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Homogenization of Noncoercive Functionals: Periodic Materials with Soft Inclusions

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Abstract. In this paper we study the asymptotic behavior, as $h\to\infty$, of the minimum points of the functionals

$$\int [f(hx, Du)+gu]\, dx,$$

where $f(x, \xi)$ is periodic in x and convex in ξ, and u is vector valued. A convergence theorem is stated without uniform coerciveness assumptions.

1. Introduction

The classical homogenization problem is the study of the behavior, as $h\to\infty$, of the minimum points on $u_0+W_0^{1,p}$ of the functionals

$$\int_\Omega [f(hx, Du)+gu]\, dx, \tag{1.1}$$

where $f(x, \xi)$ is periodic in x and convex in ξ. Many convergence results have been obtained in the scalar case $u\colon \Omega\to\mathbb{R}$ (see the extensive bibliography of [2]).

If the function f satisfies

$$f(x, \xi)\geq |\xi|^p, \tag{1.2}$$

then the scalar results may be extended to the many-dimensional case $u\colon\Omega\to\mathbb{R}^n$. Without condition (1.2), however, some convergence results for the minimum points of (1.1) have been given, in [1], [3], and [7] only in the scalar case. We deal with a many-dimensional case, in which, in addition, the function f depends on Du through the strain tensor $e(u)=(Du+{}^tDu)/2$.

Consider a foamlike periodic structure Y made of an elastic material with holes filled by a softer material. If B denotes the union of the holes, the elastic energy is given by

$$E(u)=\int_{Y\setminus B} f(e(u))\,dx+\varepsilon\int_B f(e(u))\,dx,$$

where f is a positive-definite quadratic form of the strain tensor. We study the case when both the period $1/h$ of the structure and the Young modulus ε_h of the filling material go to zero. Then, denoting the holes again by B_h, the energy is

$$E_h(u)=\int_Y f(e(u))[\mathbb{1}_{Y\setminus B_h}(x)+\varepsilon_h\mathbb{1}_{B_h}(x)]\,dx.$$

If ε_h is not too small ($\lim h^2\varepsilon_h=+\infty$), the solutions of

$$\min\left\{E_h(u)+\int_Y gu\,dx\colon u-u_0\in H_0^1(Y;\mathbb{R}^n)\right\}$$

converge in $L^2(Y)$ to the solution of

$$\min\left\{E_\infty(u)+\int_Y gu\,dx\colon u-u_0\in H_0^1(Y;\mathbb{R}^n)\right\},$$

where the homogenized functional E_∞ is an integral:

$$E_\infty(u)=\int_Y \bar{f}(e(u))\,dx$$

with $\bar{f}$ of the same type of f.

Our result (Theorem 2.2) also holds when both materials are allowed to be inhomogeneous (f depends on x), and f need not be a quadratic form in ξ, but a generic convex function such that

$$0\le f(x,\xi)\le c(1+|\xi|^p)\qquad (p>1)$$

with $f(x,\xi)\ge|\xi|^p$ only outside the holes. We remark that this general case leads to a fully nonlinear system of partial differential equations.

2. Statement of Results

In the following we denote by Y the cube $([0,1[)^n$; a function f is said to be Y-periodic if $f(x+z)=f(x)$ for all $z\in\mathbb{Z}^n$ and $x\in\mathbb{R}^n$. We denote by M_n the set of real $n\times n$ matrices, and by M_n^+, M_n^- the sets of symmetric and skew-symmetric matrices. By ${}^t\xi$ we denote the transpose of a matrix ξ. If $A\subseteq\mathbb{R}^n$, by $L^p(A;\mathbb{R}^n)$ we denote the space of functions whose n components belong to $L^p(A)$, and analogously for the spaces $W^{1,p}$ and $W_0^{1,p}$. For such functions we set

$$e(u)=(Du+{}^t(Du))/2.$$

We say that a sequence $(u_h) \subset L^p(A; \mathbb{R}^n)$ converges to u in $L_0^p(A; \mathbb{R}^n)$ if $u_h \to u$ in $L^p(A; \mathbb{R}^n)$ and $\operatorname{spt}(u_h - u)$ is compact in A for every h. Finally, we set

$$W(Y) = \{u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n): u \text{ is } Y\text{-periodic}\}.$$

Let B be an open subset of Y, with Lipschitz boundary and well contained in Y, i.e.

$$\operatorname{dist}(B, \partial Y) > 0. \tag{2.1}$$

Fix $p > 1$ and let $f: \mathbb{R}^n \times M_n^+ \to \mathbb{R}$ satisfy

$$f(\cdot, \xi) \text{ is measurable}, \qquad f(x, \cdot) \text{ is convex}, \tag{2.2}$$

$$0 \le f(x, \xi) \le c(1 + |\xi|^p), \qquad \text{with} \quad c \ge 1, \tag{2.3}$$

$$f(x, \xi) \ge |\xi|^p \qquad \text{if} \quad x \in Y \setminus B. \tag{2.4}$$

We set, for every $\xi \in M_n^+$,

$$\bar{f}(\xi) = \inf\left\{ \int_Y f(x, e(u))\, dx: u - \xi x \in W(Y) \right\}. \tag{2.5}$$

In addition, fix $g \in L^q(Y; \mathbb{R}^n)$, with $p^{-1} + q^{-1} = 1$. We define, for all $h \in \mathbb{N}$ and $u \in L^p(Y; \mathbb{R}^n)$,

$$F_h(u) = \begin{cases} \displaystyle\int_Y f(hx, e(u))\, dx & \text{if } u \in W^{1,p}(Y; \mathbb{R}^n), \\ +\infty & \text{otherwise}, \end{cases}$$

and, for all $u \in W^{1,p}(Y; \mathbb{R}^n)$, we set

$$F(u) = \int_Y \bar{f}(e(u))\, dx.$$

Finally, let (ε_h) be a sequence of nonnegative numbers such that

$$\lim_h \varepsilon_h = 0, \tag{2.6}$$

$$\lim_h h^p \varepsilon_h = +\infty. \tag{2.7}$$

We prove the following theorems in Section 3.

Theorem 2.1. *Assume only* (2.2), (2.3), *and* (2.6) *hold. Then, for every open set $A \subseteq Y$ and every $u \in W^{1,p}(A; \mathbb{R}^n)$,*

$$\int_A \bar{f}(e(u))\, dx = \Gamma^-(L^p(A; \mathbb{R}^n)) \lim_h \int_A [f(hx, e(u)) + \varepsilon_h |e(u)|^p]\, dx$$

$$= \Gamma^-(L_0^p(A; \mathbb{R}^n)) \lim_h \int_A [f(hx, e(u)) + \varepsilon_h |e(u)|^p]\, dx.$$

Theorem 2.2. *Assume* (2.1)-(2.7) *hold. Then, for every* $u \in W^{1,p}(Y;\mathbb{R}^n)$,

$$\min\left\{F(u)+\int_Y gu\,dx\colon u-u_0\in W_0^{1,p}(Y;\mathbb{R}^n)\right\}$$

$$=\lim_h \min\left\{F_h(u)+\int_Y [gu+\varepsilon_h|e(u)|^p]\,dx\colon u-u_0\in W_0^{1,p}(Y;\mathbb{R}^n)\right\}.$$

Moreover, the minimum points on $u_0+W_0^{1,p}(Y;\mathbb{R}^n)$ *of the functionals* $F_h(u)+\int_Y [gu+\varepsilon_h|e(u)|^p]\,dx$ *converge in* $L^p(Y;\mathbb{R}^n)$ *to the minimum point of* $F(u)+\int_Y gu\,dx$.

Theorem 2.3. *Let* f *be as above. Then the function* $\bar{f}$ *is convex and satisfies, for every* $\xi \in M_n^+$,

$$\frac{1}{c}|\xi|^p \le \bar{f}(\xi) \le c(1+|\xi|^p).$$

If, in addition, the function f *is p-homogeneous with respect to* ξ, *then the same is true for* $\bar{f}$. *In the case* $p=2$, *if* f *is a quadratic form in* ξ *so is* $\bar{f}$.

Theorem 2.2 may be deduced from Theorem 2.1 through the theory of Γ-limits, whose definition and main properties will be given hereafter.

Let X be a metric space, and F_h, F functionals from X to $\bar{\mathbb{R}}$. We state that

$$F(x)=\Gamma^-(X)\lim_h F_h(x)$$

if the following conditions hold:

$$F(x)\le \liminf_h F_h(x_h) \qquad \text{for every} \quad x_h\to x, \tag{2.8}$$

$$\text{there exists} \quad \bar{x}_h\to x \quad \text{such that} \quad F(x)=\lim_h F_h(\bar{x}_h). \tag{2.9}$$

The sequence (F_h) is equicoercive if, for every $c\in\mathbb{R}$, we can select from every sequence (x_h), such that $F_h(x_h)\le c$, a subsequence (x_{h_k}) which converges to some $x\in X$.

Proposition 2.4 (see Theorem 2.6 of [4]). *If the sequence* (F_h) *is equicoercive and* $F=\Gamma^-(X)\ \lim_h F_h$ *then* F *has a minimum on* X *and* $\min F=\lim_h(\inf F_h)$. *Moreover, if* $\lim_h F_h(x_h)=\lim_h(\inf F_h)$ *and* $x_h\to x$ *in* X *then* x *is a minimum point for* F.

Proposition 2.5 (see Theorem 2.3 of [4]). *If* $F=\Gamma^-(X)\lim_h F_h$ *and* $G\colon X\to\mathbb{R}$ *is continuous, then* $F+G=\Gamma^-(X)\lim_h(F_h+G)$.

The existence of the minima above, and the convergence of the minimum points, depend on the following Korn-type inequalities, which are essentially contained in [5]. Let $p>1$, and let Ω be a connected open subset of $\mathbb{R}^n$ with Lipschitz boundary.

Proposition 2.6. *If $u \in W^{1,p}(\Omega; \mathbb{R}^n)$ satisfies $e(u) = 0$ in Ω, then there exist $a \in \mathbb{R}^n$ and $\xi \in M_n^-$ such that $u(x) = \xi x + a$ in Ω.*

The affine functions with skew-symmetric gradient are called rigid displacements, and, for any $u \in L^p(\Omega; \mathbb{R}^n)$, we denote by $R_\Omega u$ its projection on the subspace of rigid displacements of Ω.

Proposition 2.7. *There exist two constants $c(\Omega)$ and $c'(\Omega)$ such that*

$$\int_\Omega (|u|^p + |Du|^p)\, dx \leq c(\Omega) \int_\Omega (|u|^p + |e(u)|^p)\, dx \quad \text{for all} \quad u \in W^{1,p}(\Omega; \mathbb{R}^n),$$

$$\int_\Omega |u|^p\, dx \leq c'(\Omega) \int_\Omega |e(u)|^p\, dx \quad \text{for all} \quad u \in W_0^{1,p}(\Omega; \mathbb{R}^n),$$

$$\int_\Omega |u - R_\Omega u|^p\, dx \leq c'(\Omega) \int_\Omega |e(u)|^p\, dx \quad \text{for all} \quad u \in W^{1,p}(\Omega; \mathbb{R}^n).$$

Moreover, $c'(\alpha\Omega) = \alpha^p c'(\Omega)$ for every $\alpha > 0$.

3. Proof of Results

In the following, if no confusion is possible, we will denote all positive constants by the same letter c, and, except in the statements, we will simply write $L^p(A)$ instead of $L^p(A; \mathbb{R}^n)$, and the same for L_0^p, $W^{1,p}$, and $W_0^{1,p}$.

3.1. *Proof of Theorem 2.1*

First, we need a compactness result with respect to Γ-convergence.

Theorem 3.1. *Let $p \geq 1$ and let $f_h: \mathbb{R}^n \times M_n \to \mathbb{R}$ satisfy*

$$f_h(\cdot, \xi) \text{ is measurable,} \qquad f_h(x, \cdot) \text{ is convex,}$$

$$0 \leq f_h(x, \xi) \leq c(1 + |\xi|^p).$$

For every open set A and every $u \in L^p(A; \mathbb{R}^n)$ set

$$F_h(u, A) = \begin{cases} \displaystyle\int_A f_h(x, Du)\, dx & \text{if } u \in W^{1,p}(A; \mathbb{R}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

Then there exists a function $\varphi: \mathbb{R}^n \times M_n \to \mathbb{R}$ satisfying the same conditions as (f_h), and a subsequence (f_{h_k}), such that for every bounded open set A and every $u \in W^{1,p}(A; \mathbb{R}^n)$

$$\int_A \varphi(x, Du)\, dx = \Gamma^-(L^p(A; \mathbb{R}^n)) \lim_k F_{h_k}(u, A)$$

$$= \Gamma^-(L_0^p(A; \mathbb{R}^n)) \lim_k F_{h_k}(u, A).$$

The proof is standard in the theory of Γ-convergence; for example, it is similar to Theorem 3.2 of [2] and we omit it. If f satisfies (2.2) and (2.3) then the functions $f_h(x, \xi) = f(hx, (\xi + {}^t\xi)/2)$ satisfy the assumptions of Theorem 3.1, hence for a suitable subsequence we have

$$\begin{aligned}\int_A \varphi(x, Du)\,dx &= \Gamma^-(L^p(A)) \lim_k F_{h_k}(u, A)\\ &= \Gamma^-(L^p_0(A)) \lim_k F_{h_k}(u, A)\end{aligned}$$

for every open set $A \subseteq Y$ and every $u \in W^{1,p}(A)$; the function $\varphi(x, \xi)$ is measurable in x and convex in ξ, and satisfies

$$0 \le \varphi(x, \xi) \le c(1 + |\xi|^p).$$

Since the period Y/h_k of the integrand f_{h_k} vanishes as $k \to \infty$, one may expect:

Proposition 3.2. *The function φ is independent of x.*

The proof is the same as Lemma 4.2 of [6]. Also, since the integrand f_{h_k} depends only on $e(u)$, it is not surprising that the same is true of φ; this depends on Lemma 3.3.

Lemma 3.3. *Let ψ be a real convex function on a vector space V. If $\psi(t\bar{x}) = 0$ for every $t \in \mathbb{R}$ then*

$$\psi(x) = \psi(x + t\bar{x}) \qquad \text{for all} \quad t \in \mathbb{R} \quad \text{and} \quad x \in V.$$

Proof. For all $s > 1$,

$$\psi(x + t\bar{x}) = \psi\left(\left(1 - \frac{1}{s}\right)\frac{s}{s-1}\,x + \frac{1}{s}\,st\bar{x}\right) \le \left(1 - \frac{1}{s}\right)\psi\left(\frac{s}{s-1}\,x\right);$$

taking the limit as $s \to +\infty$, by the continuity properties of ψ,

$$\psi(x + t\bar{x}) \le \psi(x)$$

for all $t \in \mathbb{R}$, hence the convex function $t \to \psi(x + t\bar{x})$ is constant. □

Proposition 3.4. *The function $\varphi(\xi)$ depends only on $(\xi + {}^t\xi)/2$.*

Proof. By (2.8), for every $\eta \in M_n^-$,

$$0 \le \varphi(\eta) \le \liminf_k \int_Y f(h_k x, \eta)\,dx = 0,$$

hence, by Lemma 3.3, for every $\xi \in M_n$,

$$\varphi(\xi) = \varphi\left(\xi + \frac{{}^t\xi - \xi}{2}\right) = \varphi\left(\frac{\xi + {}^t\xi}{2}\right).$$ □

We prove a representation formula for φ.

Proposition 3.5. *The function* φ *is equal to* $\bar{f}$ *of* (2.5), *and the subsequence* (F_{h_k}) *is the whole sequence* (F_h).

Proof. As in Proposition 2.6 of [7] one proves that

$$\inf\{F_h(u) \colon u - \xi x \in W(Y)\} = \bar{f}(\xi).$$

Then, by the $\Gamma^-(L_0^p(A))$ convergence,

$$\begin{aligned}
\varphi(\xi) &= \min\left\{\liminf_k F_{h_k}(u_{h_k}) \colon u_{h_k} - \xi x \in W_0^{1,p}(Y),\ u_{h_k} \to \xi x \text{ in } L_0^p(Y)\right\} \\
&\geq \inf_k \inf\{F_{h_k}(u) \colon u - \xi x \in W_0^{1,p}(Y)\} \\
&\geq \inf_k \inf\{F_{h_k}(u) \colon u - \xi x \in W(Y)\} \\
&= \bar{f}(\xi).
\end{aligned}$$

On the other hand, let $u \in \xi x + W(Y)$ and set

$$u_h(x) = \frac{1}{h} u(hx).$$

Then $u_h \in \xi x + W(Y)$ and $u_h \to \xi x$ in $L^p(Y)$, and $F_h(u_h) = F_1(u)$; by the $\Gamma^-(L^p(Y))$ convergence we have

$$\varphi(\xi) \leq \liminf_k F_{h_k}(u_{h_k}) = F_1(u),$$

whence $\varphi(\xi) \leq \bar{f}(\xi)$ since u is arbitrary. This proves the equality $\varphi = \bar{f}$, and the second assertion follows from the previous results applied to any subsequence of (F_h). □

Again following the scheme of [7], fix $\varepsilon > 0$ and apply the results above to the functionals

$$F_h^\varepsilon(u, A) = \begin{cases} \displaystyle\int_A [f(hx, e(u)) + \varepsilon |e(u)|^p]\, dx & \text{if } u \in W^{1,p}(A), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, for every $u \in W^{1,p}(A)$,

$$\Gamma^-(L^p(A)) \lim_h F_h^\varepsilon(u, A) = \Gamma^-(L_0^p(A)) \lim_h F_h^\varepsilon(u, A) = \int_A \bar{f}_\varepsilon(e(u))\, dx,$$

where

$$\bar{f}_\varepsilon(\xi) = \inf\left\{\int_Y [f(x, e(u)) + \varepsilon |e(u)|^p]\, dx \colon u - \xi x \in W(Y)\right\}. \tag{3.1}$$

We may now conclude the proof of Theorem 2.1: we have to prove (2.8) for $L^p(A)$ and (2.9) for $L_0^p(A)$; if h is large enough we have

$$F_h(u, A) \le \int_A [f(hx, e(u)) + \varepsilon_h |e(u)|^p] \, dx \le F_h^\varepsilon(u, A).$$

Then, for every $u_h \to u$ in $L^p(A)$,

$$\liminf_h \int_A [f(hx, e(u_h)) + \varepsilon_h |e(u_h)|^p] \, dx \ge \liminf_h F_h(u_h, A)$$

$$\ge \int_A \bar{f}(e(u)) \, dx, \tag{3.2}$$

while there exists a sequence $u_h^\varepsilon \to u$ in $L_0^p(A)$ such that

$$\limsup_h \int_A [f(hx, e(u_h^\varepsilon)) + \varepsilon_h |e(u_h^\varepsilon)|^p] \, dx \le \lim_h F_h^\varepsilon(u_h^\varepsilon, A)$$

$$= \int_A \bar{f}_\varepsilon(e(u)) \, dx. \tag{3.3}$$

By (2.5) and (3.1) it follows that $(\bar{f}_\varepsilon)$ is decreasing to $\bar{f}$, therefore $\int_A \bar{f}(e(u)) \, dx = \lim_\varepsilon \int_A \bar{f}_\varepsilon(e(u)) \, dx$ by the dominated convergence theorem. Then, by (3.2) and (3.3), we may choose a sequence $u_h \to u$ in $L_0^p(A)$ such that

$$\int_A \bar{f}(e(u)) \, dx = \lim_h \int_A [f(hx, e(u_h)) + \varepsilon_h |e(u_h)|^p] \, dx,$$

thus proving the assertion. □

3.2. Proof of Theorem 2.2

By Proposition 2.5, the same Γ-convergence result holds if we add $G(u) = \int_A gu \, dx$ to the sequence and to the limit functional. To apply Proposition 2.4, thus concluding the proof of Theorem 2.2, we still have to prove the equicoerciveness condition. To this aim we need an extension lemma.

Lemma 3.6. *Let $p > 1$ and let Ω, ω be bounded open subsets of $\mathbb{R}^n$ with Lipschitz boundary, such that $\omega \subseteq \Omega$. Then there exists a constant $c(\Omega, \omega)$ such that for every $u \in W^{1,p}(\omega; \mathbb{R}^n)$ there exists $\tilde{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$ such that*

$$\tilde{u} = u \quad \text{in } \omega,$$

$$\int_\Omega |e(\tilde{u})|^p \, dx \le c(\Omega, \omega) \int_\omega |e(u)|^p \, dx.$$

Moreover, $c(t\Omega, t\omega) = c(\Omega, \omega)$ for every $t > 0$.

Proof. Denote by $R_\omega u$ the projection of u on the rigid displacements of ω. Then the function $v = u - R_\omega u$ belongs to $W^{1,p}(\omega)$ and satisfies

$$e(v) = e(u) \quad \text{in } \omega.$$

Moreover, by Proposition 2.7,

$$\|v\|^p_{W^{1,p}(\omega)} \le c \int_\omega (|v|^p + |e(v)|^p)\,dx$$
$$= c \int_\omega (|u - R_\omega u|^p + |e(u)|^p)\,dx \le c \int_\omega |e(u)|^p\,dx.$$

By the regularity of ω there exists an extension $\tilde{v}$ of v such that

$$\|\tilde{v}\|_{W^{1,p}(\Omega)} \le c\|v\|_{W^{1,p}(\omega)},$$

hence

$$\int_\Omega |e(\tilde{v})|^p\,dx \le c \int_\Omega |D\tilde{v}|^p\,dx$$
$$\le c\|v\|^p_{W^{1,p}(\omega)} \le c \int_\omega |e(u)|^p\,dx. \tag{3.4}$$

On the other hand, the function $R_\omega u$ is affine, so it is defined on $\mathbb{R}^n$, and we may set, in Ω,

$$\tilde{u} = R_\omega u + \tilde{v},$$

thus obtaining

$$\tilde{u} = u \quad \text{in } \omega,$$

$$\int_\Omega |e(\tilde{u})|^p\,dx = \int_\Omega |e(\tilde{v})|^p\,dx \le c \int_\omega |e(u)|^p\,dx$$

by (3.4).

The invariance of $c(\Omega, \omega)$ is easy. □

Proposition 3.7. *Let* $u_0 \in W^{1,p}(Y; \mathbb{R}^n)$; *the functionals* $\int_Y [f(hx, e(u)) + \varepsilon_h |e(u)|^p + gu]\,dx$ *are equicoercive on* $u_0 + W_0^{1,p}(Y; \mathbb{R}^n)$.

Proof. For every $\alpha \in \mathbb{N}^n$ such that $0 \le \alpha_i < h$ for every i, we set

$$Y_{h,\alpha} = \frac{1}{h}(Y + \alpha), \qquad B_{h,\alpha} = \frac{1}{h}(B + \alpha),$$

and

$$B_h = \bigcup_\alpha B_{h,\alpha}.$$

Assume

$$\int_Y [f(hx, e(u_h)) + \varepsilon_h |e(u_h)|^p + gu_h]\,dx \le c.$$

Then, for any $s>0$ (which we will choose later), we have, by (2.4),

$$\int_{Y\setminus B_h} |e(u_h)|^p\,dx + \varepsilon_h \int_{B_h} |e(u_h)|^p\,dx \le c + \|g\|_{L^q(Y)}\|u_h\|_{L^p(Y)}$$
$$\le c + s\|u_h\|^p_{L^p(Y)}, \tag{3.5}$$

where the last constant c depends on s.

Apply Lemma 3.6 to u_h in each of the cubes $Y_{h,\alpha}$, with $\omega = Y_{h,\alpha}\setminus B_{h,\alpha}$ and $\Omega = Y_{h,\alpha}$, and call $\tilde{u}_h$ the function thus obtained. By (2.1), the function $\tilde{u}_h$ belongs to $u_0 + W_0^{1,p}(Y)$, and for every α we have $u_h - \tilde{u}_h \in W_0^{1,p}(B_{h,\alpha})$, therefore, by Proposition 2.7,

$$\int_{B_{h,\alpha}} |u_h - \tilde{u}_h|^p\,dx \le \frac{c}{h^p}\int_{B_{h,\alpha}} |e(u_h) - e(\tilde{u}_h)|^p\,dx$$
$$\le \frac{c}{h^p}\int_{B_{h,\alpha}} [|e(u_h)|^p + |e(\tilde{u}_h)|^p]\,dx. \tag{3.6}$$

Then

$$\int_Y |u_h|^p\,dx \le c\left(\int_Y |\tilde{u}_h|^p\,dx + \int_{B_h} |u_h - \tilde{u}_h|^p\,dx\right)$$
$$\le c\left(\int_Y |\tilde{u}_h|^p\,dx + \frac{1}{h^p}\int_{B_h} |e(u_h)|^p\,dx + \int_{B_h} |e(\tilde{u}_h)|^p\,dx\right).$$

Since $\tilde{u}_h - u_0 \in W_0^{1,p}(Y)$, by Proposition 2.7,

$$\int_Y |\tilde{u}_h|^p\,dx \le c\left(\int_Y (|u_0|^p + |e(u_0)|^p + |e(\tilde{u}_h)|^p)\,dx\right)$$
$$\le c + c\int_Y |e(\tilde{u}_h)|^p\,dx, \tag{3.7}$$

hence, by Lemma 3.6 and using (2.7), we have, if h is sufficiently large,

$$\int_Y |u_h|^p\,dx \le c\left(1 + \int_Y |e(\tilde{u}_h)|^p\,dx + \frac{1}{h^p}\int_{B_h} |e(u_h)|^p\,dx\right)$$
$$\le c\left(1 + \int_{Y\setminus B_h} |e(u_h)|^p\,dx + \varepsilon_h \int_{B_h} |e(u_h)|^p\,dx\right), \tag{3.8}$$

with c independent of h. If s is properly chosen we obtain, from (3.5) and (3.8),

$$\int_{Y\setminus B_h} |e(u_h)|^p\,dx + \varepsilon_h \int_{B_h} |e(u_h)|^p\,dx \le c. \tag{3.9}$$

In particular, by Lemma 3.6 and by (3.7), we have

$$\int_Y |e(\tilde{u}_h)|^p\,dx \le c,$$
$$\int_Y |\tilde{u}_h|^p\,dx \le c, \tag{3.10}$$

therefore, by Proposition 2.7, the sequence $(\tilde{u}_h)$ is relatively compact in $L^p(Y)$. But, by (3.6), (3.9), and (3.10),

$$\int_Y |u_h - \tilde{u}_h|^p \, dx = \int_{B_h} |u_h - \tilde{u}_h|^p \, dx \leq \frac{c}{\varepsilon_h h^p},$$

and, by (2.7), (u_h) is also relatively compact in $L^p(Y)$. □

With this result, it is enough to apply Proposition 2.4 to prove Theorem 2.2. □

3.3. *Proof of Theorem 2.3*

The convexity of $\bar{f}$, the inequality $\bar{f}(\xi) \leq c(1+|\xi|^p)$, and the eventual p-homogeneity are obvious consequences of (2.5). If f is a quadratic form, for any $u \in \xi x + W(Y)$ and $v \in \eta x + W(Y)$,

$$f(x, e(u+v)) + f(x, e(u-v)) = 2f(x, e(u)) + 2f(x, e(v)),$$

therefore

$$\bar{f}(\xi+\eta) + \bar{f}(\xi-\eta) \leq 2\bar{f}(\xi) + 2\bar{f}(\eta). \tag{3.11}$$

On the other hand, if $u \in (\xi+\eta)x + W(Y)$ and $v \in (\xi-\eta)x + W(Y)$, we obtain

$$2\bar{f}(\xi) + 2\bar{f}(\eta) \leq \bar{f}(\xi+\eta) + \bar{f}(\xi-\eta),$$

which, together with (3.11), implies that $\bar{f}$ is a quadratic form. We only have to prove the coercivity of $\bar{f}$.

Let $u \in \xi x + W(Y)$; applying Lemma 3.6 to u with $\omega = Y \backslash B$ we obtain a function $\tilde{u}$ such that

$$\int_Y |e(\tilde{u})|^p \, dx \leq c \int_{Y \backslash B} |e(u)|^p \, dx$$

and, in addition,

$$\tilde{u} \in \xi x + W(Y),$$

since $\tilde{u} = u$ near the boundary of Y. In particular, we have $\int_Y (D\tilde{u} - \xi) \, dx = 0$, then, by the convexity of the function $|t|^p$,

$$\int_Y |e(\tilde{u})|^p \, dx \geq |\xi|^p + p|\xi|^{p-2}\xi \int_Y (e(\tilde{u}) - \xi) \, dx = |\xi|^p,$$

so that

$$\int_{Y \backslash B} f(e(u)) \, dx \geq \int_{Y \backslash B} |e(u)|^p \, dx \geq c \int_Y |e(\tilde{u})|^p \, dx \geq c|\xi|^p,$$

and the conclusion follows by taking the infimum on $u \in \xi x + W(Y)$. □

Remark 3.8. All the results are still valid if instead of $\varepsilon_h \int_Y |e(u)|^p \, dx$ we add $\varepsilon_h \int_Y f(x, e(u)) \, dx$ to the functionals F_h, with

$$|\xi|^p \leq f(x, \xi) \leq c(1+|\xi|^p).$$

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