+\varepsilon)} + 1)\, dx \\ &\leq c\left(\fint_{B_R} (|Dv|^{p(x_0)} + 1)\, dx\right)^{1+\varepsilon} \\ &\quad + c\left(\fint_{B_{2R}} (|Du|^{p_2(1+\delta_1/4)} + 1)\, dx\right)^{(1+\varepsilon)/(1+\delta_1/4)} \\ &\overset{(4.7)}{\leq} c\left(\fint_{B_R} (|Du|^{p(x_0)} + 1)\, dx\right)^{1+\varepsilon} + C_M \leq C_M\ .\end{aligned} \tag{4.8}$$

Using the minimality of u and the two formulas above we directly obtain

$$\begin{aligned}&\fint_{B_R} [g(Du) - g(Dv)]\, dx \\ &\quad = \fint_{B_R} [f(x_0, Du) - f(x, Du)]\, dx \\ &\quad\quad + \fint_{B_R} [f(x, Du) - f(x, Dv)]\, dx \qquad [\leq 0] \\ &\quad\quad + \fint_{B_R} [f(x, Dv) - f(x_0, Dv)]\, dx \\ &\quad \leq \fint_{B_R} [f(x_0, Du) - f(x, Du)]\, dx + \fint_{B_R} [f(x, Dv) - f(x_0, Dv)]\, dx \\ &\quad \overset{(2.4)}{\leq} c\omega(R) \fint_{B_R} \left((1 + |Du|^2)^{p_2/2} + (1 + |Du|^2)^{p(x)/2}\right) \cdot \\ &\quad\quad\quad \cdot [1 + \log(1 + |Du|^2)]\, dx\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&+c\omega(R) \fint_{B_R} \left((1+|Dv|^2)^{p_2/2}+(1+|Dv|^2)^{p(x)/2}\right)\cdot \\
&\qquad \cdot[1+\log(1+|Dv|^2)]\,dx \\
&\stackrel{(4.7),(4.8)}{\leq} c(\varepsilon)\omega(R) \fint_{B_R} (1+|Du|^{p_2(1+\varepsilon/4)}+|Dv|^{p_2(1+\varepsilon/4)})\,dx \\
&\leq C_M\omega(R)\ ,
\end{aligned}
$$

so that we finally find

$$
\fint_{B_R} [g(Du)-g(Dv)]\,dx \leq \hat{C}_M R^\alpha\ . \tag{4.10}
$$

Now we consider the complete metric space $X := u + W_0^{1,1}(B_R;\mathbb{R}^N)$ endowed with the metric:

$$
\mathrm{d}(z_1,z_2) := \hat{C}_M^{-1} R^{-\alpha/4} \fint_{B_R} |Dz_1 - Dz_2|\,dx
$$

and we set

$$
\mathcal{G}(z) := \begin{cases} \fint_{B_R} g(Dz)\,dx & \text{if } z \in u + W_0^{1,p(x_0)}(B_R;\mathbb{R}^N) \\ +\infty & \text{otherwise.} \end{cases}
$$

This functional is clearly lower semicontinuous on X and moreover $\mathcal{G}(v) = \min_X \mathcal{G}$ so that, by (4.10), it also follows that

$$
\mathcal{G}(u) \leq \inf_X \mathcal{G} + \hat{C}_M R^{\alpha/2}\ ;
$$

at this point we apply lemma 3.5 above in order to find $\bar{\bar{u}} \in u + W_0^{1,p(x_0)}(B_R;\mathbb{R}^N)$ such that

$$
\fint_{B_R} |Du - D\bar{\bar{u}}|\,dx \leq \hat{C}_M R^{\alpha/4} \tag{4.11}
$$

$$
\fint_{B_R} f(x_0, D\bar{\bar{u}})\,dx \leq \fint_{B_R} f(x_0, Dw)\,dx + R^{\alpha/4} \fint_{B_R} |Dw - D\bar{\bar{u}}|\,dx := \bar{\mathcal{G}}(w) \tag{4.12}
$$

for all $w \in u + W_0^{1,1}(B_R;\mathbb{R}^N)$, so that (4.6) is proved with $\beta_2 = \alpha/4$. It remains to prove (4.5). We observe that by (4.12) the function $\bar{\bar{u}}$ minimizes the new functional $\bar{\mathcal{G}}$, so we apply lemma 3.1 to $\bar{\mathcal{G}}$, with $g(x,z) \equiv f(x_0,z) + R^{\alpha/4}|z - D\bar{\bar{u}}(x)|$, $p \equiv p(x_0)$, $q \equiv p(x_0)(1+\delta_1/4)$, $a(x) \equiv D\bar{\bar{u}} + 1 \in L^{\gamma_1}$, $\gamma \equiv \gamma_1$, so that $m \equiv \delta_1/4$ by (4.1); we obtain

$$
\begin{aligned}
\fint_{B_R} |D\bar{\bar{u}}|^{p_2(1+\varepsilon/4)}\,dx &\stackrel{(4.4)}{\leq} \fint_{B_R} (|D\bar{\bar{u}}|^{p(x_0)(1+\varepsilon)}+1)\,dx \\
&\leq c\left(\fint_{B_R} |D\bar{\bar{u}}|^{p(x_0)}\,dx\right)^{1+\varepsilon} \\
&\quad +c\left(\fint_{B_{2R}} (|Du|^{p_2(1+\delta_1/4)}+1)\,dx\right)^{(1+\varepsilon)/(1+\delta_1/4)} \\
&\quad +c\left(\fint_{B_R} (|D\bar{\bar{u}}|+1)^{1+\delta_1/4}\,dx\right)^{(1+\varepsilon)/(1+\delta_1/4)} \\
&\stackrel{(4.1),(4.7)}{\leq} C_M + c\left(\fint_{B_R} |D\bar{\bar{u}}|^{p(x_0)}+1\,dx\right)^{1+\varepsilon} \leq C_M
\end{aligned}
$$

where, in order to perform the last estimate, we used the minimality of $\bar{\bar{u}}$ and the growth conditions satisfied by $\mathcal{G}$ to have by lemma 3.5

$$\fint_{B_R} |D\bar{\bar{u}}|^{p(x_0)}\,dx \le \mathcal{G}(\bar{\bar{u}}) \le \mathcal{G}(u) \le c \fint_{B_R} (|Du|^{p(x_0)} + 1)\,dx \le C_M\ .$$

We finally interpolate p_2 between 1 and $p_2(1+\varepsilon/4)$ to obtain, using (4.11) and the previous estimates,

$$\begin{aligned}
&\fint_{B_R} |Du - D\bar{\bar{u}}|^{p_2}\,dx \\
&\le \left(\fint_{B_R} |Du - D\bar{\bar{u}}|\,dx\right)^{p_2\theta} \left(\fint_{B_R} |Du - D\bar{\bar{u}}|^{p_2(1+\varepsilon/4)}\,dx\right)^{(1-\theta)/(1+\varepsilon/4)} \\
&\le C_M R^{\theta\alpha p_2/4} \le C_M R^{\theta\alpha/4} := \check{C}_M R^{\theta\alpha/4}\ .
\end{aligned}$$

The exponent θ is independent of M, since

$$\begin{aligned}
\theta &= \left(1 - \frac{1}{p_2(1+\varepsilon/4)}\right)^{-1} \left(\frac{1}{p_2} - \frac{1}{p_2(1+\varepsilon/4)}\right) \\
&= \frac{(\epsilon/4)}{p_2(1+\epsilon/4) - 1} \ge \frac{(\epsilon/4)}{\gamma_2(1+\epsilon/4) - 1} := \bar{\theta}
\end{aligned}$$

and ε is independent of M. To conclude the proof of (4.5) it is then enough to choose $\beta_2 := \bar{\theta}\alpha/4$. □

We define the numbers $\underline{q}$, Q and β by

$$\underline{q} := \min\{2, p_2\}\ , \qquad Q := \max\{2, p_2\}\ , \qquad \beta := \frac{1}{2p_2}\min\{\beta_1, \beta_2\} \tag{4.13}$$

and we set

$$E(x_0, R) = \fint_{B(x_0,R)} |V_{p_2}(Du) - V_{p_2}((Du)_{x_0,R})|^2 dx + R^{\beta}$$

whenever $B(x_0, 4R) \subset\subset \mathcal{O}$. Roughly speaking, the quantity E (usually called excess) provides an integral measure of the oscillations of the gradient Du in a ball B_R. The next decay estimate for the quantity E is the key to the proof of theorem 2.1.

Proposition 4.2. *Let $M > 1$ and let $\mathcal{O} \subset\subset \Omega$ be an open subset related to M in the way described above. There exists a constant C_M such that for every $0 < \tau < 1/24$ there exists $\varepsilon_0 \equiv \varepsilon_0(\tau, M)$ such that if $B(x_0, 4R) \subset\subset \mathcal{O}$ and*

$$\begin{gathered}
|(Du)_{x_0,\tau R}| \le M\ , \qquad |(Du)_{x_0,R}| \le M\ , \qquad |(Du)_{x_0,4R}| \le M\ , \\
E(x_0, R) < \varepsilon_0\ , \qquad E(x_0, 4R) \le 1
\end{gathered} \tag{4.14}$$

then

$$E(x_0, \tau R) \le C_M \tau^{\beta} E(x_0, R) \tag{4.15}$$

with β defined in (4.13).

Proof. Step 1: blow-up. Arguing by contradiction we suppose there exists a sequence of balls $B(x_h, 4R_h) \subset\subset \mathcal{O}$ such that

$$\begin{gathered}
|(Du)_{x_h,\tau R_h}| \le M\ , \qquad |(Du)_{x_h,R_h}| \le M\ , \qquad |(Du)_{x_h,4R_h}| \le M\ , \\
\mu_h^2 := E(x_h, R_h) \to 0\ , \qquad E(x_h, 4R_h) \le 1
\end{gathered}$$

but

$$E(x_h, \tau R_h) \geq C(M)\tau^{\beta} E(x_h, R_h) \ , \tag{4.16}$$

where the constant $C(M)$ will be chosen later; without loss of generality we may assume that $R_h \to 0$. We immediately obtain that there exists $C_M < +\infty$ such that $(|Du|^{p_2})_{x_h,4R_h} \leq C_M$: indeed, using lemma 3.2 $(d), (f)$

$$\begin{aligned}(|Du|^{p_2})_{x_h,4R_h} &\leq c \fint_{B(x_h,4R_h)} |Du - (Du)_{x_h,4R_h}|^{p_2}\, dx + C_M \\ &\leq C_M E(x_h, 4R_h) + C_M \leq C_M \ .\end{aligned}$$

Now we apply proposition 4.1 in order to find a sequence of functions $u_h \in u + W^{1,p_2}_0(B(x_h, R_h); \mathbb{R}^N)$ such that

$$\fint_{B(x_h,R_h)} |Du - Du_h|^{p_2}\, dx \leq C_M R_h^{\beta_1} \tag{4.17}$$

$$\begin{aligned}&\fint_{B(x_h,R_h)} f(x_h, Du_h)\, dx \\ &\quad \leq \fint_{B(x_h,R_h)} f(x_h, Dw)\, dx + R_h^{\beta_2} \fint_{B(x_h,R_h)} |Dw - Du_h|\, dx\end{aligned} \tag{4.18}$$

for any $w \in u + W^{1,p_2}_0(B(x_h, R_h); \mathbb{R}^N)$ with β_1, β_2 independent of $h \in \mathbb{N}$ and M. We define

$$A_h := (Du)_{x_h,R_h} \qquad \lambda_h^2 := \fint_{B(x_h,R_h)} |V_{p_2}(Du_h) - V_{p_2}(A_h)|^2 dx + R_h^{\beta} \tag{4.19}$$

(remark that A_h is not the average of Du_h) and we rescale each function u_h in the ball $B(x_h, R_h)$ in order to have a sequence of functions defined on $B(0, 1) \equiv B_1$:

$$v_h(y) = (\lambda_h R_h)^{-1}[u_h(x_h + R_h y) - (u_h)_{x_h,R_h} - R_h A_h y]$$

for any $y \in B(0, 1)$. Then lemma 3.2 (f) yields

$$\begin{aligned}\lambda_h^{-2} \fint_{B(0,1)} |V_{p_2}(\lambda_h Dv_h(y))|^2 dy &= \lambda_h^{-2} \fint_{B(x_h,R_h)} |V_{p_2}(Du_h(x) - A_h)|^2 dx \\ &\leq C_M \lambda_h^{-2} \fint_{B(x_h,R_h)} |V_{p_2}(Du_h(x)) - V_{p_2}(A_h)|^2 dx \leq C_M\end{aligned} \tag{4.20}$$

by (4.19); so, by (d) from lemma 3.2

$$\sup_h \Big[\| |Dv_h|^{\underline{q}} \|_{L^1(B_1)} + ((p_2 - 2) \vee 0) \| \lambda_h^{p_2-2} |Dv_h|^{p_2} \|_{L^1(B_1)} \Big] \leq C_M \ .$$

Remarking that we also have $(v_h)_{0,1} = 0$, eventually selecting a subsequence we obtain that there exists $v \in W^{1,\underline{q}}(B_1)$ such that:

$$\begin{aligned} |v_h - v|^{\underline{q}} &\to 0 && \text{strongly in } L^1(B_1) \\ \lambda_h^{p_2-2}|v_h - v|^{p_2} &\to 0 && \text{strongly in } L^1(B_1) \text{ if } p_2 > 2 \\ Dv_h &\rightharpoonup Dv && \text{weakly in } L^{\underline{q}}(B_1; \mathbb{R}^{nN}) \\ x_h &\to x && \text{in } \mathbb{R}^n, \text{ with } x \in \overline{\mathcal{O}} \\ A_h &\to A && \text{in } \mathbb{R}^{nN}, \text{ with } |A| \leq M \ . \end{aligned} \tag{4.21}$$

Finally, we prove that

$$\lambda_h^2 \leq C_M \mu_h^2 \ , \tag{4.22}$$

a relation that will be useful in the sequel: in particular it implies that

$$\lambda_h^2 \to 0.$$

Using lemma 3.2 and Jensen's inequality we get

$$\begin{aligned}
\lambda_h^2 &\overset{(c)}{\leq} C_M \fint_{B(x_h,R_h)} |V_{p_2}(Du_h - A_h)|^2 \, dx + R_h^\beta \\
&\overset{(b)}{\leq} C_M \fint_{B(x_h,R_h)} |V_{p_2}(Du_h - Du)|^2 dx \\
&\quad + C_M \fint_{B(x_h,R_h)} |V_{p_2}(Du - A_h)|^2 dx + R_h^\beta \\
&\overset{(d),(f)}{\leq} C_M \fint_{B(x_h,R_h)} |Du_h - Du|^{p_2} \, dx \\
&\quad + C_M((p_2-2) \vee 0)\left(\fint_{B(x_h,R_h)} |Du_h - Du|^{p_2} \, dx\right)^{2/p_2} + C_M \mu_h^2 \\
&\overset{(4.17)}{\leq} C_M[R_h^{\beta_1} + R_h^{(2\beta_1)/p_2} + \mu_h^2] \overset{(4.13)}{\leq} C_M \mu_h^2 \, .
\end{aligned}$$

Step 2: v solves a linear system. By the minimality relation (4.18) satisfied by u_h, rescaled in B_1, it follows that v_h satisfies the Euler system

$$\fint_{B(0,1)} \langle Df(x_h, A_h + \lambda_h Dv_h), D\phi \rangle \, dy + \lambda_h (II)_h = 0 \, ,$$

where the second term is dominated by

$$\lambda_h |(II)_h| \leq R_h^{\beta_2} \fint_{B(0,1)} |D\phi| \, dy$$

for each $\phi \in C_0^1(B_1; \mathbb{R}^N)$, and also

$$\begin{aligned}
0 &= \lambda_h^{-1} \int_{B(0,1)} \langle Df(x_h, A_h + \lambda_h Dv_h) - Df(x_h, A_h), D\phi \rangle \, dy + (II)_h \\
&:= (I)_h + (II)_h \, .
\end{aligned}$$

The fact that $\lambda_h^{-1} R_h^{\beta_2} \leq R_h^{\beta_2 - \beta} \to 0$ implies $(II)_h \to 0$. In order to estimate $(I)_h$, following [AF2] we split B_1 into $E_h^+ \cup E_h^- := \{x \in B_1 : \lambda_h |Dv_h| \geq 1\} \cup \{x \in B_1 : \lambda_h |Dv_h| < 1\}$, thus we also split $(I)_h$ into $(I)_h^+ + (I)_h^-$. We immediately have that

$$|E_h^+| \leq \| \lambda_h^q |Dv_h|^q \|_{L^1(B_1)} \leq c \lambda_h^q \to 0 \, ;$$

moreover (4.20) and (d) from lemma 3.2 easily give that

$$\| \lambda_h^{p_2 - 2} |Dv_h|^{p_2} \|_{L^1(E_h^+)} \leq C_M \, .$$

Using these facts, remark 2.1 and Hölder inequality we deduce for every fixed ϕ

$$\begin{aligned}
|(I)_h^+| &\leq \lambda_h^{-1} \int_{E_h^+} |Df(x_h, A_h + \lambda_h Dv_h) - Df(x_h, A_h)| |D\phi| \, dy \\
&\leq C_M \lambda_h^{-1} \int_{E_h^+} (1 + |\lambda_h Dv_h|^{p(x_h)-1}) \, dy \\
&\leq C_M \int_{E_h^+} (\lambda_h^{-1} + \lambda_h^{p_2-2} |Dv_h|^{p_2-1}) \, dy
\end{aligned}$$

$$\begin{aligned}
&\leq C_M\lambda_h^{-1}|E_h^+| + C_M\lambda_h^{(p_2-2)/p_2}|E_h^+|^{1/p_2}\left(\int_{E_h^+}\lambda_h^{p_2-2}|Dv_h|^{p_2}\,dy\right)^{(p_2-1)/p_2}\\
&\leq C_M\lambda_h^{\underline{q}-1} + C_M\lambda_h^{(p_2-1)/p_2} \to 0
\end{aligned}$$

since $\underline{q} > 1$. As for the rest of B_1 we have

$$\begin{aligned}
(I)_h^- &= \lambda_h^{-1}\int_{E_h^-}\langle Df(x_h, A_h+\lambda_h Dv_h) - Df(x_h, A_h), D\phi\rangle\,dy\\
&= \int_{E_h^-}dy\int_0^1\langle (D^2f(x_h, A_h+s\lambda_h Dv_h) - D^2f(x_h,A_h))Dv_h, D\phi\rangle\,ds\\
&\quad+\int_{E_h^-}\langle D^2f(x_h,A_h)Dv_h, D\phi\rangle\,dy\\
&:= (III)_h + (IV)_h\ .
\end{aligned}$$

Since Dv_h is bounded in $L^{\underline{q}}$, we have $\lambda_h Dv_h \to 0$ in $L^{\underline{q}}$, thus up to another (not relabelled) subsequence we may also suppose that $\lambda_h Dv_h \to 0$ a.e. in B_1. Since D^2f is uniformly continuous on bounded sets this implies that $(III)_h \to 0$. On the other hand, since $|E_h^+| \to 0$, by (4.27) we have that

$$(IV)_h \to \int_{B_1}\langle D^2f(x,A)Dv, D\phi\rangle\,dy$$

so that connecting this with the previous estimates gives

$$\int_{B_1}\langle D^2f(x,A)Dv, D\phi\rangle\,dy = 0 \tag{4.23}$$

for any $\phi \in C_0^1(B_1,\mathbb{R}^N)$. The uniform quasiconvexity condition in (2.3), introduced in [E], implies that the matrix D^2f satisfies the following strong Legendre-Hadamard condition (see e.g. [Gia],[Giu]):

$$C_M^{-1}|\lambda|^2|\mu|^2 \leq \langle D^2f(x,A)\lambda\otimes\mu, \lambda\otimes\mu\rangle \leq C_M|\lambda|^2|\mu|^2$$

for any $\lambda \in \mathbb{R}^n,\ \mu \in \mathbb{R}^N$ and for some constant $1 \leq C_M < +\infty$. Therefore known regularity results (see for instance [AF2], and [CFM] for the case $1 < p_2 < 2$) for solutions of linear elliptic systems with constant coefficients as (4.23) give that $v \in C^\infty(B_1)$ and

$$⨍_{B_\tau}|Dv - (Dv)_\tau|^2\,dy \leq C_M\tau^2 \tag{4.24}$$

for any $\tau \leq 1/4$.

Step 3: upper bound. Keeping the notation of lemma 3.4 we define a sequence of rescaled functions:

$$f_h(z) \equiv f_{A_h,\lambda_h}(z) := \frac{f(x_h, A_h+\lambda_h z) - f(x_h,A_h) - \lambda_h Df(x_h,A_h)z}{\lambda_h^2}$$

for any $h \in \mathbb{N}$, $z \in \mathbb{R}^{nN}$ and, if $r \in (0,1]$, the corresponding functionals

$$I_h^r(w) := \int_{B_r} f_h(Dw)\,dy$$

for any $w \in W^{1,1}(B_1, \mathbb{R}^N)$. With such a notation the minimality of u_h stated in (4.18) translates, after rescaling, into:

$$I_h^r(v_h) \le I_h^r(v_h + \phi) + \lambda_h^{-2} R_h^{\beta_2} \int_{B_r} |D\phi| \, dy \tag{4.25}$$

for any $\phi \in W^{1,1}(B_1, \mathbb{R}^N)$ such that $\operatorname{spt} \phi \subset\subset B_r$. Applying lemma 3.4 with $(A, \lambda, x_0) \equiv (A_h, \lambda_h, x_h)$ we observe that the hypotheses of lemma 3.3 are verified by (3.6),(3.7) and (4.25), with $g(z) \equiv f_h(z)$, $u \equiv v_h$ and $\mu \equiv \lambda_h^{-2} R_h^{\beta_2}$, so that, with the choice of the quantities made at the beginning of the section, we easily have by (4.20)

$$\fint_{B_{1/12}} \Big| \frac{V_{p(x_h)}(\lambda_h D v_h)}{\lambda_h} \Big|^{2(1+\delta_3)} dx \le C_M \tag{4.26}$$

(remark that μ is bounded because $\lambda_h^{-2} R_h^{\beta_2} \le R_h^{\beta_2(1-1/2p_2)}$ by (4.13)). Now we want to prove that:

$$\limsup_h \, [I_h^r(v_h) - I_h^r(v)] \le 0 \tag{4.27}$$

for a.e. $r \in (0, 1/12)$. Let us consider the sequence of Radon measures given by

$$\alpha_h := \lambda_h^{-2} [|V_{p_2}(\lambda_h D v_h)|^2 + |V_{p_2}(\lambda_h D v)|^2] \llcorner dy.$$

By (4.20) and the smoothness of v it follows that

$$\sup_h \|\alpha_h\|_{BV} \le C_M$$

and so, up to not relabelled subsequences, we may suppose there exists a Radon measure α such that $\alpha_h \rightharpoonup \alpha$ weakly in the sense of measures. Moreover, since $\alpha(\partial B_t) = 0$ for all but a countable set of $t \in (0, 1)$, we may suppose without loss of generality that $\alpha(\partial B_r) = 0$. We choose $s < r$ and take $\eta \in C_0^\infty(B_r)$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on B_s, $|D\eta| \le c/(r - s)$; we then test the minimality of v_h with the function $\phi_h = (v - v_h)\eta$. We have by (3.6) and using $(a), (b)$ from lemma 3.2:

$$\begin{aligned} I_h^r(v_h) - I_h^r(v) &\le I_h^r(v_h + \phi_h) - I_h^r(v) + \lambda_h^{-2} R_h^{\beta_2} \int_{B_1} |D\phi_h| \, dy \\ &= \int_{B_r \setminus B_s} [f_h(Dv_h + D\phi_h) - f_h(Dv)] \, dx + o_h \\ &\le c\lambda_h^{-2} \int_{B_r \setminus B_s} [|V_{p(x_h)}(\lambda_h D v)|^2 \\ &\quad + |V_{p(x_h)}(\lambda_h (v - v_h) D\eta + \lambda_h \eta D v + \lambda_h (1 - \eta) D v_h)|^2] \, dy + o_h \\ &\le c\alpha_h(B_r \setminus B_s) + \frac{c\lambda_h^{-2}}{(r - s)^Q} \int_{B_r \setminus B_s} |V_{p(x_h)}(\lambda_h (v - v_h))|^2 \, dy + o_h \end{aligned} \tag{4.28}$$

where o_h will denote any quantity that vanishes as $h \to \infty$; in this case we have $o_h = \lambda_h^{-2} R_h^{\beta_2} \int_{B_1} |D\phi_h| \, dy \to 0$ since $\lambda_h^{-2} R_h^{\beta_2} \to 0$ and the norms $\|D\phi_h\|_{L^1(B_1)}$ are uniformly bounded. In order to estimate the last integral in (4.28) we observe that this vanishes, when $h \to \infty$, by (4.27) and lemma 3.2 (d) when $p_2 \ge 2$; when

$1 < p_2 < 2$ we interpolate by taking $\theta \in (0,1)$ such that $1/2 = \theta + (1-\theta)/2(1+\sigma)$, where $\sigma > 0$ is the improving constant introduced in theorem 3.2; we have:

$$
\begin{aligned}
&\int_{B_r \setminus B_s} |V_{p(x_h)}(\lambda_h(v-v_h))|^2\,dy \\
&\quad \le c\Big(\int_{B_r \setminus B_s} |V_{p(x_h)}(\lambda_h(v-v_h))|\,dy\Big)^{2\theta} \cdot \\
&\qquad \cdot\Big(\int_{B_r \setminus B_s} |V_{p(x_h)}(\lambda_h(v-v_h))|^{2(1+\sigma)}\,dy\Big)^{(1-\theta)/(1+\sigma)} \\
&\quad \le c\lambda_h^{2\theta}\left(\int_{B_r} |v-v_h|\,dy\right)^{2\theta} \cdot \\
&\qquad \cdot\Big(\int_{B_r \setminus B_s} \Big[|V_{p(x_h)}(\lambda_h(v-v_h) - \lambda_h(v-v_h)_{0,\frac{1}{3}})|^{2(1+\sigma)} \qquad (4.29)\\
&\qquad\quad + |V_{p(x_h)}(\lambda_h(v-v_h)_{0,\frac{1}{3}})|^{2(1+\sigma)}\Big]\,dy\Big)^{(1-\theta)/(1+\sigma)} \\
&\quad \le c\lambda_h^{2\theta}\left(\int_{B_r} |v-v_h|\,dy\right)^{2\theta} \cdot \\
&\qquad \cdot\left[\left(\int_{B_1} |V_{p(x_h)}(\lambda_h Dv_h)|^2\,dy\right)^{1-\theta} + \lambda_h^{2(1-\theta)}\right] \\
&\quad \overset{(4.20)}{\le} C_M \lambda_h^2 \left(\int_{B_r} |v-v_h|\,dy\right)^{2\theta}.
\end{aligned}
$$

Collecting (4.28) and (4.29), we have in any case

$$
I_h^r(v_h) - I_h^r(v) \le C_M\left[\alpha_h(B_r \setminus B_s) + \frac{1}{(r-s)^Q}\left(\int_{B_r} |v-v_h|\,dy\right)^{2\theta} + o_h\right],
$$

and (4.27) follows by letting first $h \to \infty$ and then $s \nearrow r$ (using the fact that $\alpha(\partial B_r) = 0$).

Step 4: lower bound. Our aim here is to prove that

$$
\limsup_h \lambda_h^{-2} \int_{B_{r/2}} |V_{p_2}(\lambda_h(Dv - Dv_h))|^2\,dy = 0 \tag{4.30}
$$

for all $r \in (0, 1/12)$. We begin by writing

$$
I_h^r(v_h) - I_h^r(v) = [I_h^r(v_h) - I_h^r(v+\phi_h)] + [I_h^r(v+\phi_h) - I_h^r(v)] := (I)_h + (II)_h\,,
$$

where this time $\phi_h := (v_h - v)\eta$ and η is the cut-off function defined in step 3 with $1/12 > r > s \ge r/2$. Proceeding as in step 3, we estimate in a similar fashion

$$
\begin{aligned}
|(I)_h| &\le \int_{B_r \setminus B_s} |f_h(Dv_h) - f_h(Dv + D\phi_h)|\,dy \\
&\le C_M\left[\alpha_h(B_r \setminus B_s) + \frac{1}{(r-s)^Q}\left(\int_{B_r} |v-v_h|\,dy\right)^{2\theta} + o_h\right].
\end{aligned}
$$

In order to estimate $(II)_h$ we write

$$
(II)_h = \int_{B_r} f_h(D\phi_h)\,dy + \int_{B_r} [f_h(Dv + D\phi_h) - f_h(Dv) - f_h(D\phi_h)]\,dy
$$

$$:= \quad (III)_h + (IV)_h \ .$$

Recalling that each f_h satisfies (3.7) from lemma 3.4, with $x_0 \equiv x_h$ and $\lambda \equiv \lambda_h$, we have

$$(III)_h \geq C_M^{-1} \lambda_h^{-2} \int_{B_{r/2}} |V_{p(x_h)}(\lambda_h(Dv - Dv_h))|^2 \, dy \ .$$

To estimate $(IV)_h$ we observe that by Egorov theorem we may fix $\sigma > 0$ and find $S \subset B_1$ such that $\lambda_h(|Dv_h| + |D\phi_h|) \to 0$ uniformly in S, with $|B_1 \setminus S| \leq \sigma$; so we have, linearizing around A_h in a standard way as in [AF2], [CFM], but without using the extension and selection lemmas employed in these papers,

$$\begin{aligned}
|(IV)_h| \ &\leq \ \int_{B_r \setminus S} |\cdots| \, dy + \left| \int_{B_r \cap S} (\cdots) \, dy \right| \\
&\leq \ \frac{c}{\lambda_h^2} \int_{B_r \setminus S} \left(|V_{p(x_h)}(\lambda_h Dv_h)|^2 + |V_{p(x_h)}(\lambda_h D\phi_h)|^2 + |V_{p(x_h)}(\lambda_h Dv)|^2 \right) \, dy \\
&\qquad + c \left| \int_{B_r \cap S} dy \int_0^1 \int_0^1 \langle D^2 f(A_h + s\lambda_h D\phi_h + t\lambda_h Dv_h) Dv, D\phi_h \rangle \, ds \, dt \right| \\
&:= \ (V)_h + (VI)_h \ .
\end{aligned}$$

Proceeding as in step 3, estimates (4.28),(4.29), and using the higher integrability bound (4.26) together with the smoothness of v, we have by Hölder's inequality

$$(V)_h \leq C_M |B_r \setminus S|^{\delta_3/(1+\delta_3)} + \frac{C_M}{(r-s)^Q} \left(\int_{B_r} |v - v_h| \, dy \right)^{2\theta} + o_h \ ;$$

on the other hand the uniform convergence in S, the uniform continuity of $D^2 f$ on bounded sets and the fact that $D\phi_h \rightharpoonup 0$ weakly in $L^{\underline{q}}(B_r)$ give that $(VI)_h \to 0$. Connecting all these facts we finally obtain that

$$\limsup_h \lambda_h^{-2} \int_{B_{r/2}} |V_{p(x_h)}(\lambda_h(Dv - Dv_h))|^2 \, dy \leq C_M [\sigma^{\delta_3/(1+\delta_3)} + \alpha(B_r \setminus B_s)] \ ,$$

so that by letting first $\sigma \to 0$ and then $s \nearrow r$ we find that

$$\lambda_h^{-2} \int_{B_{r/2}} |V_{p(x_h)}(\lambda_h(Dv - Dv_h))|^2 \, dy = o_h \tag{4.31}$$

for a.e. $0 < r < 1/12$ (and thus for all r by monotonicity). Finally, in order to prove (4.30) we observe that when $|z| \geq 1$ then (4.2)–(4.4) and the elementary properties of the function V imply that

$$|V_{p_2}(z)|^2 \leq c |V_{p(x_h)}(z)|^{2(1+\delta_3)}$$

with c independent of $h \in \mathbb{N}$, thus we have

$$\begin{aligned}
&\lambda_h^{-2} \int_{B_{r/2}} |V_{p_2}(\lambda_h(Dv - Dv_h))|^2 \, dy \\
&\quad \leq \lambda_h^{-2} \int_{\{\lambda_h |Dv - Dv_h| < 1\} \cap B_{r/2}} |V_{p(x_h)}(\lambda_h(Dv - Dv_h))|^2 \, dy \\
&\quad\quad + \lambda_h^{-2} \int_{\{\lambda_h |Dv - Dv_h| \geq 1\} \cap B_{r/2}} \left| V_{p(x_h)}(\lambda_h(Dv - Dv_h)) \right|^{2(1+\delta_3)} \, dy \\
&\quad \overset{(4.31)}{\leq} o_h + c \lambda_h^{2\delta_3} \int_{B_{1/12}} \left| \frac{V_{p(x_h)}(\lambda_h Dv_h)}{\lambda_h} \right|^{2(1+\delta_3)} \, dy
\end{aligned}$$

$$+c\lambda_h^{2\delta_3}\int_{B_{1/12}}\Big|\frac{V_{p(x_h)}(\lambda_h Dv)}{\lambda_h}\Big|^{2(1+\delta_3)}\,dy$$

$$\stackrel{(4.26)}{\leq}\; o_h + C_M\lambda_h^{2\delta_3}\to 0\;,$$

where we also used the smoothness of v; in this way (4.30) is completely proved.

Remark 4.1. The proof of the previous estimate relies on the possibility to control $p_2 \le p_1(1+\delta_3)$ (see (4.3)-(4.4)) and it is the main reason to blow-up the minimizer u in the open subset $\mathcal{O}$ rather than in the whole Ω. On the other hand, the open subset $\mathcal{O}$ depends on u itself (via M) and this will force us to involve in the proof of theorem 2.1 a delicate localization argument (see below).

Step 5: comparison and conclusion. We preliminarily observe that using lemma 3.2 (d)

$$\begin{aligned}
&\mu_h^{-2}\fint_{B(x_h,\tau R_h)}|V_{p_2}(Du_h-Du)|^2\,dx\\
&\quad\le c\mu_h^{-2}\fint_{B(x_h,R_h)}|Du_h-Du|^{p_2}\,dx\\
&\qquad+c\mu_h^{-2}((p_2-2)\vee 0)\left(\fint_{B(x_h,R_h)}|Du_h-Du|^{p_2}\,dx\right)^{2/p_2}\\
&\quad\stackrel{(4.17)}{\le} C_M\mu_h^{-2}[R_h^{\beta_1}+R_h^{2\beta_1/p_2}]\le C_M[R_h^{\beta_1-\beta}+R_h^{(2\beta_1/p_2)-\beta}]\stackrel{(4.13)}{\to} 0
\end{aligned}$$

and again by (d) from lemma 3.2 and (4.17), in a similar way,

$$\mu_h^{-2}\fint_{B(x_h,\tau R_h)}|V_{p_2}\big((Du_h)_{x_h,\tau R_h}-(Du)_{x_h,\tau R_h}\big)|^2\,dx = o_h\;.$$

Now, using (b) and (c) from lemma 3.2 together with the previous estimates we have (since $|(Du)_{x_h,\tau R_h}|\le M$ by assumption)

$$\begin{aligned}
&\limsup_h \mu_h^{-2}E(x_h,\tau R_h)\\
&\quad\le C_M\limsup_h\mu_h^{-2}\fint_{B(x_h,\tau R_h)}|V_{p_2}(Du-(Du)_{x_h,\tau R_h})|^2\,dx\\
&\qquad+C_M\tau^\beta\limsup_h\mu_h^{-2}R_h^\beta\\
&\quad\le C_M\tau^\beta + C_M\limsup_h\mu_h^{-2}\fint_{B(x_h,\tau R_h)}|V_{p_2}(Du-Du_h)|^2\,dx\\
&\qquad+C_M\limsup_h\mu_h^{-2}\fint_{B(x_h,\tau R_h)}|V_{p_2}(Du_h-(Du_h)_{x_h,\tau R_h})|^2\,dx\\
&\qquad+C_M\limsup_h\mu_h^{-2}\fint_{B(x_h,\tau R_h)}|V_{p_2}((Du_h)_{x_h,\tau R_h}-(Du)_{x_h,\tau R_h})|^2\,dx\\
&\quad\stackrel{(4.21),(4.22)}{\le} C_M\tau^\beta + C_M\limsup_h\lambda_h^{-2}\fint_{B_\tau}|V_{p_2}(\lambda_h(Dv_h-(Dv_h)_\tau))|^2\,dy\\
&\quad\le C_M\tau^\beta + C_M\limsup_h\lambda_h^{-2}\fint_{B_\tau}|V_{p_2}(\lambda_h(Dv_h-Dv))|^2\,dy\\
&\qquad+C_M\limsup_h\lambda_h^{-2}\fint_{B_\tau}|V_{p_2}(\lambda_h(Dv-(Dv)_\tau))|^2\,dy
\end{aligned}$$

$$+C_M \limsup_h \lambda_h^{-2} \fint_{B_\tau} |V_{p_2}(\lambda_h((Dv)_\tau - (Dv_h)_\tau))|^2\, dy$$
$$\overset{(4.24),(4.30)}{\leq} C_M(\tau^2 + \tau^\beta) \leq \hat{C}_M \tau^\beta$$

since $(Dv_h)_\tau \to (Dv)_\tau$ by $(4.21)_2$. Now the contradiction to (4.16) follows if we choose, for instance, $C(M) := 2\hat{C}_M$. □

Proof of theorem 2.1. The proof will be divided in three steps.

Step 1: Iteration. Let $M \geq 2$, $B(x_0, 16R) \subset\subset \mathcal{O} \subset\subset \Omega$ where $\mathcal{O} \equiv \mathcal{O}_M$ is as in proposition 4.2; if C_M is the constant appearing in (4.15) and $0 < \tau < 1/24$ is such that $C_M \tau^{\bar{\beta}/2} < 1/4$, then a minor modification (see remark 4.2 below) of the iteration scheme developed in [FH] shows that there exists $\eta \equiv \eta(M, \tau) \equiv \eta(M) \leq \varepsilon_0 \leq 1$, with ε_0 as in (4.14), such that if

$$\begin{aligned} &|(Du)_{x_0,\tau R}| + |(Du)_{x_0,R}| + |(Du)_{x_0,4R}| \leq M/4\,, \\ &E(x_0, R) \leq \eta\,, \qquad E(x_0, 4R) \leq 1 \end{aligned} \tag{4.32}$$

then a standard iteration procedure built upon proposition 4.2 starts and leads to

$$|(Du)_{x_0,\tau^k R}| \leq M\,, \qquad E(x_0, \tau^k R) \leq \tau^{k\bar{\beta}/2} \tag{4.33}$$

for any $k \geq 1$.

Step 2: Construction of Ω_0. We check that the inequalities (4.32) are satisfied in the set of full measure

$$\Omega_0 := \{x_0 \in \Omega \quad : \quad \limsup_{\rho\to 0} (|Du|^{p(x)})_{x_0,\rho} < +\infty \text{ and }$$
$$\limsup_{\rho\to 0} \fint_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|\, dx = 0\}$$

provided $16R < R_M$, where R_M is a radius depending on x_0 and M, smaller than the one defined before proposition 4.1. Indeed let $x_0 \in \Omega_0$ and $B(x_0, 16R) \subset\subset \mathcal{O} \subset \subset \Omega$ as before. Denoting by $c_0 > 1$ the higher integrability constant provided by theorem 3.1, let $M \geq \max\{2, 8c_0\}$ and $16R \leq R_M$ be such that for $\rho = \tau R$, for $\rho = R$ and for $\rho = 4R$

$$\begin{aligned} &(|Du|^{p(x)})_{x_0,2\rho} < \left(\frac{M}{8c_0}\right)^{1/(1+\delta_1)}\,, \qquad |(Du)_{x_0,\rho}| < M/4\,, \\ &\fint_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|\, dx < B\,, \end{aligned} \tag{4.34}$$

where $B < 1$ and θ are such that denoting by $\tilde{c}_M$ the square of the constant appearing in Lemma 3.2

$$2^{(\gamma_2)^2} M \tilde{c}_M (B^{p_2\theta} + B^{2\theta}) < \eta/4\,, \qquad 1 = \theta + (1-\theta)/(1+\delta_1/4)\,.$$

We may also suppose that R is so small that

$$(4R)^\beta < \eta/4\,,$$

where $\eta = \eta(M)$ is obtained from step 1. Using theorem 3.1 and (4.2)

$$\begin{aligned} (|Du|^{p_2(1+\delta_1/4)})_{x_0,\rho} &\leq (|Du|^{p(x)(1+\delta_1)} + 1)_{x_0,\rho} \\ &\leq c_0(|Du|^{p(x)} + 1)_{x_0,2\rho}^{(1+\delta_1)} \\ &\leq 2c_0(|Du|^{p(x)})_{x_0,2\rho}^{(1+\delta_1)} + 2c_0 \leq M/2\,. \end{aligned}$$

Then we interpolate p_2 between 1 and $p_2(1+\delta_1/4)$, using also (4.1) to estimate $p_2(1+\delta_1) \le (\gamma_2)^2$:

$$\begin{aligned} &\fint_{B(x_0,\rho)} |Du-(Du)_{x_0,\rho}|^{p_2}\,dx \\ &\quad\le \Big(\fint_{B(x_0,\rho)} |Du-(Du)_{x_0,\rho}|\,dx\Big)^{\theta p_2}\cdot \\ &\quad\cdot\Big(\fint_{B(x_0,\rho)} |Du-(Du)_{x_0,\rho}|^{p_2(1+\delta_1/4)}\,dx\Big)^{(1-\theta)/(1+\delta_1/4)} \\ &\quad\le B^{\theta p_2}2^{(\gamma_2)^2}(|Du|^{p_2(1+\delta_1/4)})_{x_0,\rho} \le 2^{(\gamma_2)^2}MB^{\theta p_2}\,. \end{aligned}$$

We are finally able to estimate, using (d) from lemma 3.2,

$$\begin{aligned} E(x_0,\rho) &= \fint_{B(x_0,\rho)} |V_{p_2}(Du)-V_{p_2}\big((Du)_{x_0,\rho}\big)|^2\,dx+\rho^\beta \\ &\le \tilde{c}_M \fint_{B(x_0,\rho)} |Du-(Du)_{x_0,\rho}|^{p_2}\,dx \\ &\quad+\frac{(p_2-2)\vee 0}{p_2-2}\Big(\tilde{c}_M\fint_{B(x_0,\rho)} |Du-(Du)_{x_0,\rho}|^{p_2}\,dx\Big)^{2/p_2}+\rho^\beta \\ &\le 2^{(\gamma_2)^2}M\tilde{c}_M[B^{p_2\theta}+B^{2\theta}]+\rho^\beta \le \eta/2<\eta \end{aligned}$$

so that also the second inequality in (4.32) holds.
Step 3: Localization and conclusion. We show that the set Ω_0 is actually open, and that if $x_0 \in \Omega_0$ and (4.32) holds for a suitable M then

$$\fint_{B(x_0,\rho)} |Du-(Du)_{x_0,\rho}|\,dx \le C_M\rho^{\bar{\beta}/4} \tag{4.35}$$

for any $0<\rho\le\tilde{R}_M$. From (4.35) the assertion of theorem 2.1 immediately follows (see [AF2],[E],[CFM]) using Campanato's integral characterization of Hölder continuity, via a standard covering argument.
Take $x_0\in\Omega_0$ and fix M such that

$$\limsup_{\rho\to 0}\,(|Du|^{p(x)})_{x_0,2\rho} \le \frac{1}{2}\Big(\frac{M}{8c_0}\Big)^{1/(1+\delta_1)}\;;$$

let R_M be as before proposition 4.2, and take $R<R_M/32$ such that (4.34) are verified: we put $\mathcal{O}_M=B(x_0,R_M)$. At this point, by step 2 we may apply step 1 to obtain (4.33), and we are ready to prove (4.35). A simple interpolation shows that it suffices to prove (4.41) only for the numbers ρ of the type $\rho=\tau^kR$, to which case we specialize henceforth. Starting from (4.33) if $p_2\ge 2$ then by (d) from lemma 3.2

$$\begin{aligned} \fint_{B(x_0,\rho)} |Du-(Du)_{x_0,\rho}|\,dx &\le \Big(\fint_{B(x_0,\rho)} |Du-(Du)_{x_0,\rho}|^2\,dx\Big)^{1/2} \\ &\le C_ME(x_0,\rho)^{1/2} \overset{(4.33)}{\le} C_M\rho^{\bar{\beta}/4}\,. \end{aligned}$$

If $1<p_2<2$ we estimate, again using (d) from lemma 2.1 and setting $S:=\{|Du-(Du)_{x_0,\rho}|\ge 1\}$:

$$\fint_{B(x_0,\rho)} |Du-(Du)_{x_0,\rho}|\,dx=\omega_n\rho^{-n}\Big[\int_{B_\rho\cap S}|\dots|\,dx+\int_{B_\rho\setminus S}|\dots|\,dx\Big]$$

$$\leq \omega_n \rho^{-n} \int_{B_\rho \cap S} |Du - (Du)_{x_0,\rho}|^{p_2}\, dx + c \fint_{B_\rho} |V_{p_2}(Du - (Du)_{x_0,\rho})|\, dx$$

$$\leq c \fint_{B_\rho} |V_{p_2}(Du - (Du)_{x_0,\rho})|^2\, dx + \left(c \fint_{B_\rho} |V_{p_2}(Du - (Du)_{x_0,\rho})|^2\, dx \right)^{1/2}$$

$$\leq C_M \left[E(x_0,\rho) + E(x_0,\rho)^{1/2} \right] \leq C_M E(x_0,\rho)^{1/2} \overset{(4.33)}{\leq} C_M \rho^{\bar{\beta}/4}\,,$$

and (4.35) is proved. Finally we observe that inequalities (4.34) hold not only in x_0 but for every x_1 in a small ball centered in x_0, and for which we may suppose that $B(x_1, 16R) \subset \mathcal{O}_M = B(x_0, R_M)$. This implies the assertion. □

Remark 4.2. The technical modification to the iteration scheme of [FH] is due to the necessity of checking the behaviour of the two quantities $E(x_0, 4\tau^k R)$ and $|(Du)_{x_0,4\tau^k R}|$; this is done by using the last inequalities above to prove that

$$E(x_0, 4\tau^{k+1}R) \leq C_M E(x_0, \tau^k R)\,.$$

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