Reinforcement of Plates in Hencky's Plasticity

E. Acerbi & D. Percivale

1. Introduction

Though the theory of elastic plates constitutes a good model to describe the deformations of thin two-dimensional bodies loaded perpendicularly to their middle plane, the hypothesis of absolute elasticity is no longer valid when, under more severe conditions of loading, the material becomes plastic.

In this paper we investigate the behaviour of a clamped plastic plate Ω surrounded by a narrow annulus of a different, softer plastic material. In particular, we are interested in the behaviour of the equilibrium solutions to this problem when both the width of the annulus approaches zero and the surrounding material becomes softer and softer.

Throughout the paper we deliberately keep to a particular case, leaving the generalizations to the last section. The strain energy is of the form

$$F_\varepsilon(u) = \int_\Omega [|D^2u| + |Du| + |u| + lu]\,dx + \varepsilon \int_{\Sigma_\varepsilon} |D^2u|\,dx,$$

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where Σ_ε is the annulus of width ε surrounding Ω, and l is a suitable body force.

We prove that the minimum points of F_ε converge in some sense to the minimum point of the limit functional

$$F_0(u) = \int_\Omega [|D^2u| + |Du| + |u| + lu]\,dx + 2\int_{\partial\Omega} |u|\,d\mathcal{H}_{n-1}(\sigma).$$

This work falls within the framework of the so-called reinforcement problems, of which some exemples may be found e.g. in [1], elastic plates, and [2], the annulus does not surround all of Ω.

2. Notation and statement

In the sequel we denote by Ω a bounded open subset of $\mathbb{R}^n$ with smooth boundary Σ, by $\nu(\sigma)$ its outward unit normal vector at the point $\sigma \in \Sigma$, and we define

$$\Sigma_\varepsilon = \{\sigma + t\nu(\sigma) : 0 < t < \varepsilon, \sigma \in \Sigma\}$$

$$\Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon.$$

When ε is small enough (which we suppose henceforth) the mapping $(\sigma, t) \mapsto \sigma + t\nu(\sigma)$ is invertible on Σ_ε, so we may freely speak of $\nu(x)$ inside Σ_ε, meaning $\nu\big(\sigma(x)\big)$.

For every function u possessing a distributional gradient Du in Σ_ε, we define the normal and tangential derivatives

$$d_\nu u = \langle Du, \nu\rangle, \qquad \delta u = Du - d_\nu u\, \nu,$$

where $\langle\ ,\ \rangle$ denotes the scalar product between two vectors in $\mathbb{R}^n$. For every open subset $A \subset \mathbb{R}^n$ we define

$$HB(A) = \{u \in W^{1,1}(A) : D^2u \text{ is a Radon measure}\},$$

the functions with bounded Hessian matrix, the properties of which are illustrated, for instance, in [4], chapter 3, section 2.3. We only remark that for functions of this class $d_\nu u$ exists on ∂A.

If f is any convex function, we set

$$f_\infty(x) = \lim_{t\to+\infty} \frac{f(tx)}{t};$$

then for every Radon measure μ, whose absolutely continuous and singular part are denoted by μ_a and μ_s respectively, we may give sense to the integral of $f(\mu)$ as

$$\int f(\mu)\, dx = \int f(\mu_a)\, dx + \int f_\infty\left(\frac{d\mu_s}{d|\mu_s|}\right) d|\mu_s|.$$

As we said above, we confine ourselves to a very special integrand, that is, let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be an even, strictly convex function with linear growth,

$$|t| \le \varphi(t) \le c(1 + |t|),$$

define on $\mathbb{R}^{n\times n}$ the convex function

$$f(\xi) = \varphi(|\xi|),$$

and take a particular function $g \in C^0(\mathbb{R}^n)$; we introduce a functional on $L^1(\mathbb{R}^n)$ by setting

$$F_\varepsilon(u) = \int_\Omega [|u| + |Du| + f(D^2u) + lu]\,dx + \varepsilon \int_{\Omega_\varepsilon\setminus\Omega} f(D^2u)\,dx$$

when $u \in HB(\Omega_\varepsilon)$, $u = 0$ outside Ω_ε, $d_\nu u = g$ on $\partial\Omega_\varepsilon$, and

$$F_\varepsilon(u) = +\infty$$

otherwise. The function $l \in L^\infty(\Omega)$ must be so small that

$$\inf\{\int_\Omega |D^2u|\,dx : u \in HB(\Omega),\ u = 0 \text{ on } \Sigma,\ \int_\Omega lu\,dx = 1\} > 1 :$$

this condition ensures that $\inf F_\varepsilon > -\infty$, see [4], chapter 3. Moreover, we take $\|l\|_{L^\infty} < 1$.

It is worth to remark that the assumption on the linear growth of $f(\xi)$ is the correct mathematical form of the strain energy for thin plates obeying the Hencky criterion of plasticity.

We study the behaviour as $\varepsilon \to 0$ of the minimizing sequences of the functionals F_ε, i.e., of the sequences (u_ε) such that

$$\lim_{\varepsilon\to 0}[F_\varepsilon(u_\varepsilon) - \inf F_\varepsilon] = 0.$$

We remark that we cannot speak of the minimum points of F_ε on $HB(\Omega_\varepsilon)$, since F_ε is not lower semicontinuous with respect to the weak* topology of HB. Instead, the greatest lower semicontinuous functional which is less than F_ε is the following (see [4], chapter 3):

$$\begin{aligned}\overline{F_\varepsilon} = &\int_\Omega [|u| + |Du| + f(D^2u) + lu]\,dx \\ &+ \varepsilon \int_\Sigma f_\infty\big((d_\nu^+u - d_\nu^-u)\nu\otimes\nu\big)\,d\mathcal{H}_{n-1}(\sigma) \\ &+ \varepsilon \int_{\Sigma_\varepsilon} f(D^2u)\,dx \\ &+ \varepsilon \int_{\partial\Omega_\varepsilon} f_\infty\big((d_\nu^-u - g)\nu\otimes\nu\big)\,d\mathcal{H}_{n-1}(\sigma)\end{aligned}$$

when $u \in HB(\Omega_\varepsilon)$ with $u = 0$ outside Ω_ε, and

$$\overline{F_\varepsilon}(u) = +\infty$$

otherwise. In the expression of $\overline{F_\varepsilon}$, we have indicated by d_ν^- and d_ν^+ the normal derivatives from inside and from outside respectively. By our assumption on f, we have $f_\infty(\xi) = c_\infty|\xi|$, where $c_\infty = \lim_{t\to\infty} \varphi(t)/t$.

Fix a particular ε_0: then a minimizing sequence of F_{ε_0} has a subsequence which converges to a minimum point of $\overline{F_{\varepsilon_0}}$, thus we may study minimizing sequences of $(\overline{F_\varepsilon})_{\varepsilon>0}$ instead of $(F_\varepsilon)_{\varepsilon>0}$.

Theorem . *Every minimizing sequence (u_ε) of (F_ε) has a subsequence which converges in $L^1(\mathbb{R}^n)$ to a minimum point of the functional*

$$F_0(u) = \int_\Omega [|u| + |Du| + f(D^2u) + lu]\, dx + 2c_\infty \int_\Sigma |u| d\mathcal{H}_{n-1}(\sigma)$$

if $u \in BV(\mathbb{R}^n) \cap HB(\Omega)$ and $u = 0$ outside Ω, and $F_0(u) = +\infty$ otherwise.

PROOF . Let $(\overline{u_\varepsilon})$ be a minimizing sequence of $(\overline{F_\varepsilon})$: then

$$\overline{F_\varepsilon}(\overline{u_\varepsilon}) \le c, \tag{1}$$

from which we deduce in particular

$$\int_\Omega [(1 - \|l\|)|\overline{u_\varepsilon}| + |D\overline{u_\varepsilon}| + |D^2\overline{u_\varepsilon}|]\, dx \le c;$$

then (see [4]) there exists $u_0 \in HB(\Omega)$ such that for a subsequence of $(\overline{u_\varepsilon})$, which we denote by the same subscript ε, we have $\overline{u_\varepsilon} \rightharpoonup u_0$ in $HB(\Omega)$, and therefore $\overline{u_\varepsilon} \to u_0$ in $L^1(\Omega)$ and

$$\begin{cases} \delta\overline{u_\varepsilon} \rightharpoonup \delta u_0 & \text{in } L^1(\Sigma) \\ \int_\Sigma |d_\nu^- \overline{u_\varepsilon}| d\mathcal{H}_{n-1}(\sigma) \le c. & \end{cases} \tag{2}$$

Also, on Σ_ε

$$D\overline{u_\varepsilon}(\sigma + t\nu) = (D\overline{u_\varepsilon})^+(\sigma) + \int_0^t d_\nu(D\overline{u_\varepsilon})\, ds,$$

so that by (1) and (2)

$$\begin{aligned}\int_{\Sigma_\varepsilon} |D\overline{u_\varepsilon}|\,dx &\le \varepsilon \int_\Sigma |(D\overline{u_\varepsilon})^+|\,d\mathcal{H}_{n-1}(\sigma) + \varepsilon \int_{\Sigma_\varepsilon} |D^2\overline{u_\varepsilon}|\,dx \\ &\le \varepsilon \int_\Sigma (|\delta \overline{u_\varepsilon}| + |d_\nu^+\overline{u_\varepsilon} - d_\nu^-\overline{u_\varepsilon}| + |d_\nu^-\overline{u_\varepsilon}|)\,d\mathcal{H}_{n-1}(\sigma) \\ &\qquad + \varepsilon \int_{\Sigma_\varepsilon} |D^2\overline{u_\varepsilon}|\,dx.\end{aligned}$$

Then

$$\int_{\mathbb{R}^n} |D\overline{u_\varepsilon}|\,dx \le c$$

and therefore we may suppose that $\overline{u_\varepsilon} \rightharpoonup u_0$ in $BV(\mathbb{R}^n)$, that u_0 is in $HB(\Omega)$ and that $u_0 = 0$ outside Ω.

We now show that u_0 is a minimum point for F_0; to this aim, we first give a name to the sets where $\overline{F_\varepsilon}$ and F_0 are finite:

$$S_\varepsilon = \{v \in L^1(\mathbb{R}^n)\colon v \in HB(\Omega_\varepsilon), v = 0 \text{ outside } \Omega_\varepsilon\}$$

$$S_0 = \{v \in L^1(\mathbb{R}^n)\colon v \in HB(\Omega), v = 0 \text{ outside } \Omega\}.$$

We will prove:

Step 1 . *For every $u \in S_0$ there exists $u_\varepsilon \rightharpoonup u$ in $BV(\mathbb{R}^n)$ such that*

$$\limsup_\varepsilon \overline{F_\varepsilon}(u_\varepsilon) \le F_0(u).$$

Step 2 . *For every $u \in S_0$ and every $u_\varepsilon \rightharpoonup u$ in $BV(\mathbb{R}^n)$*

$$F_0(u) \le \liminf_\varepsilon \overline{F_\varepsilon}(u_\varepsilon).$$

Step 1 implies that $\inf F_0 \ge \limsup_\varepsilon(\inf \overline{F_\varepsilon})$, whereas step 2 implies $F_0(u_0) \le \liminf_\varepsilon \overline{F_\varepsilon}(\overline{u_\varepsilon}) = \liminf_\varepsilon(\inf \overline{F_\varepsilon})$, and these two propositions conclude the proof of the theorem.

Proof of step 1 . We suppose at first that $u \in C^\infty(\overline{\Omega})$. Set $d(x) = \operatorname{dist}(x, \overline{\Omega})$, and define

$$u_\varepsilon(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ u^-\big(\sigma(x)\big)\left(1 - \dfrac{d(x)}{\varepsilon}\right) & \text{if } x \in \overline{\Sigma_\varepsilon} \\ 0 & \text{if } x \notin \overline{\Omega_\varepsilon}. \end{cases}$$

Recalling that all derivatives of u and ν are bounded, we obtain after some computations

$$\begin{aligned} D^2 u_\varepsilon =& D^2 u\, \mathbf{1}_\Omega \\ & - (d_\nu^- u + \frac{u^-}{\varepsilon}) \nu \otimes \nu \, d\mathcal{H}_{n-1}(\Sigma) \\ & + \frac{\text{bounded terms}}{\varepsilon} \mathbf{1}_{\Sigma_\varepsilon} \\ & + \frac{u^-}{\varepsilon} \nu \otimes \nu \, d\mathcal{H}_{n-1}(\partial\Omega_\varepsilon), \end{aligned}$$

so that

$$\begin{aligned} \overline{F_\varepsilon}(u_\varepsilon) = & \int_\Omega [|u| + |Du| + f(D^2 u) + lu]\, dx \\ & + c_\infty \left(\int_\Sigma |u^-|\, d\mathcal{H}_{n-1}(\sigma) + \int_{\partial\Omega_\varepsilon} |u^-|\, d\mathcal{H}_{n-1}(\sigma) \right) + \omega_\varepsilon, \end{aligned}$$

where $\omega_\varepsilon \to 0$ as $\varepsilon \to 0$: then step 1 is proved in the case $u \in C^\infty(\overline{\Omega})$. Now, $C^\infty(\overline{\Omega})$ is dense in S_0 with respect to the intermediate topology of HB, i.e., the topology which induces the convergence

$$\int [|u_h - u| + |Du_h - Du| + (|D^2 u_h| - |D^2 u|)]\, dx \to 0.$$

Also, f_0 is continuous with respect to the intermediate topology, and a diagonal process proves step 1 in the general case. ■

PROOF OF STEP 2 . Without loss of generality we may assume $\overline{F_\varepsilon}(u_\varepsilon) \le c$. Then inside Σ_ε

$$D^2 u_\varepsilon = \delta\delta u_\varepsilon + \nu \otimes \delta(d_\nu u_\varepsilon) + d_\nu(\delta u_\varepsilon) \otimes \nu + d_\nu u_\varepsilon\, \delta\nu + (d_\nu d_\nu u_\varepsilon)\, \nu \otimes \nu.$$

Remark that $\nu \otimes \nu$ is orthogonal to all the terms of this sum except the last; since f is convex and radially symmetric,

$$f(D^2 u_\varepsilon) \ge f\big((d_\nu d_\nu u_\varepsilon)\, \nu \otimes \nu\big). \tag{3}$$

Then

$$\varepsilon \int_{\Sigma_\varepsilon} f(D^2 u_\varepsilon)\, dx + \varepsilon c_\infty \int_\Sigma |d_\nu^+ u_\varepsilon - d_\nu^- u_\varepsilon|\, d\mathcal{H}_{n-1}(\sigma)$$

$$
\begin{aligned}
&+\varepsilon c_\infty \int_{\partial\Omega_\varepsilon} |d_\nu^- u_\varepsilon - g| \, d\mathcal{H}_{n-1}(\sigma) \\
\geq \varepsilon \int_{\Sigma_\varepsilon} f\big((d_\nu d_\nu u_\varepsilon)\, \nu\otimes\nu\big)\, dx &+ \varepsilon c_\infty \int_\Sigma |d_\nu^+ u_\varepsilon| \, d\mathcal{H}_{n-1}(\sigma) \quad (4) \\
&+\varepsilon c_\infty \int_{\partial\Omega_\varepsilon} |d_\nu^- u_\varepsilon| \, d\mathcal{H}_{n-1}(\sigma) - \omega_\varepsilon
\end{aligned}
$$

with $\omega_\varepsilon \to 0$ as $\varepsilon \to 0$, since we may repeat here what we did to obtain (2) from (1). We may now consider for a. e. $\sigma \in \Sigma$ the problem

$$
\min\Big\{ \int_0^\varepsilon f\big(v''(t)\nu(\sigma)\otimes\nu(\sigma)\big)\, dt + c_\infty\big(|v'(0)| + |v'(\varepsilon)|\big) : \\
v(0) = u_\varepsilon^-(\sigma),\ v(\varepsilon) = 0,\ v \in HB(0,\varepsilon)\Big\}. \tag{5}
$$

Call $v_{\sigma,\varepsilon}$ its solution, and define in Σ_ε

$$
v_\varepsilon\big(\sigma + t\nu(\sigma)\big) = v_{\sigma,\varepsilon}(t).
$$

By the assumption on φ (the function used to define f), we have that φ' is an odd, strictly increasing function, whose inverse we call ψ. Also, it is not restrictive to assume φ to be of class C^∞, otherwise, we may set

$$
\varphi^\delta(t) = \frac{1}{\pi} \int \frac{\varphi(t-s)}{1+s^2}\, ds;
$$

then φ^δ is a very regular approximating sequence, we obtain the result on the related functionals F_ε^δ, then we pass to F_ε by a diagonal argument.

From the Euler equation of (5) we get

$$
v''_{\sigma,\varepsilon}(t) = \psi[a_\varepsilon(\sigma) + t b_\varepsilon(\sigma)],
$$

for suitable functions a_ε, b_ε, so that v is very regular with respect to t, and

$$
\varphi'\big(v''_{\sigma,\varepsilon}(0)\big) + c_\infty \frac{v'_{\sigma,\varepsilon}(0)}{|v'_{\sigma,\varepsilon}(0)|} = 0,
$$

$$
\varphi'\big(v''_{\sigma,\varepsilon}(\varepsilon)\big) - c_\infty \frac{v'_{\sigma,\varepsilon}(\varepsilon)}{|v'_{\sigma,\varepsilon}(\varepsilon)|} = 0.
$$

Precisely,

$$a_\varepsilon(\sigma) = -c_\infty \frac{v'_{\sigma,\varepsilon}(0)}{|v'_{\sigma,\varepsilon}(0)|},$$

$$\varepsilon b_\varepsilon(\sigma) = c_\infty \left(\frac{v'_{\sigma,\varepsilon}(0)}{|v'_{\sigma,\varepsilon}(0)|} + \frac{v'_{\sigma,\varepsilon}(\varepsilon)}{|v'_{\sigma,\varepsilon}(\varepsilon)|} \right),$$

so that

$$|v''_{\sigma,\varepsilon}(t)| \le \psi(3c_\infty).$$

Then from

$$0 = v_{\sigma,\varepsilon}(\varepsilon) = u_\varepsilon^-(\sigma) + \varepsilon v'_{\sigma,\varepsilon}(0) + \int_0^\varepsilon (\varepsilon - t) v''_{\sigma,\varepsilon}(t)\, dt$$

we deduce

$$\varepsilon v'_{\sigma,\varepsilon}(0) = -u_\varepsilon^-(\sigma) + \alpha_\varepsilon$$

with $|\alpha_\varepsilon| \le \frac{3}{2} c_\infty \varepsilon^2$. In addition, we have

$$v'_{\sigma,\varepsilon}(\varepsilon) = v'_{\sigma,\varepsilon}(0) + \beta_\varepsilon$$

with $|\beta_\varepsilon| \le 3c_\infty \varepsilon$, and hence

$$\varepsilon v'_{\sigma,\varepsilon}(\varepsilon) = -u_\varepsilon^-(\sigma) + \alpha_\varepsilon + \varepsilon\beta_\varepsilon.$$

Then by (4)

$$\begin{aligned}
\varepsilon \int_{\Sigma_\varepsilon} & f(D^2 u_\varepsilon)\, dx + c_\infty \int_\Sigma |d_\nu^+ u_\varepsilon - d_\nu^- u_\varepsilon|\, d\mathcal{H}_{n-1}(\sigma) \\
& + \varepsilon c_\infty \int_{\partial\Omega_\varepsilon} |d_\nu^- u_\varepsilon - g|\, d\mathcal{H}_{n-1}(\sigma) \\
& \quad \ge \varepsilon \int_{\Sigma_\varepsilon} f\big((d_\nu d_\nu v_\varepsilon)\, \nu \otimes \nu\big)\, dx + \varepsilon c_\infty \int_\Sigma |d_\nu^+ v_\varepsilon|\, d\mathcal{H}_{n-1}(\sigma) \\
& \qquad + \varepsilon c_\infty \int_{\partial\Omega_\varepsilon} |d_\nu^- v_\varepsilon|\, d\mathcal{H}_{n-1}(\sigma) - \omega_\varepsilon \\
& \quad \ge c\varepsilon^2 \min f + 2c_\infty \int_\Sigma |u_\varepsilon^-|\, d\mathcal{H}_{n-1}(\sigma) - \omega_\varepsilon.
\end{aligned}$$

Since in $\overline{F_\varepsilon}$ the integral over Ω is semicontinuous, letting $\varepsilon \to 0$ we prove step 2. ∎

3. Generalizations

If we studied the functionals

$$F_\varepsilon^\alpha(u) = \int_\Omega [|u| + |Du| + f(D^2u) + lu]\,dx + \varepsilon^\alpha \int_{\Sigma_\varepsilon} f(D^2u)\,dx,$$

with $\alpha > 0$, we would have obtained in the limit:
$\to$ if $\alpha < 1$,

$$F_0^\alpha(u) = \begin{cases} \int_\Omega [|u| + |Du| + f(D^2u) + lu]\,dx & \text{if } u \in HB(\Omega),\ u^- = 0 \text{ on } \Sigma \\ +\infty & \text{otherwise;} \end{cases}$$

$\to$ if $\alpha > 1$,

$$F_0^\alpha(u) = \begin{cases} \int_\Omega [|u| + |Du| + f(D^2u) + lu]\,dx & \text{if } u \in HB(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Another remark: the annulus Σ_ε need not be of uniform thickness. Take for instance a smooth function $h : \Sigma \to]0, +\infty[$, thus $0 < h_1 \le h(\sigma) \le h_2$, and let

$$\widetilde{\Sigma_\varepsilon} = \{\sigma + t\nu(\sigma) \colon 0 < t < \varepsilon h(\sigma)\}.$$

Substituting $\widetilde{\Sigma_\varepsilon}$ for Σ_ε, we obtain the limit

$$\widetilde{F_0}(u) = \int_\Omega [|u| + |Du| + f(D^2u) + lu]\,dx + 2c_\infty \int_\Sigma \frac{|u^-|}{h}\,d\mathcal{H}_{n-1}(\sigma).$$

Also, one may easily study the case when f depends on x, provided

$$|f(x,\xi) - f(y,\xi)| \le \omega(|x-y|)(1 + |\xi|)$$

with $\omega(t) \to 0$ as $t \to 0$: the result is unchanged, except in that c_∞ depends on σ, so the limit is

$$\int_\Omega [|u| + |Du| + f(D^2u) + lu]\,dx + 2\int_\Sigma c_\infty(\sigma)|u^-|\,d\mathcal{H}_{n-1}(\sigma).$$

We may also drop the restrictive assumption of radial symmetry: to this aim, remark that (except for the obvious substitution of $f_\infty(\xi)$ for $c_\infty|\xi|$)

the only place where it was used is formula (3); define for all $z \in \mathbb{R}$ and $\sigma \in \Sigma$

$$f_0(\sigma, z) = \min_{A\in\mathbb{R}^{n\times n}} \{f(A - \langle A\nu(\sigma), \nu(\sigma)\rangle\, \nu(\sigma) \otimes \nu(\sigma) + z\,\nu(\sigma) \otimes \nu(\sigma))\}.$$

The function f_0 is strictly convex for every σ,

$$|z| \le f_0(\sigma, z) \le c(1 + |z|),$$

and it is as smooth with respect to σ as $\nu(\sigma)$. Moreover clearly $f(A) \ge f_0(\sigma, \langle A\nu, \nu\rangle)$: thus in (4)

$$\varepsilon \int_{\Sigma_\varepsilon} f(D^2 u_\varepsilon)\, dx \ge \varepsilon \int_{\Sigma_\varepsilon} f_0(\sigma(x), d_\nu d_\nu u_\varepsilon)\, dx.$$

Accordingly, problem (5) must be changed into

$$\min\Bigl\{\int_0^\varepsilon f_0\bigl(\sigma, v''(t)\bigr)\, dt + c_\infty\bigl(|v'(0)| + |v'(\varepsilon)|\bigr)\Bigr\},$$

and $\psi(\sigma, z)$ will be for each σ the inverse function of $f_0'(\sigma, z)$. The estimates on a_ε, b_ε still hold, and again $v''_{\sigma,\varepsilon}$ is bounded uniformly with respect to ε. The rest of the proof is unchanged.

Finally, the various remarks above may be put together to obtain formally more complex problems, whose difficulty may be increased by loosening the regularity of Σ, g, h, f, or by taking a more involved integrand.

4. References

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Authors' addresses

Emilio Acerbi
Dipartimento di Matematica
Politecnico
Torino, Italy

Danilo Percivale
S.I.S.S.A.
Strada Costiera, 11
Trieste, Italy